Logic Intro

Arguments and proofs

Definition: *Argument*

A mathematical argument is a sequence of statements. All statements but the final one are called premises (or assumptions or hypotheses). The final statement is the conclusion.

P_i

For premises $P_1, P_2, \dots P_n$ and conclusion C, we often write arguments in the following form:

C

$$\left.\begin{array}{l} P_1 \\ P_2 \\ \vdots \\ P_n \\ \hline C \end{array}\right\} \text{argument} \qquad [P_1 \wedge P_2 \wedge \dots \wedge P_n] \to C$$

where the premises and conclusion are separated by a horizontal bar to help keep the distinction in mind.

An argument is valid if the conclusion is true whenever the premises are true. Another word for a valid argument is a proof.

- In a valid argument the truth of the premises forces the truth of the conclusion.
- Given a valid argument, there is *no guarantee at all that the conclusion is true*. Before you can be sure that the conclusion is true you have to know that the premises are true.

Testing Validity

P	Q	$P \to Q$
1	0	0

Here is an easy way to test whether an argument *containing no quantifiers* is valid.

Construct a truth table for P_1, P_2, ..., P_n and C. If there is a row with $P_1, P_2, \ldots, P_n$ all 1 and C having 0, then the argument is invalid.
To see this, note that the row we found is a situation where all the premises are true, but the conclusion is false.
Otherwise the argument is valid.

$$[\underset{1}{P_1} \wedge \underset{1}{P_2} \wedge \cdots \wedge \underset{1}{P_n}] \to \underset{0}{C} \quad \text{invalid!}$$

If Alex got an A then she passed ENGR123.
Alex got an A.

Alex passed ENGR123.

More formally:

$$\begin{array}{l} P \rightarrow Q \\ P \\ \hline Q \end{array}$$

Valid!

P = "Alex got an A"

Q = "Alex passed Engr 123"

If Alex got an A then she passed ENGR123.
Alex got an A.

Alex passed ENGR123.

More formally:

$$\frac{\begin{array}{l} P \to Q \\ P \end{array}}{Q}$$

Valid!

$$([P \to Q] \land P) \to Q$$

If Alex got an A then she passed ENGR123.
Alex got an A.

Alex passed ENGR123.

More formally:

$$\begin{array}{l} P \to Q \\ P \\ \hline Q \end{array}$$

Valid!

modus ponens

If Alex got an A then she passed ENGR123.
Alex passed ENGR123.

Alex got an A.

More formally:

$$\frac{\begin{array}{l} P \to Q \\ Q \end{array}}{P}$$

Invalid!

If Alex got an A then she passed ENGR123.
Alex passed ENGR123.

Alex got an A.

More formally:

$$\begin{array}{l} P \to Q \\ Q \\ \hline P \end{array}$$

Invalid!

$$\begin{array}{ccccccccc} (& [P & \to & Q] & \wedge & Q &) & \to & P \\ & 0 & 1 & 1 & 1 & 1 & & \underline{0} & 0 \end{array}$$

If Alex got an A then she passed ENGR123.
Alex passed ENGR123.

Alex got an A.

More formally:

$$\frac{P \to Q \qquad Q}{P}$$

Invalid!

All human beings are mortal
Socrates is a human being

Socrates is mortal

Socrates

$P(x) =$ "x is a human being"

$Q(x) =$ "x is mortal"

More formally:

$\forall x, P(x) \to Q(x)$
$P(\text{Socrates})$

$Q(\text{Socrates})$

Valid!

All human beings are mortal
Socrates is a human being

Socrates is mortal

More formally:

$\forall x [P(x) \to Q(x)]$
$P(\text{Socrates})$

$Q(\text{Socrates})$

Valid!

All human beings are mortal
Socrates is a human being

Socrates is mortal

More formally:

$$\frac{\forall x,\ P(x) \to Q(x) \qquad P(\text{Socrates})}{Q(\text{Socrates})}$$

Valid!

Some rules of inference

- *Modus ponens*

$$\begin{array}{l} P \\ P \to Q \\ \hline Q \end{array}$$

- *Modus tollens*

$$\begin{array}{l} P \to Q \\ \neg Q \\ \hline \neg P \end{array}$$

$\equiv \neg Q \to \neg P$ Contrapositive

- ~~*Or-elimination*~~

$$\cancel{\begin{array}{l} P \vee Q \\ \neg P \\ \hline Q \end{array}}$$

- *And-elimination*

$$\begin{array}{l} P \wedge Q \\ \hline P \end{array}$$

Some rules of inference

- *Transitivity*

$$\frac{P \rightarrow Q \qquad Q \rightarrow R}{P \rightarrow R}$$

- ~~*Or-introduction*~~

$$\cancel{\frac{P}{P \vee Q}}$$

- *Contrapositive*

$$\frac{P \rightarrow Q}{\neg Q \rightarrow \neg P}$$

- ~~*Implies-introduction*~~

$$\cancel{\frac{Q}{P \rightarrow Q}}$$

e.g.

$$\frac{P \rightarrow Q \qquad \neg R \rightarrow \neg Q}{P \rightarrow R} \equiv \neg R \rightarrow \neg P$$

$$(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$$

Lewis Carroll

The king of these logical deductions was Lewis Carroll (actual name Charles Dodgson) who wrote a book called "Symbolic Logic" and had lots of fun examples of varying degrees of complexity.

$$\frac{\begin{array}{l}\text{All lions are fierce}\\ \text{Some lions do not drink coffee}\end{array}}{\text{Some fierce creatures do not drink coffee}}$$

Valid!

$$\frac{\begin{array}{l}\text{Some pigs can fly}\\ \text{Only birds can fly}\end{array}}{\text{Some birds are pigs}}$$

Valid!