Week 11

Probability

Reading: Croft — Chapter 28

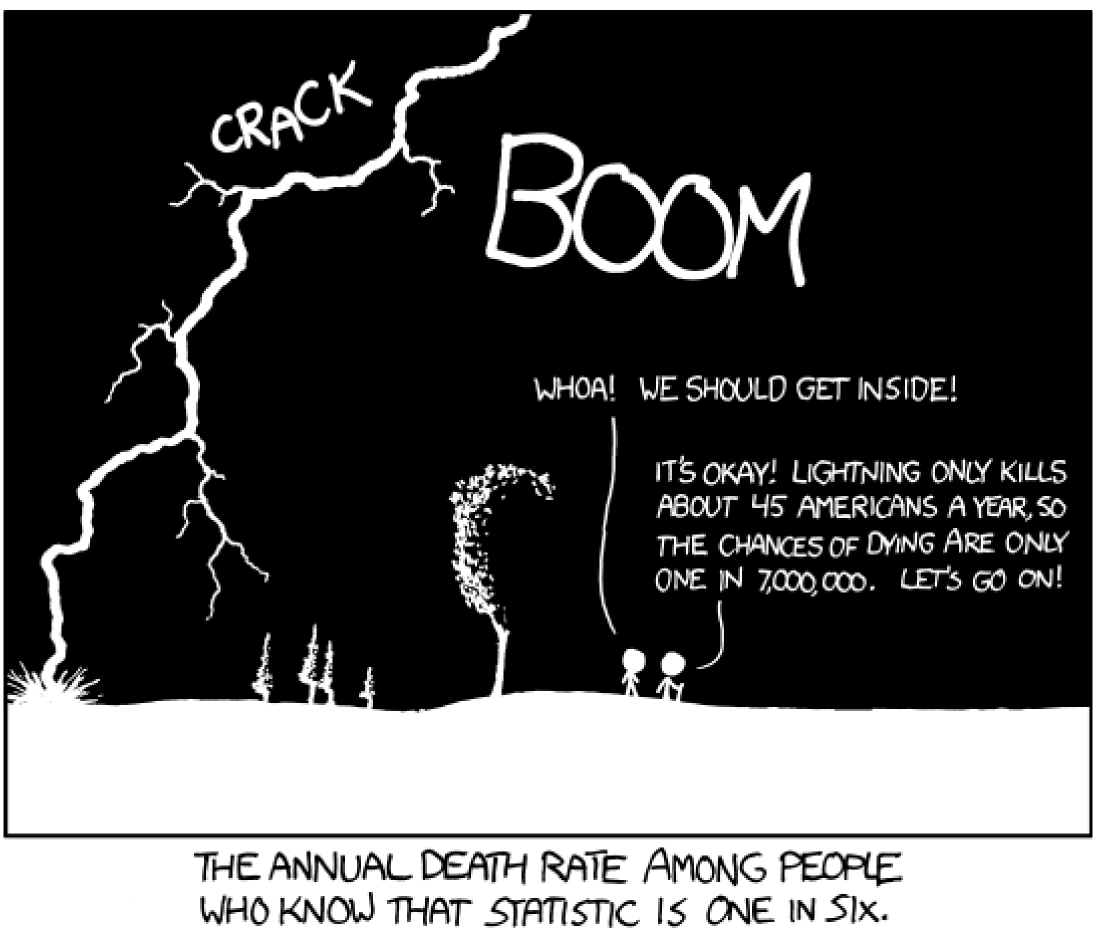


Image from http://xkcd.com/795/

11.1 What is Probability?

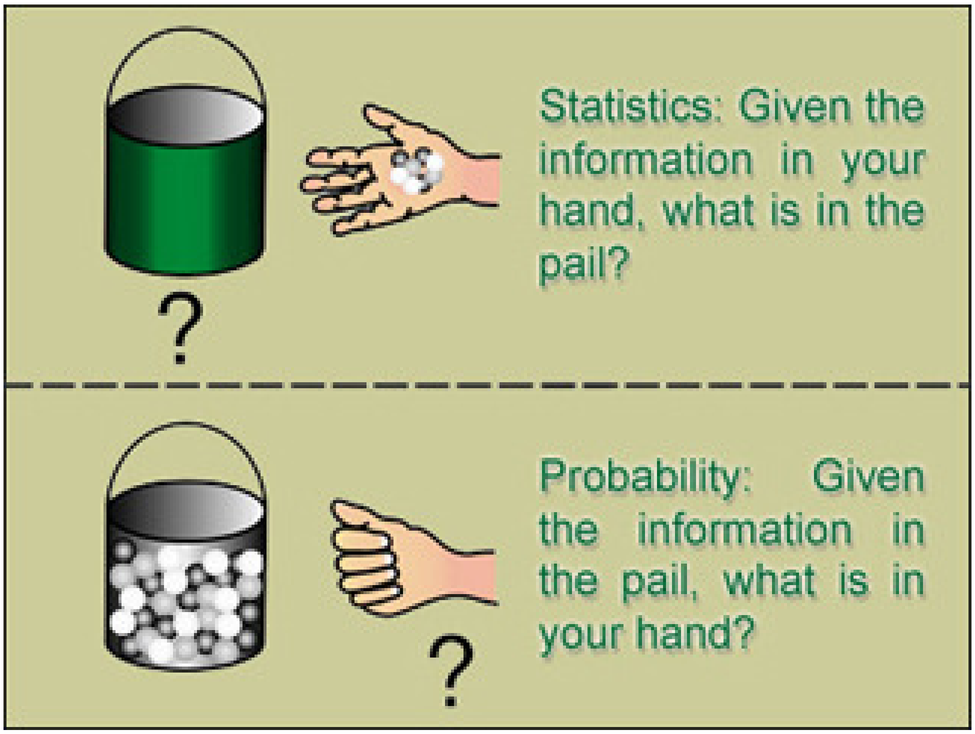


Image from http://ocw.mit.edu/courses/sloan-school-of-management/15-082j-network-optimization-spring-2003/chp_probability_stat.jpg

11.1.1 Sample Space and Events

Reading: Croft — §28.2

- An **experiment** (or **trial**) is any operation or procedure whose *outcome* cannot be predicted with certainty.
- The **sample space**, S, consists of *all possible outcomes* associated with the experiment.
- An **event**, E, is some subset of the sample space, i.e., $E \subseteq S$. A **simple event** is an event consisting of a single outcome.

Example 11.1 Consider the experiment of tossing a coin twice in a row. The sample space is a head or a tail for each toss, i.e.,

$$S = \{HH, HT, TH, TT\}$$

Obtaining at least one head is the event

$$E = \{HH, HT, TH\}$$

Example 11.2 Consider the experiment of tossing a single die. The sample space is

$$S = \{1, 2, 3, 4, 5, 6\}$$

Obtaining an even number is the event

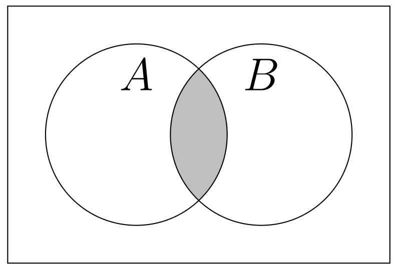
$$E = \{2, 4, 6\}$$

Recall (Croft §5.2) —

Suppose A and B are events, i.e., $A \subseteq S$ and $B \subseteq S$.

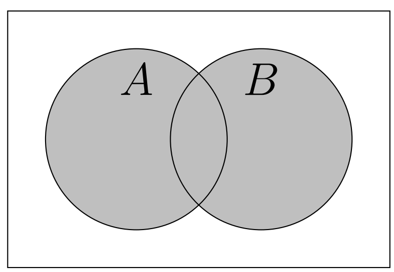
- The **intersection** of A and B is the set of outcomes in A **and** B, i.e., $A \cap B$.

$A \cap B$



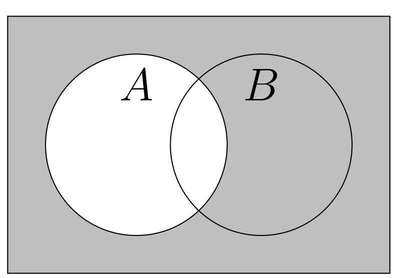
- The **union** A and B is the set of outcomes in A **or** B or both, i.e., $A \cup B$.

$A \cup B$



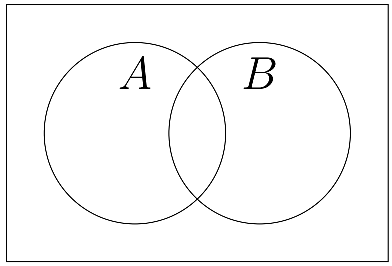
- The **complement** of A is the set of outcomes in S that are **not** in A, i.e., $\overline{A}$.

$\overline{A}$

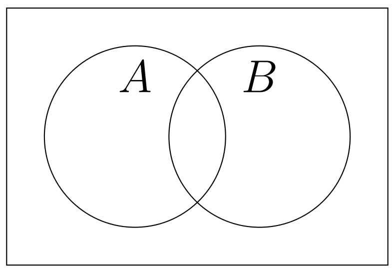


Practice Problem. Shade the following sets on the Venn diagram.

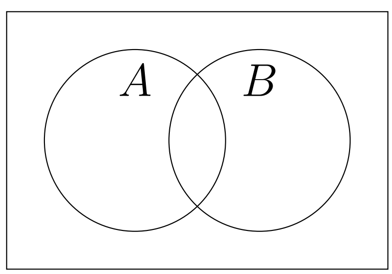
$$A \cap \overline{B}$$



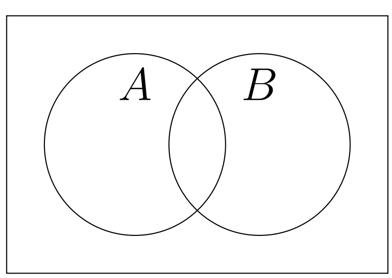
$$\overline{A} \cap B$$



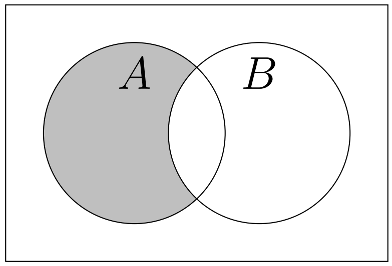
$$\overline{A \cup B}$$



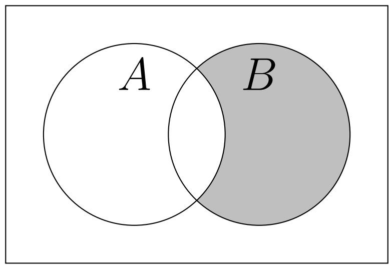
$$\overline{A \cap B}$$



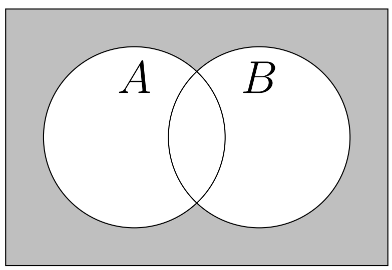
$$A \cap \overline{B}$$



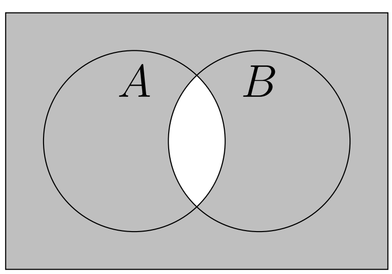
$$\overline{A} \cap B$$



$$\overline{A \cup B}$$



$$\overline{A \cap B}$$



11.1.2 Definition of Probability

Reading: Croft — §28.2

- When an event is **impossible** we say the probability of it happening is 0.

  Example — it is impossible to live without oxygen and so the probability of doing this is 0.

- When an event is **certain** we say the probability of it happening is 1.

  Example — it is certain that a metal bar will sink when placed in water and so the probability of this happening is 1.

- Most events are **neither impossible nor certain**.

  The probability of such events lies between 0 and 1.

  Events which are likely to happen have probabilities close to 1.

  Events which are unlikely to happen have probabilities close to 0.

  An event which is as likely to happen as not has a probability of $\frac{1}{2}$, e.g., throwing a head with a fair coin.

- Events A and B are **mutually exclusive** if

$$A \cap B = \emptyset$$

i.e., they have no common outcomes.

■ *Axioms of Probability*

Let $P(A)$ denote the probability of the event A in a finite sample space S.

Axiom 1. $0 \leq P(A) \leq 1$ for each event A in S

Axiom 2. $P(S) = 1$

Axiom 3. If A and B are mutually exclusive events in S, then

$$P(A \cup B) = P(A) + P(B)$$

Question —

How do we know the probability of an event?

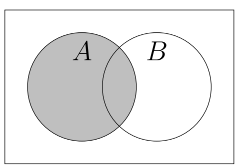
Equally Likely Outcomes

- Suppose an experiment has n equally likely outcomes, and an event E occurs if any one of k of these outcomes occurs as an outcome of the experiment.
- We define the probability of event E as

$$P(E) = \frac{k}{n}$$

If all outcomes are equally likely, the **probability** of event A is

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } S}$$



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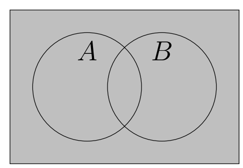


Example 11.3 Suppose a box of 100 items contains 5 defective items. If one item is **randomly selected** then

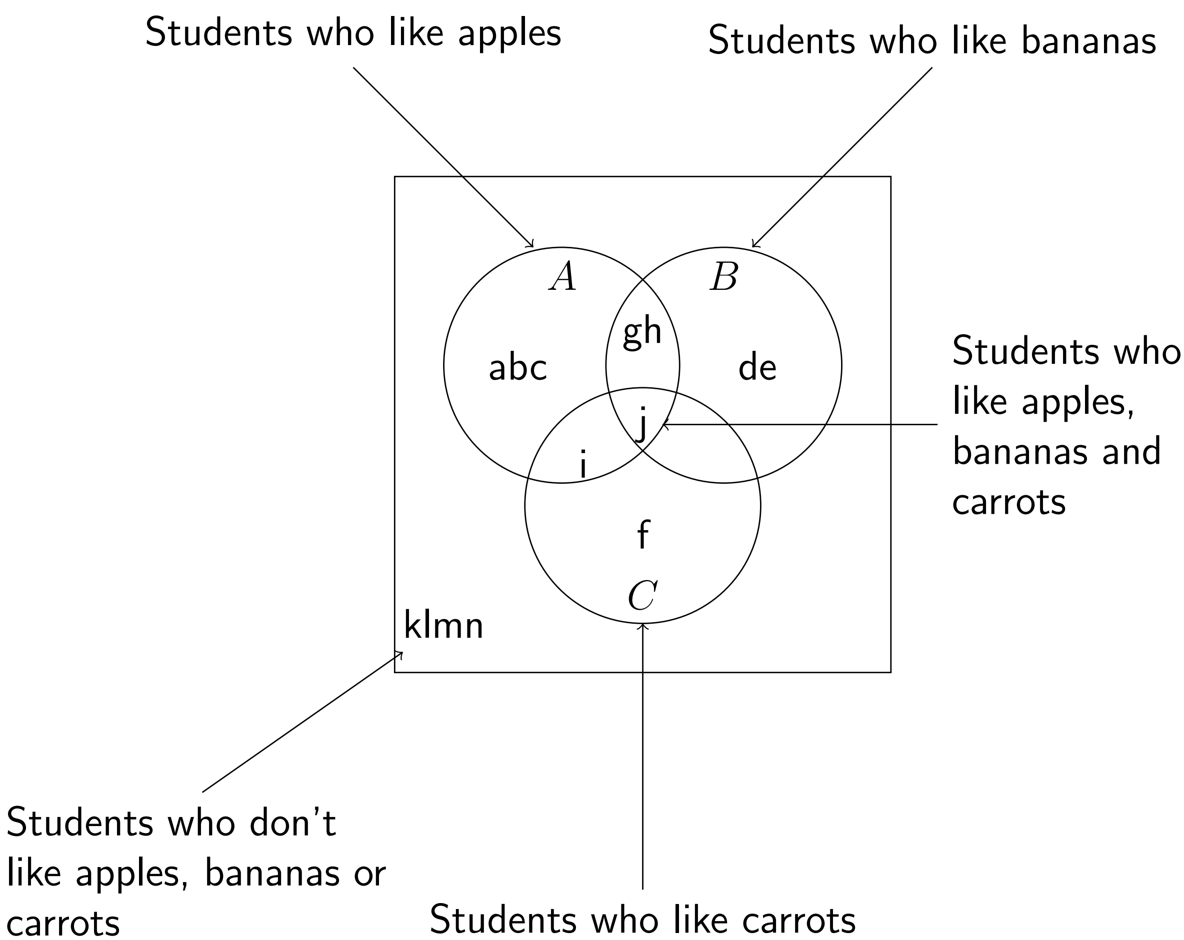
$$P(\text{item is defective}) = \tfrac{5}{100} = 0.05$$

Example 11.4 A fair die is tossed. The event A is defined as "the number obtained is a multiple of 3".

Here the sample space is $S = \{1, 2, 3, 4, 5, 6\}$ and $A = \{3, 6\}$. So

$$P(A) = \tfrac{2}{6} = \tfrac{1}{3}$$

Example 11.5 Suppose there are a group of 14 students. For simplicity, assume their names are a, b, c, . . . , n. Some like to eat apples, some like to eat bananas and some like to eat carrots, as shown below.



If one student is **randomly selected**, what is the probability that the student

- likes apples

$$P(A) = \frac{|A|}{|S|} = \frac{7}{14} = 0.5$$

- likes bananas

$$P(B) = \frac{|B|}{|S|} = \frac{5}{14}$$

- likes carrots

$$P(C) = \frac{|C|}{|S|} = \frac{3}{14}$$

- likes apples and bananas

$$P(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{3}{14}$$

- likes bananas and carrots

$$P(B \cap C) = \frac{|B \cap C|}{|S|} = \frac{1}{14}$$

- likes apples, bananas and carrots

$$P(A \cap B \cap C) = \frac{|A \cap B \cap C|}{|S|} = \frac{1}{14}$$

- does not like apples

$$P(\overline{A}) = \frac{|\overline{A}|}{|S|} = \frac{7}{14} = 0.5$$

- likes bananas or likes carrots

$$P(B \cup C) = \frac{|B \cup C|}{|S|} = \frac{7}{14}$$

- likes bananas but does not like apples

$$P(B \cap \overline{C}) = \frac{|B \cap \overline{C}|}{|S|} = \frac{2}{14}$$

- likes bananas and carrots but does not like apples

$$P(B \cap C \cap \overline{A}) = \frac{|B \cap C \cap \overline{A}|}{|S|} = \frac{0}{14} = 0$$

Practice Problem. A fair coin is tossed twice. The sample space is

$$S = \{HH, HT, TH, TT\}$$

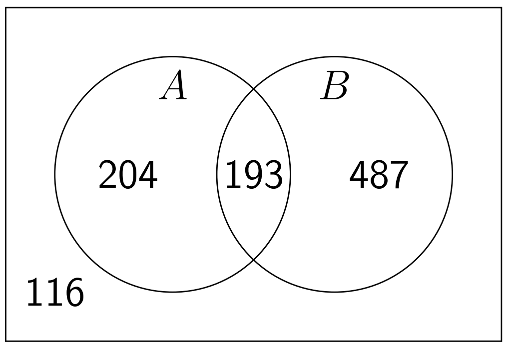
Find the probability that the outcome has

- no heads
- exactly one head
- two heads

Relative Frequency

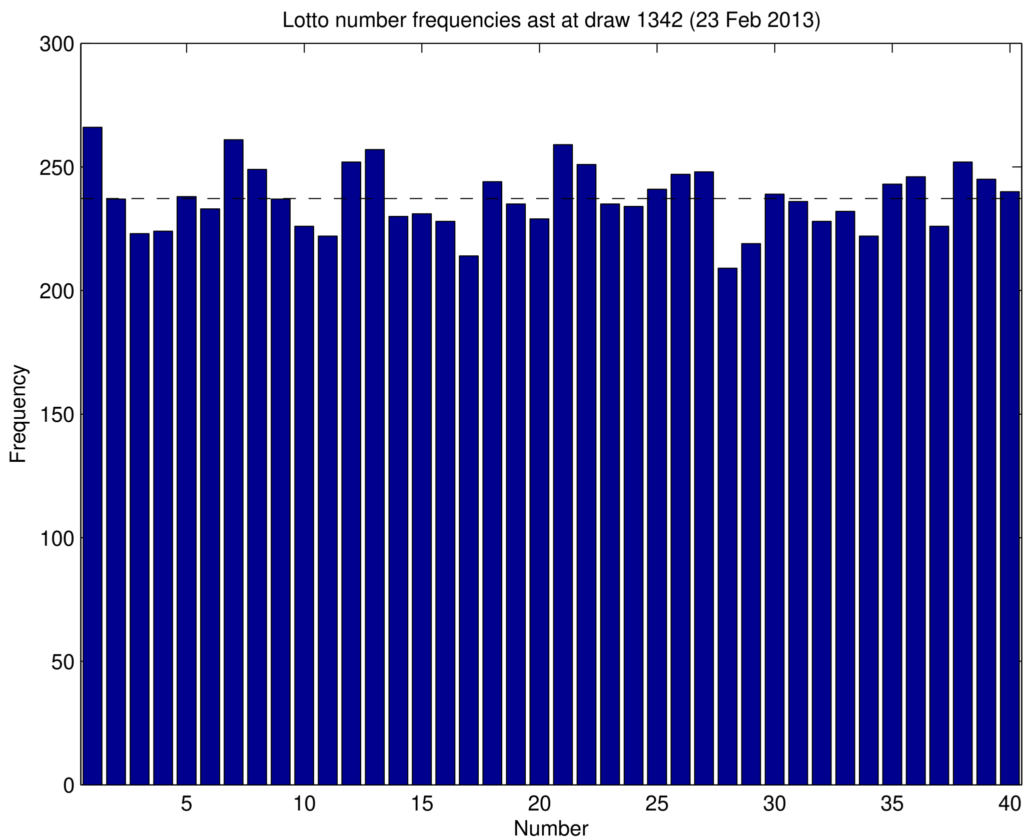
- If an experiment is repeated many times, the probability of an event is the proportion of times the event occurs in n repetitions of the experiment.

Example 11.6 Suppose an experiment is repeated 1000 times and we count how many times events A only occurs, B only occurs, A and B both occur, and neither A nor B occur.



$$
\begin{aligned}
P(A) &= \tfrac{204+193}{1000} = 0.397 \\
P(B) &= \tfrac{487+193}{1000} = 0.68 \\
P(A \cup B) &= \tfrac{204+193+487}{1000} = 0.884 \\
P(A \cap B) &= \tfrac{193}{1000} = 0.193
\end{aligned}
$$

Example 11.7 Lotto results (as at draw #1342, 23 February 2013).



Data from `https://mylotto.co.nz/downloads/`

Number $\boxed{1}$ appears 266 times out of a total of 9488 number occurrences. If a single ball were to be drawn from the lotto ball machine at random then

$$P(\text{number } \boxed{1} \text{ is drawn}) = \frac{266}{9488} = 0.028035 \ (6dp)$$

11.1.3 Some Rules of Probability

Reading: Croft — §28.3–28.4

Axiom 3 (see page 1108) can be extended from two mutually exclusive events to n mutually exclusive events.

- Events $A_1, A_2, \ldots, A_n$ are **mutually exclusive** if the occurrence of any one of them implies that none of the others can occur, i.e.,

$$A_i \cap A_j = \emptyset \text{ for every } i \text{ and } j \text{ where } i \neq j$$

- Events $A_1, A_2, \ldots, A_n$ are **exhaustive** if it is certain that at least one of them occurs, i.e.,

$$A_1 \cup A_2 \cup \cdots \cup A_n = S$$

■ *Addition Law of Probability*

If $A_1, A_2, \ldots, A_n$ are n mutually exclusive events within a sample space, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

Example 11.8 A machine makes chips which are then classified as one of the following: top quality, standard quality or substandard. An examination of 3000 chips showed 2700 were top quality, 240 were standard quality and 60 were substandard. A chip is selected at random.

- $P(\text{chip is top quality}) = \frac{2700}{3000} \; = \; 0.9$
- $P(\text{chip is standard quality}) = \frac{240}{3000} \; = \; 0.08$
- $P(\text{chip is substandard}) = \frac{60}{3000} \; = \; 0.02$
- $P(\text{chip is top quality or standard quality}) = 0.9 + 0.08 = 0.98$ since the events are mutually exclusive
- $P(\text{chip is standard quality or substandard}) = 0.08 + 0.02 = 0.1$ since the events are mutually exclusive

Example 11.9

An Irish rugby club has 40 players, of whom 7 are called O'Brien, 6 are called O'Connell, 4 are called O'Hara, 8 are called O'Neill and there are 15 others.

The captain of the team is chosen at random, determine the probability that the captain is

(i) called either O'Brien or O'Connell

(ii) is not called either O'Hara or O'Neill

Solution.
The sample space consists of the 40 players, each of whom is equally likely to be selected as captain.

- Let B be the event "the captain is an O'Brien".
- Let C be the event "the captain is an O'Connell".
- Let H be the event "the captain is an O'Hara".
- Let N be the event "the captain is an O'Neill".

Events B, C, H and N are mutually exclusive, since a player cannot have two surnames.

(i) $P(B \cup C)$

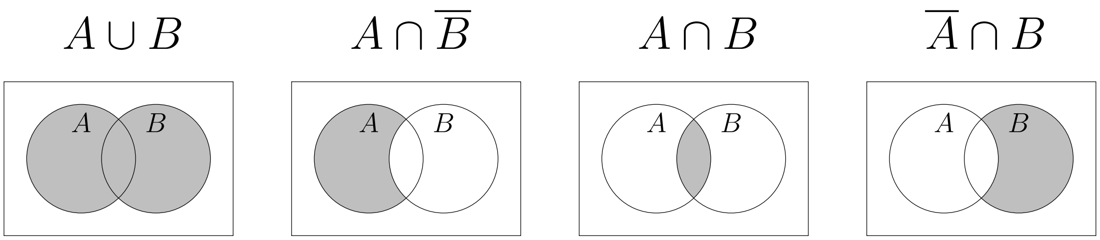
(ii) $P(\text{neither } H \text{ nor } N) = 1 - P(H \cup N) = 1 - (P(H) + P(N)) = 1 - (\frac{4}{40} + \frac{8}{40}) = \frac{28}{40}$

$P(B \cup C) = P(B) + P(C) = \frac{7}{40} + \frac{6}{40} = \frac{13}{40}$

Question —

What is $P(A \cup B)$ when A and B are **not** mutually exclusive?

Recall — (from page 1105)



We can see that

$$(A \cup B) \;=\; (A \cap \overline{B}) \cup (A \cap B) \cup (\overline{A} \cap B)$$

i.e., $A \cup B$ can be expressed as the union of *three mutually exclusive events*.

$$\begin{aligned}
P(A \cup B) &= P(A \cap \overline{B}) + P(A \cap B) + P(\overline{A} \cap B) \\
&= P(A \cap \overline{B}) + P(A \cap B) + P(\overline{A} \cap B) \\
&\qquad\qquad + P(A \cap B) - P(A \cap B) \\
&= \Big(P(A \cap \overline{B}) + P(A \cap B)\Big) \\
&\qquad + \Big(P(\overline{A} \cap B) + P(A \cap B)\Big) - P(A \cap B) \\
&= P(A) + P(B) - P(A \cap B)
\end{aligned}$$

Example 11.10 For the sample space S it is known that

$$P(A) = 0.5 \qquad \text{and} \qquad P(B) = 0.6$$

Determine the minimum and maximum possible values of $P(A \cap B)$.

Solution.

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ P(A \cup B) &= 0.5 + 0.6 - P(A \cap B) \\ P(A \cup B) &= 1.1 - P(A \cap B) \end{aligned}$$

- Since $P(A \cup B) \leq 1$ the minimum value of $P(A \cap B)$ is 0.1. Note that when $P(A \cap B) = 0.1$, we have $P(A \cup B) = 1$.

- Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$ the maximum value of $P(A \cap B)$ is $\min\{P(A), P(B)\} = 0.5$.

Question —

What is $P(\overline{A})$?

Since A and $\overline{A}$ are complementary we have

- $A \cap \overline{A} = \emptyset$, i.e., A and $\overline{A}$ are mutually exclusive, so
$$P(A \cup \overline{A}) = P(A) + P(\overline{A})$$
- $A \cup \overline{A} = S$, so
$$P(A \cup \overline{A}) = 1$$
- Therefore
$$P(A) + P(\overline{A}) = 1$$

■ For any event A

$$P(\overline{A}) = 1 - P(A)$$

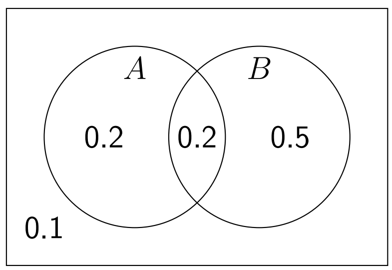
Practice Problem. The events A and B are such that

$$P(A) = 0.4 \qquad P(\overline{B}) = 0.3 \qquad P(A \cap B) = 0.2$$

Determine $P(A \cup B)$ and $P(\overline{A} \cap \overline{B})$.

Solution.

$$\begin{aligned} P(B) &= 1 - P(\overline{B}) = 1 - 0.3 = 0.7 \\ P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ P(A \cup B) &= 0.4 + 0.7 - 0.2 = 0.9 \\ P(\overline{A} \cap \overline{B}) &= 1 - P(A \cup B) = 0.1 \end{aligned}$$



11.1.4 Permutations and Combinations

Reading: Croft — §29.9

Suppose a sample of two items is selected from four items, and the

order of selection *is* *important.*

For simplicity, assume the items are the letters a, b, c and d.
Then there are 12 possible **ordered** choices

$$(a, b), (a, c), (a, d), (b, a), (b, c), (b, d),$$

$$(c, a), (c, b), (c, d), (d, a), (d, b), (d, c)$$

When the order of selection is important, we say there are "12 **permutations**" possible when selecting two from four.

■ When k objects are selected from a set of n distinct objects, there are

$$^{n}P_k \;=\; \frac{n!}{(n-k)!} \;=\; n \times (n-1) \times \cdots \times (n-k+1)$$

permutations (ordered choices) possible.

Recall — "n factorial" $= n! = n \times (n-1) \times \cdots \times 2 \times 1$

Suppose a sample of two items is selected from four items, and the

order of selection **is not** *important*.

Considering the ordered choices (the 12 permutations), there are two possible ways of ordering each pair of two letters.

So there are 6 possible **unordered** choices

$$\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}$$

When order of selection is **not** important, we say there are "6 **combinations**" possible when selecting two from four.

■ When k objects are selected from a set of n distinct objects, there are

$$^{n}C_k = \binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{n \times (n-1) \times \cdots \times (n-k+1)}{k \times (k-1) \times \cdots \times 2 \times 1}$$

combinations (unordered choices) possible.

Note — These are precisely the numbers in **Pascal's Triangle** and the **Binomial Theorem** (see Croft §6.4).

$$\begin{aligned}(a+b)^n \;=\; & \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \dots \\ & + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n +\end{aligned}$$

Note — when $n=4$ and $k=2$ $\quad {}^4P_2 = 12 \quad \binom{4}{2} = 6$

Example 11.11 When $n = 4$ and $k = 2$

$$\text{number permutations} = {}^4P_2 = \frac{4!}{(4-2)!} = \frac{24}{2} = 12$$

$$\text{number combinations} = \binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{24}{2 \times 2} = 6$$

Example 11.12 A pack of 52 playing cards (all different) is shuffled. Determine the probability that the top card in the pack is the Ace of Spades, the next is the Ace of Hearts, and the next is the Ace of Diamonds.

Solution.
Order $\boxed{\text{is}}$ important. Here $n = 52$ and $k = 3$.

The number of ordered possibilities for the first three cards in order is

$$^{52}P_3 \;=\; \frac{52!}{(52-3)!} = 52 \times 51 \times 50 = 132600$$

Only one of these corresponds to the event described, so

$$P(\text{Ace of } \spadesuit \text{ then Ace of } \heartsuit \text{ then Ace of } \diamondsuit) \;=\; \frac{1}{132600}$$

Example 11.13 A pack of 52 playing cards (all different) is shuffled. Determine the probability that the top three cards in the pack are the Ace of Spades, the Ace of Hearts and the Ace of Diamonds.

Solution.
Order $\boxed{\text{is } \underline{\textbf{not}}}$ important. Here $n = 52$ and $k = 3$.

The number of unordered possibilities for the first three cards in order is

$$\binom{52}{3} \;=\; \frac{52!}{(52-3)!\,3!} = \frac{52 \times 51 \times 50}{3 \times 2 \times 1} = \frac{132600}{6} = 22100$$

Only one of these corresponds to the event described, so

$$P(\text{Cards are } \{\text{Ace of } \spadesuit, \text{Ace of } \heartsuit, \text{Ace of } \diamondsuit\}) \;=\; \frac{1}{22100}$$

Recap

- *Experiment* (or *trial*) — any operation or procedure whose outcome cannot be predicted with certainty.
- *Sample space*, S — set of all possible outcomes associated with the experiment.
- *Event* — some subset of the sample space.
- Suppose, an experiment has n equally likely outcomes, and an event E occurs if any one of k of these outcomes is an outcome of the experiment. The classic definition of probability defines the probability of the event E as $P(E) = \frac{k}{n}$.
- If an experiment is repeated many times, the probability of an event is the proportion of times the event occurs in n repetitions.
- Suppose the sample space S is finite.

  Axiom 1. $0 \leq P(A) \leq 1$ for each event A in S

  Axiom 2. $P(S) = 1$

  Axiom 3. If A and B are mutually exclusive events in S, then

  $$P(A \cup B) = P(A) + P(B)$$

  Mutually exclusive events have empty intersections.
- For any events A and B

  $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
- For any event A we have $P(\overline{A}) = 1 - P(A)$
- ${}^nP_k = \frac{n!}{(n-k)!}$ *permutations* (ordered choices) of k from n
- ${}^nC_k = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$ *combinations* (unordered) of k from n

11.2 Conditional Probability

Reading: Croft §28.6

Example 11.14

- If a single fair die is tossed, then

$$P(\text{face 2 turns up}) = \frac{1}{6}$$

- If a single fair die is tossed, and it is known that the face that turns up is an even number, then

$$P(\text{face 2 turns up}) = \frac{1}{3}$$

- If a single fair die is tossed, and it is known that the face that turns up is an odd number, then

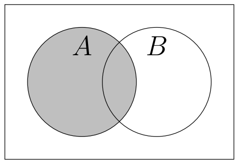
$$P(\text{face 2 turns up}) = 0$$

Notes —

- The probability of an event is **conditioned** by what other events we know to have occurred.
- The **sample space** that a probability is calculated with respect to has a great bearing upon the probability.

The **probability** of event A is

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } S}$$



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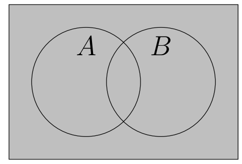
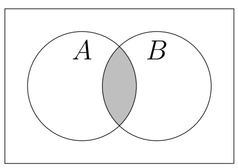


The **probability** of event A **conditional** on knowing that event B has occurred is

$$P(A\,|\,B) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } B}$$



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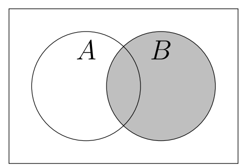

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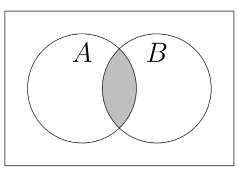

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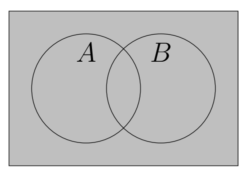

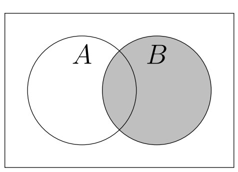

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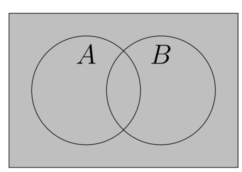


■ *Conditional probability of "A given B"*

$$P(A\,|\,B) = \frac{P(A \cap B)}{P(B)}$$

Example 11.15

An electronic display is equally likely to show any of the digits 1,2,3,4,5,6,7,8,9.

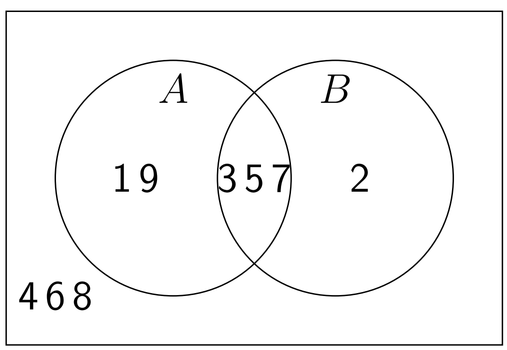
Determine the probability that it shows a prime number (i.e. one of 2, 3, 5 and 7):

(i) given no knowledge about the number

(ii) given the information that the number is odd

Solution.
Let A be the event "an odd number".

Let B be the event "a prime number".



(i) $P(B) = \frac{|B|}{|S|} = \frac{4}{9}$

(ii) $P(B \mid A) = \frac{|B \cap A|}{|A|} = \frac{3}{5}$

11.2.1 Independent Events

Two events are **independent** if the occurrence of either event does not change the probability of the other event occurring.

■ Events A and B are **independent** if

$$P(A \,|\, B) = P(A)$$

(or equivalently)

$$P(B \,|\, A) = P(B)$$

Notes —

- Physically independent events are always statistically independent.
- Since $P(A \,|\, B) = \dfrac{P(A \cap B)}{P(B)}$ we have

$$P(A \cap B) = P(A \,|\, B) \times P(B)$$

- If A and B are independent then $P(A \,|\, B) = P(A)$ so ...

■ *Multiplication Law of Probability*

If A and B are **independent** events then

$$P(A \cap B) \;=\; P(A) \times P(B)$$

Example 11.16 Two events A and B are such that

$$P(A) = 0.5 \qquad P(B) = 0.4 \qquad P(A\,|\,B) = 0.3$$

(a) Are A and B independent?

(b) Find the value of $P(A \cap B)$.

Solution.

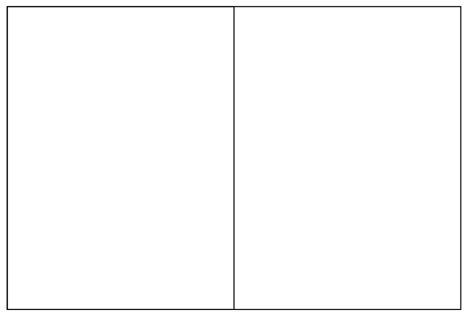
(a) Since $P(A) \neq P(A\,|\,B)$ the events A and B are **not** independent.

(b) $P(A \cap B) = P(A\,|\,B) \times P(B) = 0.4 \times 0.3 = 0.12$

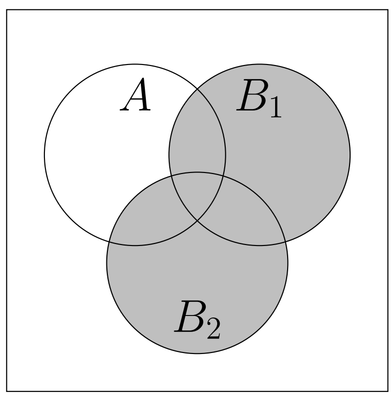
11.2.2 Law of Total Probability

Suppose $S = B_1 \cup B_2$ where $B_1 \cap B_2 = \emptyset$

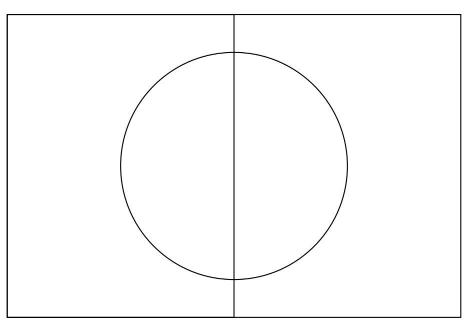
i.e., B_1 and B_2 are *mutually exclusive* and *exhaustive* (see page 1114).



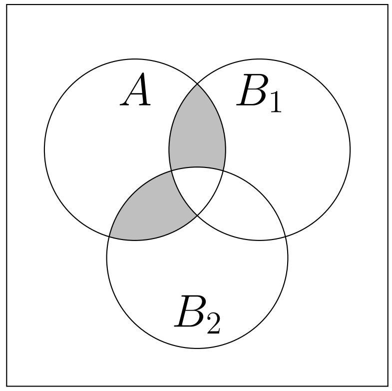
$S = B_1 \cup B_2$



Now consider event A.



$$A = (A \cap B_1) \cup (A \cap B_2)$$



It is clear that

$$A = (A \cap B_1) \cup (A \cap B_2)$$

and that

$$(A \cap B_1) \cap (A \cap B_2) = \emptyset$$

i.e., A can be expressed as the union of two mutually exclusive events $(A \cap B_1)$ and $(A \cap B_2)$.

Therefore

$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) \\ &= P(A \,|\, B_1) \times P(B_1) + P(A \,|\, B_2) \times P(B_2) \end{aligned}$$

Example 11.17

- Of those students who do well in Physics, 80% also do well in Mathematics.
- Of those students who do not do well in Physics, only 30% do well in Mathematics.
- If 40% of students do well in Physics, what proportion do well in Mathematics?

Solution.

Let A be the event "does well in Mathematics".

Let B_1 be the event "does well in Physics".

Let B_2 be the event "does not do well in Physics".

B_1 and B_2 are mutually exclusive and exhaustive events
So

$$\begin{aligned} P(A) &= P(A \mid B_1) \times P(B_1) + P(A \mid B_2) \times P(B_2) \\ &= 0.8 \times 0.4 + 0.3 \times 0.6 \\ &= 0.32 + 0.18 \\ &= 0.5 \end{aligned}$$

Therefore half the students do well in Mathematics.

Law of Total Probability

Suppose that $B_1, B_2, \ldots, B_n$ are mutually exclusive and exhaustive events, i.e.,

- $B_i \cap B_j = \emptyset$ for every i and j where $i \neq j$
- $B_1 \cup B_2 \cup \cdots \cup B_n = S$

Then

$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_n) \\ &= P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) \\ &\qquad + \ldots + P(A \mid B_n)\, P(B_n) \end{aligned}$$

11.2.3 Tree Diagrams

By definition

$$P(B \mid A) \;=\; \frac{P(B \cap A)}{P(A)}$$

Since $B \cap A = A \cap B$ this can be rearranged as

$$P(A \cap B) \;=\; P(A) \times P(B \mid A)$$

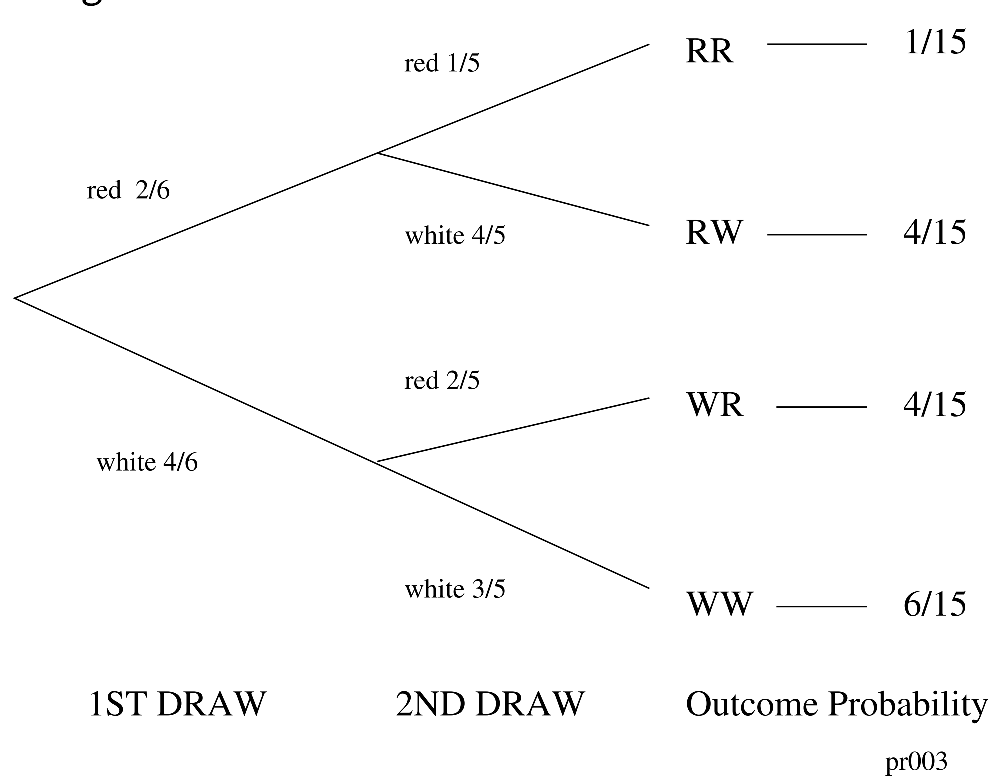
Repeating this idea gives the **chain rule** for conditional probabilities

$$P(A \cap B \cap C) \;=\; P(A) \times P(B \mid A) \times P(C \mid A \cap B)$$

Example 11.18 A bucket contains 2 red balls and 4 white balls. Two balls are drawn in sequence, without replacement. What is the probability that the second ball drawn is red?

Solution.

A *tree diagram* is shown below.



- The probabilities along the branches are the conditional probabilities for the next stage, given the results of the previous stages.
- Multiplying the probabilities along each branch gives

  (by the **chain rule**)

  the probability of the outcome that branch represents.
- Then add the probabilities contributing to the event of interest.

11.2.4 Bayes' Theorem

Thomas Bayes
18th century British clergyman and mathematician
`http://www.bayesian.org/bayesian/bayes.html`

Idea of conditional probability —

Given that event B has happened in the past, what is the probability that event A will occur?

Reverse question —

Given that event A has just occurred, what is the probability that it was preceded by the event B?

- Since $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ we have

$$P(A \cap B) \;=\; P(A \mid B)\, P(B)$$

- Since $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)}$ we have

$$P(B \cap A) \;=\; P(B \mid A)\, P(A)$$

- Since $P(A \cap B) = P(B \cap A)$ we have

$$P(A \mid B)\, P(B) \;=\; P(B \mid A)\, P(A)$$

■ *Bayes' Theorem*

$$P(B \mid A) \;=\; \frac{P(A \mid B) \times P(B)}{P(A)}$$

Example 11.19

An engineer has a fair coin and a double-headed coin. She chooses one of the coins at random and tosses it. She obtains a head. Determine the probability that the coin that she tossed was double-headed.

Solution.
Let A be the event that "a head is obtained".

Let B_1 be the event that "the fair coin was chosen".

Let B_2 be the event that "the double-headed coin was chosen".

B_1 and B_2 are mutually exclusive and exhaustive events.

We know

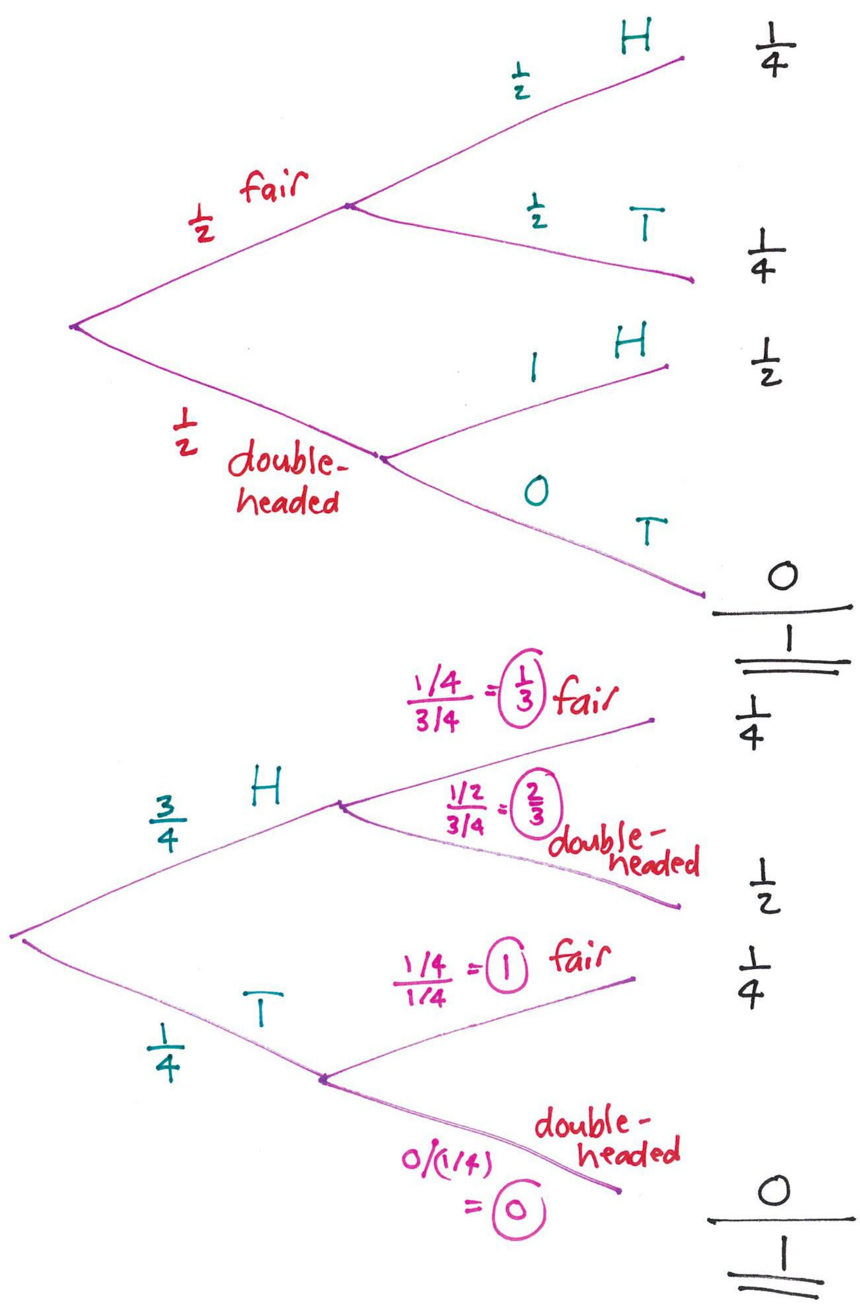
$P(B_1) = \frac{1}{2} \qquad P(B_2) = \frac{1}{2} \qquad P(A \mid B_1) = \frac{1}{2} \qquad P(A \mid B_2) = 1$

$$\begin{aligned} P(A) &= P(A \mid B_1) \times P(B_1) + P(A \mid B_2) \times P(B_2) \\ &= \tfrac{1}{2} \times \tfrac{1}{2} + 1 \times \tfrac{1}{2} = \tfrac{3}{4} \end{aligned}$$

$$\begin{aligned} P(\text{coin was double-headed}) &= P(B_2 \mid A) \\ &= \frac{P(A \mid B_2) \times P(B_2)}{P(A)} \\ &= \frac{1 \times \frac{1}{2}}{\frac{3}{4}} = \tfrac{2}{3} \end{aligned}$$

► *A good thing about Bayes' Theorem is that, once the events have been carefully defined, we do not need to think!*

► We can often perform all the calculations on two probability trees.



Recap

- The *conditional probability* of event A, given that event B is known to have occurred, is
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$
then we can also write
$$P(A \cap B) = P(A \mid B) \times P(B)$$

- Events A and B are *independent* if $P(A \mid B) = P(A)$
(or equivalently) $P(B \mid A) = P(B)$
(or equivalently) $P(A \cap B) = P(A) \times P(B)$

- *Law of total probability* —
If $B_1, B_2, \ldots, B_n$ are mutually exclusive and exhaustive then
$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_n) \\ &= P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) \\ &\quad + \ldots + P(A \mid B_n)\, P(B_n) \end{aligned}$$

- *Chain rule* — multiply probability along the branches in tree diagrams
$$\begin{aligned} P(A \cap B) &= P(A) \times P(B \mid A) \\ P(A \cap B \cap C) &= P(A) \times P(B \mid A) \times P(C \mid A \cap B) \end{aligned}$$

- *Bayes' Theorem*
$$P(B \mid A) = \frac{P(A \mid B) \times P(B)}{P(A)}$$

11.3 Application — Games of Chance and Decisions Trees

Question —

What do we **expect** to happen if we know the probabilities concerning an event?

Example 11.20 A fair coin is flipped. If it lands heads, you are paid \$5. If it lands tails, you pay \$5.

If you perform this experiment many times, what do you expect your winnings to be?

Winning	$\$5$ if heads	$-\$5$ if tails
Probability	$\frac{1}{2}$	$\frac{1}{2}$

Conclusion —

$$\text{expected winnings} \;=\; 5(\tfrac{1}{2}) - 5(\tfrac{1}{2}) \;=\; 0$$

■ *Expected Value in a Game of Chance*

If you win amount a_1 with probability p_1, amount a_2 with probability p_2, ..., and amount a_n with probability p_n, your **expected value** (or expected winnings) is

$$a_1\,p_1 + a_2\,p_2 + \ldots + a_n\,p_n$$

Notes —

- If your expected winnings are zero

  the game is called **fair**.

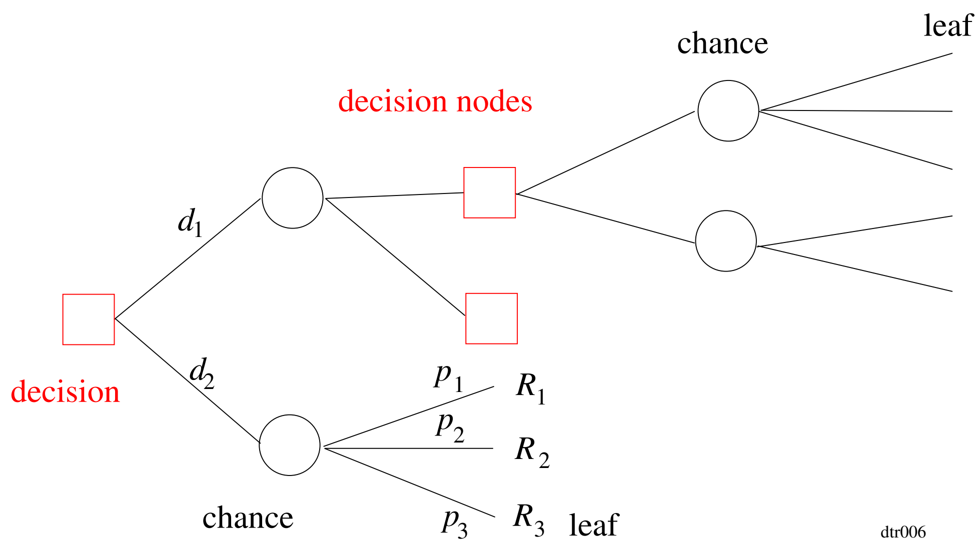
Example 11.21 A game is played as follows. You pay \$1 to play. A coin is flipped four times. If four tails or four heads i obtained, you get your \$1 back plus \$5 more. Otherwise you forfeit your \$1. Is this a fair game?

Solution.

- Win \$5 with probability $\frac{2}{16}$
- Win $-\$1$ with probability $\frac{14}{16}$
- expected winnings $= 5 \times \frac{2}{16} + (-1) \times \frac{14}{16} = -\frac{1}{4}$
- no the game is not fair

Question —

What **action** do we take as a result of knowing expected values?



- A **decision tree** is a diagram used to display the structure of a decision problem.
- It is constructed from **square** nodes representing decision points and **circular** nodes representing lotteries (or chance events).
- The nodes are connected by **arcs** to form a tree with a decision node at the **root** and values at each **leaf** (the ends of branches not leading to nodes).
- The nodes are located (left to right) in the **order** that decisions have to be made and uncertain variables resolved.
- There is a **value** associated with every leaf.

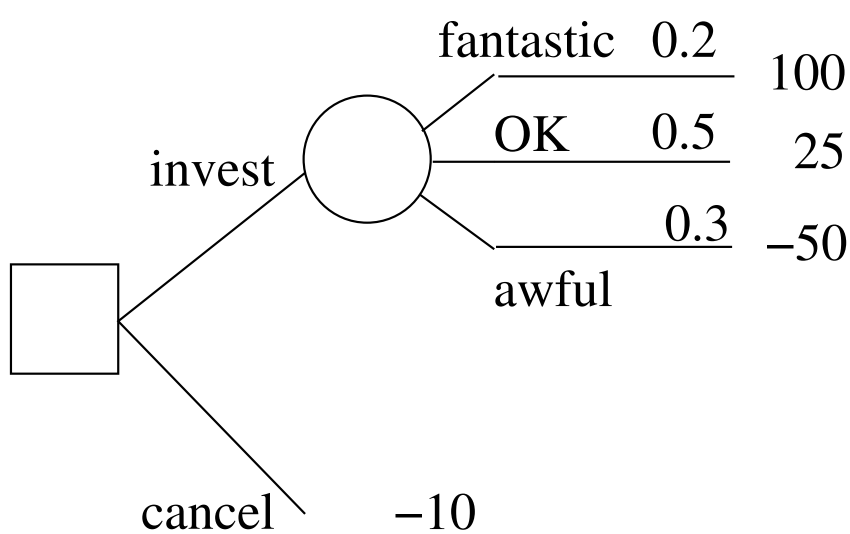
Example 11.22 We can either "invest" or "cancel".

If we "cancel" there is a cost of \$10.

If we "invest", three **outcomes** can occur:

- "fantastic" with probability 0.2 and reward $\$100$
- "OK" with probability 0.5 and reward $\$25$
- "awful" with probability 0.3 and reward $-\$50$

What action should we take?



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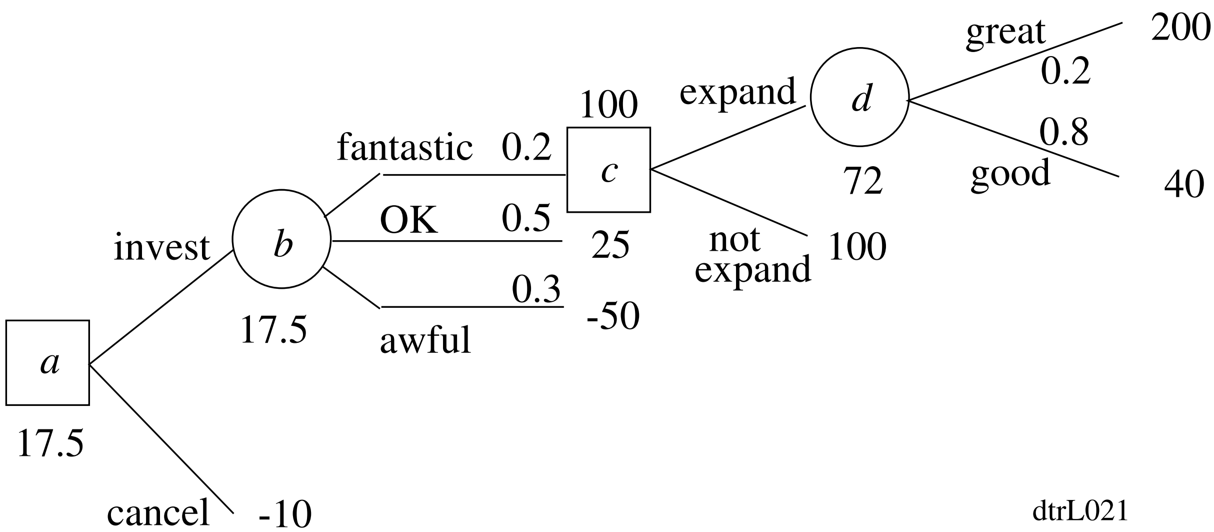
▶ A decision tree is **evaluated** by "folding the tree back".

Practice Problem. If we "invest" and the market is "fantastic" we may "expand" the project. If we "expand" we may have further "great" results (with probability 0.2 and return of $\$200$) or "good" results (probability 0.8 and return of $\$40$). If we do "not expand" we will get a return of $\$100$, as before. If the market is not "fantastic" there will be no expansion.

What action should we take?



Recap

- *Expected value* in a game of chance —

  If you win amount a_1 with probability p_1, amount a_2 with probability p_2, $\ldots$, and amount a_n with probability p_n, your **expected value** (or expected winnings) is

  $$a_1\,p_1 + a_2\,p_2 + \ldots + a_n\,p_n$$

- *Decision trees* —

  – squares nodes are decisions (choice is made by the decision maker)

  – circular nodes are chance nodes (outcome is result of an experiment involving uncertainty)

  – a decision tree is *evaluated* by "folding the tree back".