2018

Student ID: ....

Question 2. DISCRETE RANDOM VARIABLES [13 marks]

(a) [4 marks] The discrete random variable, X, takes on the values in the set $\{-1, 1, 2, 3, 5\}$ with the following probabilities: $P(X = -1) = \frac{1}{8}$, $P(X = 1) = \frac{1}{4}$, $\cancel{P(2)} = \frac{3}{8}$, $P(X = 3) = \frac{1}{8}$, $P(X = 5) = \frac{1}{8}$.

$P(x=2) = \frac{3}{8}$

(i) Find the probability that X^2 is less than 2.

$$P(X^2 < 2) = P(-\sqrt{2} < X < \sqrt{2})$$
$$= P(-1) + P(1) = \frac{1}{8} + \frac{1}{4} = 3/8$$

(ii) Find $E(X)$

$$\sum_{\text{all } x} x P(X=x) = (-1)\left(\tfrac{1}{8}\right) + 1\left(\tfrac{1}{4}\right) + 2(3/8) + 3(1/8) + 5(1/8)$$
$$= \frac{15}{8}$$

(iii) Find $\text{Var}(X)$

$$E(X^2) = \sum_{\text{all } x} x^2 P(X=x) = 1\left(\tfrac{1}{8}\right) + 1\left(\tfrac{1}{4}\right) + 4(3/8) + 9\left(\tfrac{1}{8}\right) + 25(1/8)$$
$$= 49/8$$

$$\text{Var}(X) = \frac{49}{8} - \left(\frac{15}{8}\right)^2$$

(b) [3 marks]

If X is a Poisson random variable with the rate parameter $\lambda = 3$, what is the mean and the variance of Y and Z where $Y = -3X + 2$ and $Z = 5(X - 1)$?

$E(Y) = -3(3) + 2 = -7$ $\quad$ $\text{Var}(Y) = 9\,\text{Var}(X) = 27$

$E(Z) = 5(3) - 5 = 10$ $\quad$ $\text{Var}(Z) = 25\,\text{Var}(X) = 75$

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(c) [2 marks]

In a quiz of 5 multiple choice questions with 4 answers for each question, suppose that you are just guessing. Denote by X the number of correct answers you get. What is the distribution of X (using notation with specified parameters)? What is the probability of getting 4 correct answers (or $P(X = 4)$)?

$X \sim binom(5, 0.25)$

$P(X=4) = \binom{5}{4} 0.25^4 0.75 = 5 \times \frac{1}{4^4} \times \frac{3}{4} = \frac{15}{4^5} = \frac{15}{1024}$

(d) [2 marks]

A mass contains 10000 atoms of a radioactive substance. The probability that a given atom will decay in a one-minute time period is 0.0005. Let Y represent the number of atoms that decay in one minute. Answer the following questions using notation with specified parameters:

(i) What is the distribution of Y?

$binom(10{,}000, 0.0005)$

(ii) What distribution can we use to **approximate** the distribution of Y to improve computational efficiency?

$Po(\lambda)$ $\lambda = np$ $\lambda = 5 \Rightarrow Po(5)$

(e) [2 marks] Given the fact that for a Poisson random variable X with parameter λ (in notation we write $X \sim Poi(\lambda)$), we have

$$\sum_{k=0}^{\infty} P(X = k) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = 1.$$

Prove that

$$E(X) = \sum_{k=0}^{\infty} k\frac{e^{-\lambda}\lambda^k}{k!} = \lambda.$$

SAME AS THE TEST

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Question 3. CONTINUOUS RANDOM VARIABLES [13 marks]

(a) [6 marks] Let the random variable X have the probability density function (pdf), $f(x)$, where:

$$f(x) = \begin{cases} \frac{k}{x^2}, & 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find k.

$$\int_1^2 kx^{-2}\,dx = 1 \Rightarrow \left[-kx^{-1}\right]_1^2 = \left(-\frac{k}{2}\right) - \left(-k\right) = 1$$

$$\Rightarrow \frac{k}{2} = 1 \Rightarrow k = 2$$

(ii) Find $P\left(\frac{4}{3} < X < \frac{3}{2}\right)$.

$$\int_{4/3}^{3/2} 2x^{-2}\,dx = \left[-2x^{-1}\right]_{4/3}^{3/2} = \left(-2\left(2/3\right)\right) - \left(-2\left(3/4\right)\right)$$

$$= -\frac{4}{3} + \frac{6}{4} = -\frac{8}{6} + \frac{9}{6} = \frac{1}{6}$$

(iii) Find $E(X)$.

$$\int_1^2 x \times 2x^{-2}\,dx = \int_1^2 2x^{-1}\,dx = \left[2\log x\right]_1^2$$

$$= 2\log 2 - 2\log 1$$

$$= 2\log 2 = \log 4$$

(b) [4 marks]

Let the random variable X have the cumulative distribution function (cdf), $F(x)$, where:

$$F(x) = \begin{cases} 1 - \frac{1}{x^2}, & x \geq 1, \\ 0, & x < 1. \end{cases}$$

(i) Find the probability density function $f(x)$ of X.

$$f(x) = F'(x) = \begin{cases} 2x^{-3} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

NOTE THESE ARE CRITICAL

(ii) Find $P(X \geq 2)$.

$$\int_2^\infty f(x)\,dx \quad \underline{\text{or}} \quad 1 - F(2) = 1 - \left(1 - \frac{1}{2^2}\right)$$

$$= \frac{1}{4}$$

(c) [3 marks] Are the following statements regarding continuous random variables and distributions TRUE or FALSE?

(i) Every exponential random variable has range $[0, \infty)$.

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(ii) Exponential distribution is the only continuous distribution with memory-less property.

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(iii) The bell-shaped curve of $\mathcal{N}(1, 1)$ is wider than that of $\mathcal{N}(0, 4)$.

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Question 4. CONFIDENCE INTERVALS [11 marks]

(a) [6 marks] Let $X_1, X_2, \ldots, X_n$ be a simple random sample drawn from a distribution with mean μ and variance σ^2.

(i) Prove that $E(\bar{X}) = \mu$ where $\bar{X}$ is the sample mean.

See notes

(ii) Prove that $Var(\bar{X}) = \frac{\sigma^2}{n}$.

See notes

(iii) Explain the significance of $E(\bar{X}) = \mu$ and $Var(\bar{X}) = \frac{\sigma^2}{n}$ in terms of estimation.

$$\left.\begin{array}{l} E(\bar{X}) = \mu \quad \text{UNBIASED} \\ Var(\bar{X}) = \sigma^2/n \quad \text{VAR. DROPS WITH } n \end{array}\right\} \text{together} = \text{consistency}$$

(b) [1 mark] Give a form of a 95% confidence interval (CI) for μ based on the simple random sample $X_1, X_2, \ldots, X_n$ when σ is known.

$$\bar{X} \pm 1.96\, \sigma/\sqrt{n}$$

(c) [2 marks] Give a form of a 95% confidence interval (CI) with specific numbers for μ based on a simple random sample of size 400 for which the sample mean and variance are $\bar{x} = 10$ and $s = 4$.

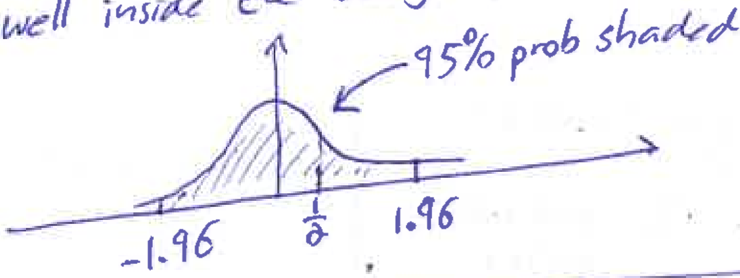
$$10 \pm 1.96 \times \frac{4}{\sqrt{400}} = 10 \pm \frac{1.96 \times 4}{20}$$

$$10 \pm 0.392$$

(d) [2 marks] Consider a discrete distribution with mean 10 and variance 16. Consider a simple random sample of size 400 drawn from this distribution. Why is an individual observation with value $x = 12$ a reasonable typical value whereas a sample mean of value $\bar{x} = 12$ is very unusual.

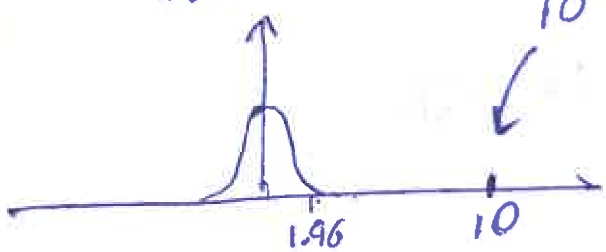
$$\frac{12 - 10}{4} = \frac{x - \mu}{\sigma} = \frac{1}{2}$$

$\frac{1}{2}$ is well inside the body of an $N(0,1)$ distⁿ



$$\frac{12 - 10}{4/\sqrt{400}} = \frac{2}{4/20} = 10$$

10 is well outside the normal range of values



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