Logic Intro

Arguments and proofs

Definition: *Argument*

A mathematical argument is a sequence of statements. All statements but the final one are called premises (or assumptions or hypotheses). The final statement is the conclusion.

For premises $P_1, P_2, \ldots P_n$ and conclusion C, we often write arguments in the following form:

$$\begin{array}{l} P_1 \\ P_2 \\ \vdots \\ P_n \\ \hline C \end{array}$$

where the premises and conclusion are separated by a horizontal bar to help keep the distinction in mind.

An argument is valid if the conclusion is true whenever the premises are true. Another word for a valid argument is a proof.

- In a valid argument the truth of the premises forces the truth of the conclusion.
- Given a valid argument, there is *no guarantee at all that the conclusion is true*. Before you can be sure that the conclusion is true you have to know that the premises are true.

Testing Validity

Here is an easy way to test whether an argument *containing no quantifiers* is valid.

Construct a truth table for P_1, P_2, ..., P_n and C. If there is a row with $P_1, P_2, \ldots, P_n$ all 1 and C having 0, then the argument is invalid.
To see this, note that the row we found is a situation where all the premises are true, but the conclusion is false.
Otherwise the argument is valid.

If Alex got an A then she passed ENGR123.
Alex got an A.

Alex passed ENGR123.

More formally:

$P \rightarrow Q$
P

Q

Valid!

If Alex got an A then she passed ENGR123.
Alex got an A.

Alex passed ENGR123.

More formally:

$P \to Q$
P

Q

Valid!

If Alex got an A then she passed ENGR123.
Alex got an A.

Alex passed ENGR123.

More formally:

$$\frac{P \to Q \qquad P}{Q}$$

Valid!

If Alex got an A then she passed ENGR123.
Alex passed ENGR123.

Alex got an A.

More formally:

$P \rightarrow Q$
Q

P

Invalid!

If Alex got an A then she passed ENGR123.
Alex passed ENGR123.

Alex got an A.

More formally:

$P \rightarrow Q$
Q

P

Invalid!

If Alex got an A then she passed ENGR123.
Alex passed ENGR123.

Alex got an A.

More formally:

$$\frac{P \to Q \qquad Q}{P}$$

Invalid!

All human beings are mortal
Socrates is a human being

Socrates is mortal

More formally:

$\forall x,\ P(x) \rightarrow Q(x)$
$P(\text{Socrates})$

$Q(\text{Socrates})$

Valid!

All human beings are mortal
Socrates is a human being

Socrates is mortal

More formally:

$$\frac{\forall x,\ P(x) \rightarrow Q(x) \qquad P(\text{Socrates})}{Q(\text{Socrates})}$$

Valid!

All human beings are mortal
Socrates is a human being

Socrates is mortal

More formally:

$$\frac{\forall x,\ P(x) \to Q(x) \qquad P(\text{Socrates})}{Q(\text{Socrates})}$$

Valid!

Some rules of inference

- *Modus ponens*

$$\frac{\begin{array}{l} P \\ P \rightarrow Q \end{array}}{Q}$$

- *Modus tollens*

$$\frac{\begin{array}{l} P \rightarrow Q \\ \neg Q \end{array}}{\neg P}$$

- *Or-elimination*

$$\frac{\begin{array}{l} P \vee Q \\ \neg P \end{array}}{Q}$$

- *And-elimination*

$$\frac{P \wedge Q}{P}$$

Some rules of inference

- *Transitivity*

$$\frac{P \rightarrow Q \qquad Q \rightarrow R}{P \rightarrow R}$$

- *Or-introduction*

$$\frac{P}{P \vee Q}$$

- *Contrapositive*

$$\frac{P \rightarrow Q}{\neg Q \rightarrow \neg P}$$

- *Implies-introduction*

$$\frac{Q}{P \rightarrow Q}$$

Lewis Carroll

The king of these logical deductions was Lewis Carroll (actual name Charles Dodgson) who wrote a book called "Symbolic Logic" and had lots of fun examples of varying degrees of complexity.

All lions are fierce
Some lions do not drink coffee

Some fierce creatures do not drink coffee

Valid!

Some pigs can fly
Only birds can fly

Some birds are pigs

Valid!

Every country has a capital
Wellington is the capital of a country
Wellington is in NZ

Wellington is the capital of NZ

Invalid!

Strategy

As a general strategy, try working backwards from the conclusion and forwards from the premises until your paths of reasoning meet somewhere in the middle. Here are some specific techniques for manipulating statements.

- Replace an implication with its contrapositive.
- Use De Morgan's Law to rewrite a conjunction or a disjunction.
- Use De Morgan to rewrite a negation of a conjunction or a disjunction.
- Try using any of the other tautological equivalences to rewrite a statement.
- Take a coffee break to clear your head.

Above all, be *persistent* (come back from that coffee break and go back to work)!

Your turn

All hummingbirds are brightly coloured
No large birds live on honey
Birds that do not live on honey are dull in colour

Hummingbirds are small

Animals, that do not kick, are always unexcitable
Donkeys have no horns
A buffalo can always toss one over a gate
No animals that kick are easy to swallow
No hornless animal can toss one over a gate
All animals are excitable, except buffaloes

Varieties of Proofs

There are many proof techniques. We consider three here. We will consider another, *proof by induction*, in a couple weeks.

Direct Proof

Goal: To prove $P \rightarrow Q$

Strategy: Assume P

$\vdots$

Q

Theorem

If x is even and y is even then $x + y$ is also even.

Theorem

If n is an odd integer, then so is n^2.

Proof by contrapositive

Goal: To prove $P \rightarrow Q$

Strategy: Assume $\neg Q$

$\vdots$

$\neg P$

Theorem

Suppose that $n \in \mathbb{N}_+$ *and* d *divides* n.
If n *is odd then* d *is odd.*

Theorem

If $3n + 2$ *is odd, then so is* n.

Proof by contradiction

Goal: To prove $P \to Q$

Strategy: Assume P and $\neg Q$

$\vdots$

R

$\vdots$

$\neg R$

Conclusion: $(P \land \neg Q) \to (R \land \neg R)$ which is a contradiction.
Therefore, P cannot is true while Q is false.
So $P \to Q$.

Theorem

If $n^3 + 5$ is odd then n is even.

Theorem

$\sqrt{2}$ is irrational.

False Proofs

Proof by Blatant Assertion A string of statements with “clearly...," “obviously...," “it is easily shown that...," and “as any fool can plainly see..."

Proof by Seduction “If you will just agree to believe this, the rest is easy”

Proof by Intimidation “You better believe this to follow the rest."

Proof by Misconception An example of this is the Freshman’s Conception of the Limit Process: "2 equals 3 for large values of 2." Once introduced, any conclusion is reachable.

Proof by Obfuscation A long list of lemmas and abstruse notation.

Proof by Confusion This is a more refined form of proof by obfuscation. The long list of lemmas should be arranged into circular patterns of reasoning.

Proof by Exhaustion This is a modification of an inductive proof. Instead of going to the general case after proving the first one, prove the second case, then the third, then the fourth, and so on, until a sufficiently large n is achieved whereby the nth case is being propounded to a soundly sleeping audience.

False Proofs

It's quite easy to make silly mistakes in proofs that lead to some obviously wrong conclusions:

- $2 = -2$
- $1 = 2$
- $4 \leq 1$