School of Mathematics and Statistics
Te Kura Mātai Tatauranga

ENGR 123 **Tutorial 4**

1 *Truth tables*
Construct truth tables for the following propositions, and determine whether each is a tautology, contradiction, or contingent:

(a) $\neg Q \wedge (P \to Q)$

Solution: Construct the truth table below

P	Q	$P \to Q$	$\neg Q$	$\neg\mathbf{Q} \wedge (\mathbf{P} \to \mathbf{Q})$
0	0	1	1	**1**
0	1	1	0	**0**
1	0	0	1	**0**
1	1	1	0	**0**

The proposition is a contingent since the final column, corresponding to the compound proposition $\neg Q \wedge (P \vee Q)$, contains both zeros and ones.

(b) $(P \to (Q \vee R)) \to (\neg Q \to (Q \to P))$

Solution: from the truth table below, we see that the compound proposition is a tautology.

P	Q	R	$P \to (Q \vee R)$	$\neg Q \to (Q \to P)$	$(\mathbf{P} \to (\mathbf{Q} \vee \mathbf{R})) \to (\neg\mathbf{Q} \to (\mathbf{Q} \to \mathbf{P}))$
0	0	0	1	1	**1**
0	0	1	1	1	**1**
0	1	0	1	1	**1**
0	1	1	1	1	**1**
1	0	0	0	1	**1**
1	0	1	1	1	**1**
1	1	0	1	1	**1**
1	1	1	1	1	**1**

2 *Translations*

- Using the propositional variable e to mean *she majors in engineering* and m to mean *she majors in maths*, and d to mean *she is doing a double major* at university, turn the following sentences into compound propositions:

  (a) She majors in either Engineering or Maths at university, but not both
  Solution: e xor m, which is $(e \wedge \neg m) \vee (\neg e \wedge m)$

  (b) She majors in neither Engineering nor Maths, but is working on a double major at university *(1 mark)*
  Solution: $d \wedge \neg(e \vee m)$

- Using the same variables, turn the following propositions into English:

(a) $e \to \neg d$ *(1 mark)*

Solution: Majoring in Engineering implies she is not doing a double major

(b) $\neg d \wedge m$ *(1 mark)*

Solution: Not working on a double major but is majoring in Maths

3 *Arguments*

(a) Rewrite the following statement and it negation formally. Rewrite the negation in English.

Let $T(x, y) =$ "x trusts y".

(i) Somebody trusts everybody.
Solution: $\exists x\, \forall y\, T(x, y)$.
The negation is $\forall x\ \exists y\ \neg T(x, y)$ or, equivalently in English, "everybody distrusts somebody."

(b) Rewrite the following arguments formally. State which are valid/invalid:

(i) All kiwis ride bicycles.
All bicycle riders eat curry.

All kiwis eat curry.

Solution:
$\forall x$ kiwi$(x) \to$ rides bicycles(x)
$\forall x$ rides bicycles$(x) \to$ eats curry(x) **Valid**

$\forall x$ kiwi$(x) \to$ eats curry(x)

(ii) All kokakos sing.
Sam sings.

Sam is a kokako.

Solution:
$\forall x$ kokako$(x) \to$ sings(x)
sings(Sam) **Invalid**

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kokako(Sam)

(iii) All kakapo are birds.
Max is a bird.

Max is a kakapo.

Solution:
$\forall x$ kakapo$(x) \to$ bird(x)
bird(Max) **Invalid**

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kakapo(Max)

(c) Rewrite the following argument to show that the conclusion follows logically. That is, reorder the premises, and rewrite statements as "if-then's" or contrapositives where necessary.

1. No students who can't do maths are in ENGR123.
2. All hard-working students can think logically.
3. Only ENGR123 students code.
4. Only students who can't do maths are lazy.

Only students who think logically can code.

Solution:

Let $s =$ "students"

3. $\forall s \neg$ engr$(s) \to \neg$ code (s)
cp. $\forall s$ code$(s) \to$engr(s)
1. $\forall s$ engr$(s) \to$maths (s)
4. $\forall s$ maths$(s) \to \neg$ lazy (s)
2. $\forall s \neg$ lazy$(s) \to$ logic (s)

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cp. $\forall s \neg$ logic$(s) \to \neg$ code (s)

4 *Relations*
Let C be the set of all cities in New Zealand. Determine whether the relation R on C is reflexive, symmetric, antisymmmetric and/or transitive where (a, b) in R iff

(a) there is no road between city a and city b

Solution: symmetric.
It isn't reflexive, there can be a road from city x to itself.
It isn't antireflexive, two different cities can have roads between each other.
It isn't transitive, consider a bypass.

(b) everyone who has visited city a has also visited city b

Solution: reflexive; transitive
It isn't symmetric, it is possible that it only holds in one direction, everyone who visited city x also visited city y, but some of those that visited city y didn't visit city x.
It isn't antisymmetric, it is possible that the cities are different, but everyone who has visited one also visited the other and vice-versa.

(c) there is a highway with interchanges to both city a and city b

Solution: symmetric
It isn't reflexive, the city may not be off a highway interchange.
It isn't antisymmetric, two different cities could share the same highway interchange.
It isn't transitive, think of interchanges on either side of city y, here x and y share an interchange, as do y and z, but x and z don't.

(d) there is at least one road between city a and city b

Solution: reflexive (if all city have roads!); symmetric
It isn't antisymmetric, different cities can have roads between them. It isn't transitive, to get from x to z, you may have to go through y.

In each case, for the properties that don't hold, explain why e.g. it isn't symmetric because ...

5 *Equivalence relations*
Let R be the relation on the set of people who have visited a particular city such that xRy iff person x and person y left the city using the same road. Show that R is an equivalence relation.

Solution:

<u>reflexive:</u> xRx because person used same road as self

<u>symmetric:</u> if x and y used same road then so did y and x

<u>transitive:</u> easy

6 *Partitions*
Which of the following are partitions of the set of all students at Victoria?

(a) the set of students whose family lives in Wellington, the set of students whose family lives elsewhere in New Zealand, and the set of students whose family lives overseas.

Solution: Not a partition. Family could be split between the places, some in Wellington, some elsewhere etc ....

(b) the set of students who can speak a foreign language, the set of students who can play a musical instrument, the set of students who can't do either.

Solution: Not a partition. First two subsets are not disjoint.

(c) the set of students who are New Zealanders, and the set of students who are not.

Solution: Is a partition. All people are in one group, and only one. I would argue that citizens with dual (NZ) passports are New Zealanders, with extra options!

(d) the set of students who study engineering, the set of students who study science, the set of students who study arts.

Solution: Not a partition. Students studying e.g, law, are not in any subset.

7 *Proofs*
Claim: for all $n \in \mathbb{N}$, if $2n+1$ is a multiple of 3 then n^2+1 is a multiple of 3.

Incorrect Proof: by contrapositive. Assume that $2n+1$ is not a multiple of 3. It follows that

- If $n = 3k+1$ for $k \in \mathbb{N}$ then $n^2+1 = 9k^2+6k+2$ is not a multiple of 3.
- If $n = 3k+2$ for $k \in \mathbb{N}$ then $n^2+1 = 9k^2+12k+5$ is not a multiple of 3.
- If $n = 3k+3$ for $k \in \mathbb{N}$ then $n^2+1 = 9k^2+18k+10$ is not a multiple of 3.

In all cases, we have shown that n^2+1 is not a multiple of 3, and so we have proved the claim.

(a) Provide a counterexample to the claim.

Solution: Let $n = 1$. Then $2n+1 = 3$ is a multiple of 3, but $n^2+1 = 2$ is not a multiple of 3.

(b) Explain clearly where the "proof" made logical errors.

Solution: Contrapositive of $P \to Q$ is $\neg Q \to \neg P$. So "theorem" does not use contrapositive (it assumes $\neg P$ instead of $\neg Q$). Also, the third case ($n = 3k+3$) *is* a multiple of 3.

8 *More proofs*
Use a *proof by contradiction* to show that if x is rational and y is irrational, then $x+y$ is irrational.

Suppose the premise is true and the conclusion is false.

That is, assume $x = \frac{p}{q}$, y is irrational and $x+y$ is *not* irrational.

Then $x+y$ is rational, and $x+y = \frac{r}{s}$.

$$y = (x+y) - x = \frac{r}{s} - \frac{p}{q} = \frac{rq-ps}{sq}$$

which is a fraction, so y is rational.

But y is irrational.

That is the contradiction, y is both rational and irrational.

9 *Induction*
Use induction to show that $4n < 2^n$ for all $n \geq 5$.

Solution:

Base case ($n = 5$): $4 \times 5 = 20 < 2^5 = 32$

Inductive hypothesis ($n = k$): Assume that $4k < 2^k$ for some $k \geq 5$

Inductive step ($n = k+1$): Check that $4(k+1) < 2^{k+1}$.

Observe that

$$\begin{aligned}
2^{k+1} &= 2 \times 2^k && \text{by definition} \\
&> 2 \times 4k && \text{by inductive hypothesis} \\
&> 4k + 4 && \text{since } 8k > 4k+4 \text{ for } k \geq 5 \\
&= 4(k+1)
\end{aligned}$$