EEEN203
Circuit Analysis
Systems Analysis using the Laplace Transform

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Revision 98

Impedance of an inductor in the s-domain

Recall that an for an inductor we have

$$v(t) = L = \frac{\mathrm{d}i}{\mathrm{d}t}$$
$$\implies V(s) = sLI(s) - Li(0^-)$$

Consider for a moment the case where $i(0^-) = 0$. We then have

$$\frac{V(s)}{I(s)} = Z(s) = sL$$

This looks just like our previous treatment, where we had $Z(\mathrm{j}\omega) = \mathrm{j}\omega L$, but with $s = \mathrm{j}\omega$. That is, we have generalised our previous method that dealt with sinusoidal frequencies to one that can cope with a wider range of signals.

Inductor Initial conditions

However, remember that we have the additional term arising from the initial condition. This produces an extra voltage in series with the inductor, having magnitude $Li(0^-)$.

So, while we were previously content to replace an inductor with an impedance of sL, we now also need to add a voltage source of $-Li(0^-)$ in *series* with the inductor.

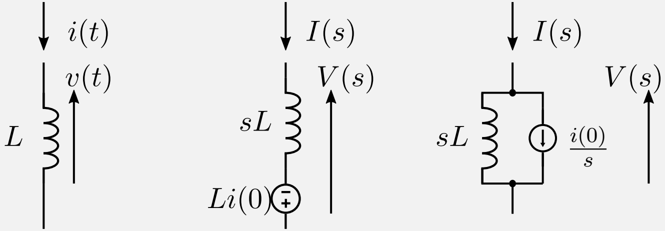
Inductor Initial conditions

In some circumstances you might prefer to represent an inductors's initial condition by a current source, rather than a voltage source. We can see how to do this by returning to $V(s) = sLI(s) - Li(0^-)$ and rearranging.

$$\begin{aligned} V(s) &= sLI(s) - Li(0^-) \\ \implies sLI(s) &= V(s) + Li(0^-) \\ I(s) &= \frac{V(s)}{sL} + \frac{i(0^-)}{s} \end{aligned}$$

That is, the inductor can be represented in the s-domain as an inductor with impedance sL in *parallel* with a current source with magnitude $\frac{i(0^-)}{s}$.

Inductor Initial conditions



Impedance of a capacitor in the s-domain

Similarly, for a capacitor,

$$i(t) = C\frac{\mathrm{d}v}{\mathrm{d}t} \Longleftrightarrow I(s) = sCV(s) - Cv(0^-)$$

Thus for a capacitor with initial conditions we need to add a *parallel* current source with magnitude $-Cv(0^-)$.

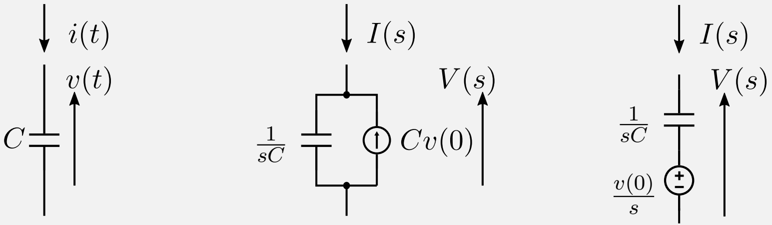
We would often prefer to deal with the initial voltage using a voltage source. We can rearrange to find

$$V(s) = \frac{I(s)}{sC} - \frac{v(0^-)}{s}$$

That is the capacitor's initial condition can be represented by a series voltage source with magnitude $-\frac{v(0^-)}{s}$.

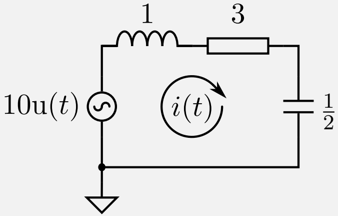
Notice that both inductors and capacitors behave sensibly in the limit of zero initial conditions (ie, a series voltage source becomes zero volts, and a parallel current source goes to zero amps.)

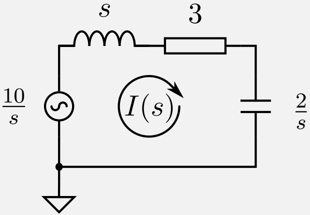
Capacitor Initial conditions



Example One – Zero initial conditions

Consider the RLC circuit shown in the figure. Let's find the current flowing through the circuit, if we assume zero initial conditions. We begin by redrawing the circuit in the s-domain.





We can write the total impedance of the loop as

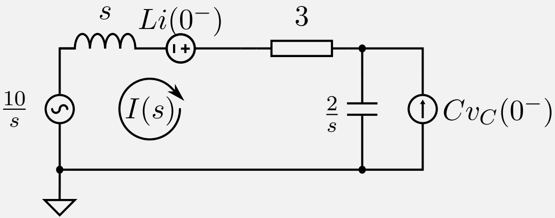
$$Z(s) = 3 + s + \frac{2}{s} = \frac{s^2 + 3s + 2}{s}$$

Example One – Zero initial conditions

$$\text{Hence, } I(s) = \frac{V(s)}{Z(s)}$$
$$= \frac{10}{s}\frac{s}{s^2+3s+2}$$
$$= \frac{10}{(s+1)(s+2)}$$
$$= \frac{10}{s+1} - \frac{10}{s+2}$$
$$\rightsquigarrow i(t) = 10\left[\mathrm{e}^{-t} - \mathrm{e}^{-2t}\right]\mathrm{u}(t)$$

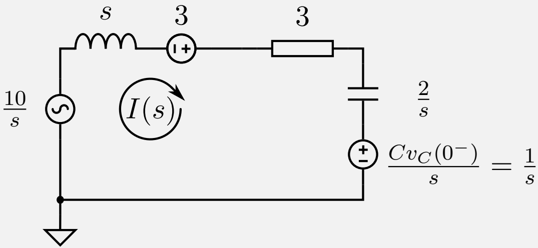
Example Two – Nonzero initial conditions

Let's reconsider the same circuit, but with an initial inductor current of 3 A and an initial capacitor voltage of 1 V. This produces a voltage source of $L = 1 \times 3$ V and $\frac{1}{2} \times 1$ A respectively.



When calculating the current we will also need to consider the extra current source. This is awkward, so we will convert the capacitor and current source combination to a capacitor and voltage source combination. This is simply a generalisation of the Norton to Thévenin conversion that we have previously seen.

Example Two – Nonzero initial conditions

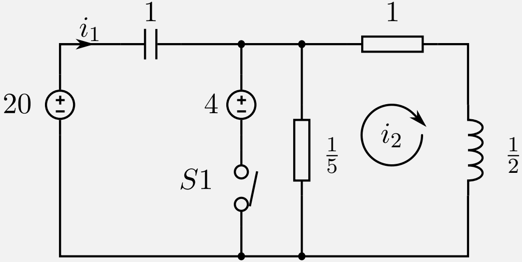


$$
\begin{aligned}
I(s) &= \left(\frac{10}{s} + 3 - \frac{1}{s}\right)\frac{s}{(s+1)(s+2)} \\
&= \frac{9+3s}{(s+1)(s+2)} \\
&= \frac{6}{s+1} - \frac{3}{s+2} \\
\rightsquigarrow i(t) &= \left[6\mathrm{e}^{-t} - 3\mathrm{e}^{-2t}\right]\mathrm{u}(t)
\end{aligned}
$$

Notice that $i(0) = 3$ A, so our solution matches the initial condition.

Example Three – Multiple Loops

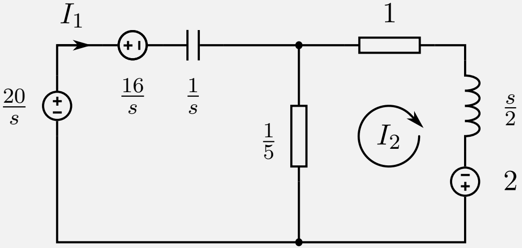
Consider the circuit shown in the figure. The switch $S1$ is intially closed for a sufficiently long time that the circuit reaches steady state. The switch is then opened at $t = 0$. Calculate the two currents $i_1(t)$ and $i_2(t)$.



Let's begin by considering the initial conditions. When $S1$ is closed, 4V is applied across the series combination of $1\ \Omega$ and 0.5 H. The inductor is a short at dc, so we will have $L(0^-) = i_2(0^-) = 4$ A.

When the switch is closed we have 16 V across the capacitor, so $v_C(0^-) = 16$ V.

Let's redraw the circuit in the s-domain, including only those components that matter for $t > 0$ and including the sources modelling the initial conditions. The diagram already includes the effect of converting the capacitor initial condition to a series voltage source.



We can write equations for the loop currents.

$$\begin{cases} I_1\left(\frac{1}{s}+\frac{1}{5}\right) - I_2\frac{1}{5} & = \frac{4}{s} \\ I_2\left(\frac{6}{5}+\frac{s}{2}\right) - I_1\frac{1}{5} & = 2 \end{cases}$$

Example Three – Multiple Loops

We can write this in a matrix form

$$\begin{bmatrix} \frac{1}{s} + \frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{6}{5} + \frac{s}{2} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{4}{s} \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{5+s}{5s} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{12+5s}{10} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{4}{s} \\ 2 \end{bmatrix}$$

and solve using Cramer's rule.

$$I_1(s) = \frac{\begin{vmatrix} \frac{4}{s} & -\frac{1}{5} \\ 2 & \frac{12+5s}{10} \end{vmatrix}}{\begin{vmatrix} \frac{5+s}{5s} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{12+5s}{10} \end{vmatrix}} \qquad I_2(s) = \frac{\begin{vmatrix} \frac{5+s}{5s} & \frac{4}{2} \\ -\frac{1}{5} & 2 \end{vmatrix}}{\begin{vmatrix} \frac{5+s}{5s} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{12+5s}{10} \end{vmatrix}}$$

Example Three – Multiple Loops

$$\begin{cases} I_1(s) & = \dfrac{24(s+2)}{s^2+7s+12} = \dfrac{24(s+2)}{(s+3)(s+4)} = \dfrac{-24}{s+3} + \dfrac{48}{s+4} \\ I_2(s) & = \dfrac{4(s+7)}{s^2+7s+12} = \dfrac{4(s+7)}{(s+3)(s+4)} = \dfrac{16}{s+3} - \dfrac{12}{s+4} \end{cases}$$

$$\rightsquigarrow \begin{cases} i_1(t) & = \left(-24\mathrm{e}^{-3t} + 48\mathrm{e}^{-4t}\right)\mathrm{u}(t) \\ i_2(t) & = \left(16\mathrm{e}^{-3t} + 12\mathrm{e}^{-4t}\right)\mathrm{u}(t) \end{cases}$$

Writing transfer functions in the s-domain

Just as we were previously able to write transfer functions directly with $\mathrm{j}\omega$, we can now do so using s.

Remember that transfer functions are *by definition* taken with no initial conditions, so this is an identical procedure to that we have used before, but now with no need to carry j around.

Impulse Response

An extremely useful tool for characterising circuits (or other systems) is the impulse response. We simply apply an impulse to a system and observe the output.

Consider a system having transfer function $G(s)$. We drive the system with an input $x(t) = \delta(t)$ and look at the output $y(t)$.

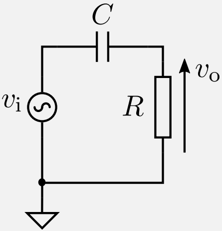
$$\begin{aligned} Y(s) &= G(s)X(s) \\ &= G(s)\mathcal{L}\left\{\delta(t)\right\} \\ &= G(s) \\ \implies y(t) &= g(t) \end{aligned}$$

That is, the impulse response of a system is the inverse Laplace transform of its transfer function.

Impulse Response $\quad g(t) \Longleftrightarrow G(s) \quad$ Transfer function

Impulse response of a high pass filter

Consider the RC high pass filter shown in the figure. We wish to find the transfer function and the impulse response. Let's find the transfer function directly by using the voltage divider rule.



$$G(s) := \frac{V_\mathrm{o}}{V_\mathrm{i}} = \frac{R}{R + \frac{1}{sC}} = \frac{s}{s + \frac{1}{RC}}$$

So if we apply an input $v_\mathrm{i} = \delta(t) \implies V_\mathrm{i} = 1$, we get

$$V_\mathrm{o} = \frac{s}{s + \frac{1}{RC}} V_\mathrm{i} = \frac{s + \frac{1}{RC}}{s + \frac{1}{RC}} - \frac{\frac{1}{RC}}{s + \frac{1}{RC}} = 1 - \frac{1}{RC}\frac{1}{s + \frac{1}{RC}}$$

$$v_\mathrm{o} = \delta(t) - \frac{1}{RC}\mathrm{e}^{-\frac{t}{RC}}\mathrm{u}(t)$$

LPF Step response from impulse response

A low pass filter has impulse response $g(t) = a\mathrm{e}^{-at} \quad a \in \mathbb{R}_{++}$. For example, $a = \frac{1}{RC}$ for a passive RC filter.

What is the filter's step response $y(t)$? Now, the transfer function $G(s)$ is $\mathcal{L}\left\{g(t)\right\}$.

$$
\begin{aligned}
G(s) &= \frac{a}{s+a} \\
\implies Y(s) &= G(s)\mathcal{L}\left\{\mathrm{u}(t)\right\} \\
&= \frac{a}{s(s+a)} \\
&= \frac{1}{s} - \frac{1}{s+a} \qquad \text{Using partial fractions expansion.} \\
y(t) &= \left(1 - \mathrm{e}^{-at}\right)\mathrm{u}(t)
\end{aligned}
$$