Victoria University of Wellington
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MATH 161 — Discrete Mathematics

8. Mathematical Induction

Induction is one of the most powerful proof techniques in the whole of mathematics. In fact you already understand induction. The difficulty students have with induction is more to do with understanding the formalism. That gets sorted out with practice. The moral is: *practice lots of examples.*

Scientific Induction. Before we begin mathematical induction it's worth starting with another type of induction that we use in day-to-day life and forms part of scientific reasoning. The idea is that if something has happened regularly in the past, then we believe that it will continue to do so in the future. Here is an example.

The sun has risen every morning since the earth began. Therefore the sun will rise tomorrow.

We use that type of induction all the time. But there are no guarantees. Consider the inductive chicken. Every morning the chicken hears the farmer and believes that she is going to be fed corn. This has always worked in the past. But one fateful day ...

Mathematical Induction. This type of induction is subtly different. In mathematics we can sometimes *prove* that if something works on a given instance, then it will work on the next instance. This means that if it works in the beginning, it will work forever.

Sound confusing? Consider an example. When only eight years old, the great German mathematician **Karl Gauss** worked out how to quickly add the numbers from 1 to 100. It turns out that

$$1 + 2 + 3 + \cdots + 100 = 5050.$$

How did he do it? In fact he discovered the following general rule.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

It turns out that this rule works for any natural number n. But how can we prove that? Lets check some. $1 = \frac{1(1+1)}{2}$, $1+2 = \frac{2(2+1)}{2} = 3$, $1+2+3 = \frac{3(3+1)}{2} = 6$, $1+2+3+4 = \frac{4(4+1)}{2}$, ...

☹ Boring, boring. That's never going to work. There are infinitely many numbers. You'll never be able to check them all. ☹

That does seem to be a good point. A much better way is to use mathematical induction. The key idea is to show that if the formula works for some natural number k, then it has to work for the next natural number. OK here we go.

We *assume* that, for some number k, we have

$$1+2+3+\cdots+k = \frac{k(k+1)}{2}.$$

Now consider the next number which is $k+1$. We need to prove that if the above works, then the formula also works for $k+1$. In other words we need to prove that

$$1+2+3+\cdots+k+(k+1) = \frac{(k+1)((k+1)+1)}{2},$$

after simplifying, we see that we need to prove that

$$1+2+3+\cdots+k+(k+1) = \frac{(k+1)(k+2)}{2}$$

All right here goes. We are assuming that $1+2+\cdots+k = \frac{k(k+1)}{2}$. Hence we can say

$$\begin{aligned}
1+2+\cdots+k+(k+1) &= (1+2+\cdots+k)+k+1 \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1)+2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}.
\end{aligned}$$

We have proved that if the formula works for the instance k, then it also works for the next instance $k+1$. Now we have set up a chain reaction. Our instance k can be any instance. So if it works for k, it will will work for $k+1$, and then for $k+2$ and so on to infinity. Therefore it works forever.

☹ You've forgotten that chain reactions have to get started. ☹

Indeed. We need to show that it works for some small k to initiate the chain reaction. But we already showed it works for $k = 1$. So that sets off the reaction and it works for all natural numbers.

The General Setup. We start with a statement $P(n)$ that we wish to show is true for all natural numbers n. To do this, it suffices to do the following:

(a) **Base Step:** Show that $P(1)$ is true.
(b) **Induction Step:** Show that $P(k + 1)$ is true whenever $P(k)$ is true. In other words, **assume** that $P(k)$ is true, and use this assumption to **prove** that $P(k + 1)$ is true.

Using this style, we can do our previous example as follows. $P(n)$ is the statement that $1 + 2 + \cdots + n = n(n + 1)/2$.

For the base step, we have

$$P(1) \text{ is } 1 = \frac{1(1+1)}{2}$$

which is true.

For the induction step, we assume that $P(k)$ is true, that is, assume

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2} \qquad *$$

Now we need to show that $P(k + 1)$ is true. What is $P(k + 1)$? It says

$$1 + 2 + \cdots + k + (k + 1) = \frac{(k+1)((k+1)+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

Now we know what $P(k + 1)$ is, let's prove it.

$$1 + 2 + \cdots + k + (k + 1) = (1 + 2 + \cdots + k) + (k + 1)$$

Now for the crucial step. By the induction assumption, ie by $*$,

$$\begin{aligned}(1 + 2 + \cdots + k) + (k + 1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

We now know that our formula holds for all natural numbers. This is pretty awesome. Using induction we have managed to prove an *infinite number of statements*; one for every natural number.

Doing the same example twice was probably overkill, but there is no harm in that. To prove something by induction, you always need to do prove a *base case* and an *inductive step.*

The secret to learning how to prove things by induction is to practice lots of examples.

Inductive proofs don't just come in one type. Here is a different type. Recall that $a|b$ means that a *divides* b. Thus $3|6$ and $3|9$ etc.

Theorem 8.1. *For every natural number n, $6|(n^3+5n)$.*

Proof. Let $P(n)$ be the statement that $6|(n^3+5n)$. When $n=1$, $n^3+5n = 1+5\times 1 = 6$. Since $6|6$ we see that $P(1)$ is true.

Now assume that $P(k)$ is true, that is, assume that $6|(k^3+5k)$. To show that $P(k+1)$ is true we need to show that $6|((k+1)^3+5(k+1))$. Now,

$$\begin{aligned}(k+1)^3+5(k+1) &= k^3+3k^2+3k+1+5k+5\\ &= (k^3+5k)+3k^2+3k+6\\ &= (k^3+5k)+3(k^2+k)+6\end{aligned}$$

Now *by the induction assumption* $6|(k^3+5k)$. Also, k^2+k is always even (Why?), so $6|3(k^2+k)$, and, of course, $6|6$. Thus we get $6|((k+1)^3+5(k+1))$, and $P(n)$ is true for all n. □

- Induction proofs do not have to be written written using the $P(n)$ stuff.
- Our base case does not have to be $n=1$.

Recall that $n! = n(n-1)(n-2)\cdots 3.2.1$.

In the next theorem we start with $n=4$.

Theorem 8.2. $2^n < n!$ *for all* $n \geq 4$.

Proof. Now $2^4 = 16$ and $4! = 4.3.2 = 24$, so the result holds when $n=4$. This establishes the base case.

Assume that $k \geq 4$ and that the theorem holds when $n=k$, that is, assume that $2^k < k!$.

Now $2^{k+1} = 2^k 2$. But, by the induction assumption, $2^k < k!$. Also, $2 < k+1$. Hence,

$$2^{k+1} = 2^k 2 < k!(k+1) = (k+1)!$$

and the theorem follows by induction. □

A slightly different technique is

Strong Induction Consider the statement $P(n)$. If

(a) $P(1)$ is true, and
(b) whenever $P(1), P(2), \ldots, P(k-1)$ are true, then $P(k)$ is true,

then $P(n)$ is true for all natural numbers n.

In other words, if it suits us, we can assume that the result holds for all natural numbers less than k.

Fibonacci Sequences: This dates from 1202, after the Italian mathematician Fibonacci who first studied it. The sequence goes

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$

It is given by the function $\mathcal{F} : \mathbb{N} \to \mathbb{N}$, where $\mathcal{F}(1) = \mathcal{F}(2) = 1$, and for all $n > 1$,

$$\mathcal{F}(n) = \mathcal{F}(n-2) + \mathcal{F}(n-1).$$

Theorem 8.3. *For all $n \geq 1$, $\mathcal{F}(n) < 2^n$.*

Proof. Evidently, the result holds for $n = 1$, and $n = 2$. (Why does our base case here have to look at both $n = 1$ and $n = 2$?)

Say $k > 2$, and assume that $\mathcal{F}(n) < 2^n$ for all $n < k$. Then

$$\mathcal{F}(k) = \mathcal{F}(k-2) + \mathcal{F}(k-1).$$

By the induction assumption $\mathcal{F}(k-2) < 2^{k-2}$ and $\mathcal{F}(k-1) < 2^{k-1}$. Hence

$$\begin{aligned} \mathcal{F}(k) < 2^{k-2} + 2^{k-1} & \\ &= 2^{k-2}(1+2) \\ &< 2^{k-2}4 \\ &= 2^k \end{aligned}$$

and the result follows by *strong induction.* □

- Logically there is no real difference between ordinary induction and strong induction. Either technique is perfectly valid and you should simply use whichever one will work for a given problem.