COMP307/AIML420 INTRODUCTION TO ARTIFICIAL INTELLIGENCE

Tutorial on probability

Independence of Events

- Let A and B be events
- For independence any of the following is sufficient:
  - $P(A|B) = P(A)$
  - $P(B|A) = P(B)$
  - $P(A, B) = P(A)\,P(B)$
- Show that $P(B|A) = P(B)$ if $P(A|B) = P(A)$ is true:
  - $P(B|A) = P(A|B)\frac{P(B)}{P(A)} = P(A)\frac{P(B)}{P(A)} = P(B)$
- Show that $P(B|A) = P(B)$ if $P(A, B) = P(A)P(B)$ is true:
  - $P(B|A) = \frac{P(A,B)}{P(A)} = \frac{P(A)\,P(B)}{P(A)} = P(B)$

Independence Example 1

$\boldsymbol{X}$	$\boldsymbol{Y}$	$P(\boldsymbol{X}, \boldsymbol{Y})$
0	0	0.1
0	1	0.2
1	0	0.3
1	1	0.4

- X and Y are random variables
- Are events $X = 0$ and $Y = 1$ independent?

Independence Example 1

X	Y	$P(X,Y)$
0	0	0.1
0	1	0.2
1	0	0.3
1	1	0.4

- X and Y are random variables
- Are events $X = 0$ and $Y = 1$ independent?
- $P(X = 0) = \sum_y P(X = 0, Y = y) = 0.3$ (sum rule)
- $P(X = 0 \mid Y = 1) = P(X = 0, Y = 1) / P(Y = 1) = 0.2 / 0.6 = 1/3$ (product rule)
- No, not independent (but not so far off).

Independence Example 2

X	Y	$P(X,Y)$
0	0	0.1
0	1	0.1
1	0	0.4
1	1	0.4

- X and Y are random variables
- Are events $X = 0$ and $Y = 1$ independent?
- $P(X = 0) = \sum_y P(X = 0, Y = y) = 0.2$ (sum rule)
- $P(X = 0 \mid Y = 1) = P(X = 0, Y = 1) / P(Y = 1) = 0.1 / 0.5 = 0.2$ (product rule)
- Yes, independent

Independence Example 2

$\boldsymbol{X}$	$\boldsymbol{Y}$	$P(\boldsymbol{X}, \boldsymbol{Y})$
0	0	0.1
0	1	0.1
1	0	0.4
1	1	0.4

- X and Y are random variables
- Are events $X = 1$ and $Y = 0$ independent?
- $P(X = 1)$ = 0.8
- $P(X = 1 \mid Y = 0) = P(X = 1, Y = 0) / P(Y = 0) = 0.4 / 0.5 = 0.8$
- Yes, independent

Independence of *Random Variables*

- (Not discussed in class, but useful)
- The independence of random variables requires that $P(X = x, Y = y) = p(X = x)P(Y = y)$ for all possible combinations of x and y
- Very commonly used
- In example 2 the random variables are independent:

X	Y	$P(X,Y)$
0	0	0.1
0	1	0.1
1	0	0.4
1	1	0.4

X	$P(X)$
0	0.2
1	0.8

Y	$P(Y)$
0	0.5
1	0.5

Independence Example 3

X	Y	$P(X, Y)$
0	0	0.05
0	1	0.05
1	0	0.15
1	1	0.25
2	0	0.2
2	1	0.3

- X and Y are random variables
- $P(X = 1) = 0.4$
- $P(X = 1 \mid Y = 0) = P(X = 1, Y = 0) / P(Y = 0) = 0.15 / 0.4 = 3/8$
- Events $P(X = 1)$ and $P(Y = 0)$ are *not* independent
- $P(X = 2) = 0.5$
- $P (X = 2 \mid Y = 0) = P(X = 2, Y = 0) / P(Y = 0) = 0.2 / 0.4 = 0.5$
- Events $P(X = 2)$ and $P(Y = 0)$ are independent
- Random variables X and Y are not independent

Three Events

- Let A, B and C be events
- Note that {event A happens *or* event B happens} is just another event (we can call it event D)
- Hence the rules for two variables carry over

- Conditional independence is particularly relevant:
  - Let A and B be conditionally independent given C:
    - $P(A, B \,|C) = P(A \,|C)P(B|C)$
    - $P(A, B, C) = P(A, B|C)P(C) = P(A|C)P(B|C)\, P(C)$
    - Does *not* imply $P(A, B) \; = \; P(A)P(B)$

Conditional Independence Example

X	Y	Z	$P(X,Y,Z)$
0	0	0	0.05
0	0	1	0.05
0	1	0	0.2
0	1	1	0.2
1	0	0	0.1
1	0	1	0.1
1	1	0	0.2
1	1	1	0.1

Y	$P(Y\|X=0)$
0	0.2
1	0.8

Z	$P(Z\|X=0)$
0	0.5
1	0.5

X	$P(X)$
0	0.5
1	0.5

- For the example (for any y and z)

  $$P(Y=y, Z=z|X=0) = P(Y=y|X=0)P(Z=z|X=0)$$

  - Events Y and Z independent conditional on event $X=0$ happening
- But not independent conditional on event $X=1$:
  - $P(Y,Z|X=1) \neq P(Y|X=1)P(Z|X=1)$

    P(Y=1| X=1) =P(Y=1, X=1)/P(x=1) = 0.3 / 0.5 = 0.6

    P(Y=1| Z=0, X=1) = P(Y=1, Z=0|X=1) / P(Z=0|X =1) =

    (P(Y=1. Z=0, X=1)/ P(X=1))/ P(Z=0, X=1) / P(X=1)) =

    (P(Y=1, Z=0, X=1) / p(Z=0, X=1) = 0.2/0.3 = 2/3
  - $P(Y,Z) \neq P(Y)P(Z)$

Slides with examples of probability theory

- Many courses use the same examples
- Some useful introductory-level slides are
  - University of Chicago
    - Good introduction and nice examples
    - We discuss some examples; slides 17-23, 24-27, 30-37
  - Stony Brook
    - Nice basic material and good random variable examples
    - We briefly go through all slides, emphasing random variables
  - Stanford
    - Nice motivation; looks a bit more intimidating.
  - Illinois
    - Different examples
    - Links to decision theory, utility theory
  - Trinity College
    - A nice example of conditional independence, such independence is relevant for Bayesian networks later on