School of Mathematics and Statistics
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ENGR 123 **Assignment 3**

1 For each relation on the given set, determine whether it is reflexive, symmetric, antisymmetric and/or transitive.

In each case where the relation doesn't have a property, give a short reason why or an example e.g. it isn't reflexive because ...

(a) On the set of people, $(a, b) \in R$ iff a and b share a parent in common. *(4 marks)*

Solution: Reflexive, Symmetric.
It is not anti-symmetric, being siblings doesn't make you the same.
It is not transitive, could share different parents.

(b) On the set of 2×2 matrices, $(a, b) \in R$ iff a and b have different top-left entries (so they differ in row one, column one). *(4 marks)*

Solution: Symmetric.
It is not reflexive, the same matrix will have identical entries everywhere!
It is not anti-symmetric, for example I_2 and the zero matrix.
It is not transitive, for example I_2, the zero matrix and I_2 again.

2 Give a direct proof that shows if x is odd, then x^2 is odd. *(4 marks)*

Solution: Suppose x is odd.
Then $x = 2k + 1$ for some $k \in \mathbb{Z}$. It follows that

$$\begin{aligned} x^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

So $x^2 = 2m + 1$ (where $m = 2k^2 + 2k$), hence x^2 is odd.

3 For the equivalence relation on $\{0, 1, 2, 3, 4\}$ defined as

$$\{(x, y) : x - y \text{ is a multiple of } 3\}$$

(a) Write it out as a list of ordered pairs *(2 marks)*

Solution: The relation as a list of ordered pairs is

$$\{(0, 0), (0, 3), (3, 0), (3, 3), (1, 1), (1, 4), (4, 1), (4, 4), (2, 2)\}$$

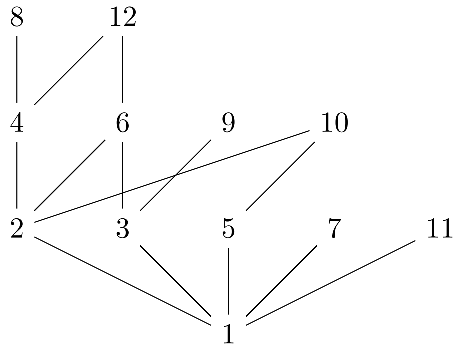
(b) List the cells in the associated partition. *(2 marks)*

Solution: The cells are

$$\{0, 3\}, \{1, 4\} \text{ and } \{2\}$$

4 Let $A = \{1, 2, \ldots, 12\})$. Draw the Hasse diagram of the divisibility relation on A i.e. (x, y) iff x is a factor of y iff y is a multiple of x. *(2 marks)*

Solution:



5 Use induction to show that $\frac{n!}{2^n} > 1$ for all $n \geq 4$. *(6 marks)*

Solution:

Base case $(n = 4)$: $\frac{4!}{2^4} = \frac{24}{16} = \frac{3}{2} > 1$

Induction hypothesis $(n = k)$: Assume that $\frac{k!}{2^k} > 1$ for some $k \geq 4$.

Inductive step $(n = k + 1)$: Start with the left-hand side

$$\begin{aligned} \frac{(k+1)!}{2^{k+1}} &= \frac{(k+1)k!}{2 \cdot 2^k} && \text{defns} \\ &= \frac{k+1}{2} \cdot \frac{k!}{2^k} && \text{algebra} \\ &> \frac{k+1}{2} \cdot 1 && \text{IH} \\ &\geq 1 && \text{for } k \geq 1 \end{aligned}$$

which is the desired result!

6 Consider the following pseudocode:

Algorithm 1: ReverserHelper(L, S)

```
input: sequences L = [l_1, ..., l_m] and S = [s_1, ..., s_n]
if (Length(L) = 0) then
    return S
L' = [l_2, ..., l_m]
S' = [l_1, s_1, ..., s_n]
return ReverserHelper(L', S')
```

(a) Use induction on $\mathtt{m}$ to show that *(4 marks)*

$$\mathtt{ReverserHelper}([\mathtt{l}_1,\ldots,\mathtt{l}_\mathtt{m}],[\mathtt{s}_1,\ldots,\mathtt{s}_\mathtt{n}]) = [\mathtt{l}_\mathtt{m},\mathtt{l}_{\mathtt{m}-1},\ldots,\mathtt{l}_1,\mathtt{s}_1,\mathtt{s}_2,\ldots,\mathtt{s}_\mathtt{n}]$$

for any sequences $[\mathtt{l}_1,\ldots,\mathtt{l}_\mathtt{m}]$ and $[\mathtt{s}_1,\ldots,\mathtt{s}_\mathtt{n}]$

Solution:

By induction on $\mathtt{m}$, the length of the first list.

Base case: If $\mathtt{m} = 1$ then $\mathtt{ReverserHelper}([\mathtt{l}_1],[\mathtt{s}_1,\ldots,\mathtt{s}_\mathtt{n}])=[\mathtt{l}_1,\mathtt{s}_1,\ldots,\mathtt{s}_\mathtt{n}]$ (check this yourself using the algorithm!)

Inductive hypothesis: Assume algorithm works for length($\mathtt{L}$) $\leq \mathtt{m}$

Inductive step: Check when $\mathtt{L} = [\mathtt{l}_1,\ldots,\mathtt{l}_\mathtt{m},\mathtt{l}_{\mathtt{m}+1}]$.

$$\begin{aligned}&\mathtt{ReverserHelper}([\mathtt{l}_1,\mathtt{l}_2,\ldots,\mathtt{l}_\mathtt{m},\mathtt{l}_{\mathtt{m}+1}],[\mathtt{s}_1,\ldots,\mathtt{s}_\mathtt{n}])\\ &= \mathtt{ReverserHelper}(\underbrace{[\mathtt{l}_2,\ldots,\mathtt{l}_\mathtt{m},\mathtt{l}_{\mathtt{m}+1}]}_{\mathtt{L}'},\underbrace{[\mathtt{l}_1,\mathtt{s}_1,\ldots,\mathtt{s}_\mathtt{n}]}_{\mathtt{S}'})\end{aligned}$$

by design of the algorithm, where length($\mathtt{L}'$) $= \mathtt{m}$. By the inductive hypothesis, (in particular since the length of $\mathtt{L}'$ is $\mathtt{m}$, it follows that

$$\mathtt{ReverserHelper}(\mathtt{L}',\mathtt{S}') = [\mathtt{l}_{\mathtt{m}+1},\mathtt{l}_\mathtt{m},\ldots,\mathtt{l}_2,\mathtt{l}_1,\mathtt{s}_1,\ldots,\mathtt{s}_\mathtt{n}]$$

as required. QED

(b) Explain how to use the previous result to show that if $\mathtt{L} = [\mathtt{l}_1,\ldots,\mathtt{l}_\mathtt{m}]$ and

$\mathtt{Reverse(L)} := \mathtt{ReverseHelper}(\mathtt{L},[\,])$

then $\mathtt{Reverse(L)}$ returns $[\mathtt{l}_\mathtt{m},\ldots,\mathtt{l}_1]$. *(2 marks)*

Solution:

Follows immediately by choosing $\mathtt{S} = []$ to be the empty string.