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MATH 161 Notes — Relations

A **relation** is a set of ordered pairs.

For example, $R = \{(1, p), (2, 2), (q, 1)\}$ is a relation.

A relation **from set** A **to set** B is a subset of $A \times B$. In other word a relation in which the first element of each ordered pair is always in A and the second is always in B.

For example, let W be the set of all women and P be the set of all people, then

$$R = \{(x, y) : x \text{ is an aunt of } y\}$$

is a relation from W to P.

Note how much order can count in a relation. My guess is that if Helen is an aunt of Roger, then Roger is *not* an aunt of Helen.

Intuitively, we think of the elements of the ordered pair as being *connected* or *related* in some way. Hence the terminology.

A relation **on a set** A is a relation from A to A.

Let P be the set of people again. Then

$$R = \{(x, y) : x \text{ likes } y\}$$

is a relation on P.

Does order count in this relation?

Notation: If $(x, y) \in R$, we often write xRy and read it as "x is related to y".

Is this style more soothing to you psychologically? Anyhow you need to be comfortable with both notations.

If $(x, y) \notin R$, we write $x \not R y$.

There are various methods for **visualising relations**. We will discuss three in the lectures.

Some basic relations follow.

The **identity relation** in a set A is given by

$$id_A = \{(a, a) : a \in A\} = \{(x, y) : x = y\}.$$

The **empty relation** from A to B is given by

$$\emptyset \subseteq A \times B$$

It is indeed a pretty odd relation, but it crops up, so we have to know about it. For example, in a set S of people we can define the relation B by $(x, y) \in B$ if x and y have the same birthday. It is quite possible that this relation is empty.

The **universal relation** from A to B is given by $R = A \times B$, ie aRb for all $a \in A$, $b \in B$.

Inverse of a Relation: If $R \subseteq A \times B$, then

$$R^{-1} = \{(b, a) : (a, b) \in R\} \subseteq B \times A$$

is the **inverse of** R.

In other words, $aR^{-1}b$ if and only if bRa.

For example, if R is the relation “is a parent of”, then R^{-1} is “is a child of”.

Inverse relations are fundamental and crop up in many situations.

Theorem 7.1. $(R^{-1})^{-1} = R$

Composing Relations If $R \subseteq A \times B$, and $S \subseteq B \times C$, then the **composite** relation $RS \subseteq A \times C$ is defined by $a(RS)c$ if and only if there is some $b \in B$ such that aRb and bRc. More formally,

$$RS = \{(a, c) : \exists b \in B(aRb \wedge bSc)\}.$$

Does the above definition sound complicated? It’s not. For example, if R is the relation “is a brother of” and P is the relation “is a parent of”, then RP is the relation “is an uncle of”. Applying the definition we see that Bill is an uncle of Jane if and only if there exists some person x such that Bill is a brother of x and x is a parent of Jane.

It is quite common to take the composite of a relation with itself. With P as above, we see that $xPPy$ if and only if x is a grandparent of y. Sometimes we would say xP^2y.

Theorem 7.2. $(RS)^{-1} = S^{-1}R^{-1}$

Properties of a relation on a set. Let R be a relation on A, then R is

(1) **reflexive** if aRa for all $a \in A$,
(2) **irreflexive** if, for all $a \in A$, $a \not R a$,
(3) **symmetric** if whenever aRb, then bRa,
(4) **antisymmetric** if, whenever aRb, then $b \not R a$ unless $a = b$,
(5) **transitive** if, whenever aRb and bRc, then aRc.

Theorem 7.3. *Let R be a relation on A. Then*

(1) *R is reflexive if and only if $id_A \subseteq R$;*
(2) *R is symmetric if and only if $R^{-1} \subseteq R$, if and only if $R^{-1} = R$.*
(3) *R is antisymmetric if and only if $R \cap R^{-1} \subseteq id_A$.*
(4) *R is transitive if and only if $RR \subseteq R$.*

The properties that a relation has depends on the relation, and there is a *huge* variety of possible relations. But some types of relations are of particular importance.

Equivalence Relations are relations that are reflexive, symmetric, and transitive. We often use a symbol such as $\sim$ to denote an equivalence relation.

Some examples

(1) A any set. Then id_A is an equivalence relation, a and b are related if and only if $a = b$.
(2) Let A be the set of lines in the plane, and say $a \sim b$ if a is parallel to b.
(3) Let A be the set of all people, and say $a \sim b$ if a and b have the same age.
(4) Let A be the set of all propositions. Then equivalence of propositions as defined in Section 1 is an equivalence relation.

It is impossible to overestimate the importance of equivalence relations. The idea behind them is that given an equivalence relation $\sim$, then $a \sim$

b means that a and b are the same, or indistinguishable, or equivalent in some respect.

For the relations above we have (1) identifies "exactly the same", (2) identifies "same direction", (3) identifies "same age", and (4) identifies "same truth table".

Let $\sim$ be an equivalence relation on A, and $a \in A$. Then the set

$$[a]_\sim = \{x \in A : a \sim x\}$$

is called the **$\sim$-equivalence** class of A.

- $[a]_\sim$ is just the set of things that are equivalent to a.
- If there is no danger of confusion we usually drop the $\sim$ from the notation above — yet more laziness.

For the relations above we have (1) $[a] = \{a\}$ (you are only equal to yourself) (2) $[a]$ is the set of all lines parallel to a (3) $[a]$ is the set of all people with the same age as a.

Theorem 7.4. *For an equivalence relation $\sim$ on A,*

(1) *$a \in [a]$ for all $a \in A$,*
(2) *$[a] = [b]$ if and only if $a \sim b$,*
(3) *if $[a] \neq [b]$, then $[a] \cap [b] = \emptyset$.*

A **partition** of a set A is a collection $\mathcal{C}$ of subsets of A called **cells** such that

(1) Any two cells are **disjoint**, that is, if C_1, C_2 are cells, then $C_1 \cap C_2 = \emptyset$.
(2) Each element of A belongs to some cell, that is, for all $a \in A$, there is a cell C such that $a \in C$.

Say $A = \{a, b, c, d, e, f\}$, and $\mathcal{C} = \{\{a\}, \{b, d\}, \{c, e, f\}\}$. Then $\mathcal{C}$ is a partition of A.

The point of all this is that there is a fundamental relationship between equivalence relations and partitions.

Theorem 7.5. *If $\sim$ is an equivalence relation, then*

$$\mathcal{C} = \{[a] : a \in A\}$$

is a partition of A.

Theorem 7.6. *If $\mathcal{C}$ is a partition of A, and $\sim$ is defined by $a \sim b$ if and only if a and b are in the same cell, then $\sim$ is an equivalence relation on A. Moreover, the equivalence classes of $\sim$ are the cells of $\mathcal{C}$.*

Partial Orders: Another use of relations (quite different from equivalence relations) is to make *comparisons*, eg "bigger than," "more intelligent than," or "contains". Specifically, the relation R is a **partial order** if it is reflexive, transitive, and antisymmetric.

Examples: The following are partial orders.

(1) The relation $\leq$ on $\mathbb{R}$.
(2) Let A be a set and consider the power set of A, ie $\mathcal{P}(A)$. Subsets C and D are related if $C \subseteq D$.
(3) Divisibility on $\mathbb{N}$, ie n is related to m if $n|m$. Note that $n|m$ means that n *divides* m.

For an arbitrary partial order, we typically use a symbol like $\preccurlyeq$. Let $\preccurlyeq$ be a partial order on A. Elements a and b are **comparable** if, either $a \preccurlyeq b$ or $b \preccurlyeq a$; otherwise they are **incomparable**. A partial order is a **total order** if every pair of elements is comparable.

Which of the partial orders above are total orders?

To give a diagrammatic representation of partial orders on finite sets we use **Hasse diagrams**. These have the following properties.

(1) The vertices are labelled by members of A.
(2) If $a \preccurlyeq b$ and $a \neq b$, then a is *below* b.
(3) If $a \preccurlyeq b$, then there is an edge joining a to b if and only if there is no element c such that $a \preccurlyeq c$ and $c \preccurlyeq b$.