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ENGR 123 **Tutorial 4 extras**

1 Consider the following pseudocode:

Algorithm 1: `Average(A)`

```
  input: sequence A = (a_1, ..., a_n) of n distinct numbers, where n is a power of two
1 if (n = 1) then
2 |   return a_1
3 else
4 |   c = X(a_1, ..., a_{n/2})
5 |   d = X(a_{n/2+1}, ..., a_n)
6 |   return (c + d)/2
```

(a) What does algorithm `Average(A)` do?

Solution: `X(A)` finds the average of a sequence $\mathtt{A} = (\mathtt{a}_1, \ldots, \mathtt{a}_\mathtt{n})$.

(b) Let $E(n)$ be the number of arithmetic operations `Average()` performs on a list of length n (where $n = 2^k$ is a power of two). Write down a formula for $E(n)$ and prove your answer using induction.

Solution:

Theorem. $\mathtt{E}(\mathtt{n}) = 2\mathtt{n} - 2$.

Proof. By induction on $\mathtt{n}$.

<u>Base case:</u> If $\mathtt{n} = 1$ then no arithmetic operations were performed, so $\mathtt{E}(1) = 0$.

This agrees with the formula, whose calculation states $2(1) - 2 = 0$ operations.

<u>Inductive hypothesis:</u> Assume true for $\mathtt{n} = 2^\mathtt{k}$.

That is, $E(n) = 2n - 2$

<u>Inductive step:</u> Check for $\mathtt{n} = 2^{\mathtt{k}+1}$.

$$\begin{aligned}
\mathtt{E}(2^{\mathtt{k}+1}) &= \underbrace{\mathtt{E}(2^\mathtt{k}))}_{\text{line 4}} + \underbrace{\mathtt{E}(2^\mathtt{k}))}_{\text{line 5}} + \underbrace{2}_{\text{line 6}} \\
&= 2 \times 2^\mathtt{k} - 2 + 2 \times 2^\mathtt{k} - 2 + 2 \\
&= 2^{\mathtt{k}+1} + 2^{\mathtt{k}+1} - 2 \\
&= 2 \times 2^{\mathtt{k}+1} - 2
\end{aligned}$$

as required.

(c) Is it possible to design a more efficient algorithm that performs the same task (that is, one which uses fewer comparisons)? Why or why not?

Solution. Yes, just sum up the values and divide by the length of the sequence. That will take $(n-1)$ additions and 1 division, or n arithmetic operations in total.

2 Show by induction that $n(n+1)$ is even for all $n \geq 0$.

Base Case : $(n = 0)$
Evaluating, we see that $n(n+1) = 0(0+1) = 0$ which is even.

Induction Hypothesis : $(n = k)$
We assume that $k(k+1)$ is even.

That is, $k(k+1) = 2t$ for some $t \in \mathbb{Z}$.

Induction Step : $(n = k+1)$
We'll try to show that $(k+1)(k+1+1)$ is even.

$$\begin{aligned}
(k+1)(k+2) &= (k+1)k + 2(k+1) && \text{expand} \\
&= 2t + 2(k+1) && \text{by IH} \\
&= 2(t+k+1) && \text{factorise}
\end{aligned}$$

Which is a multiple of 2, so this is even!