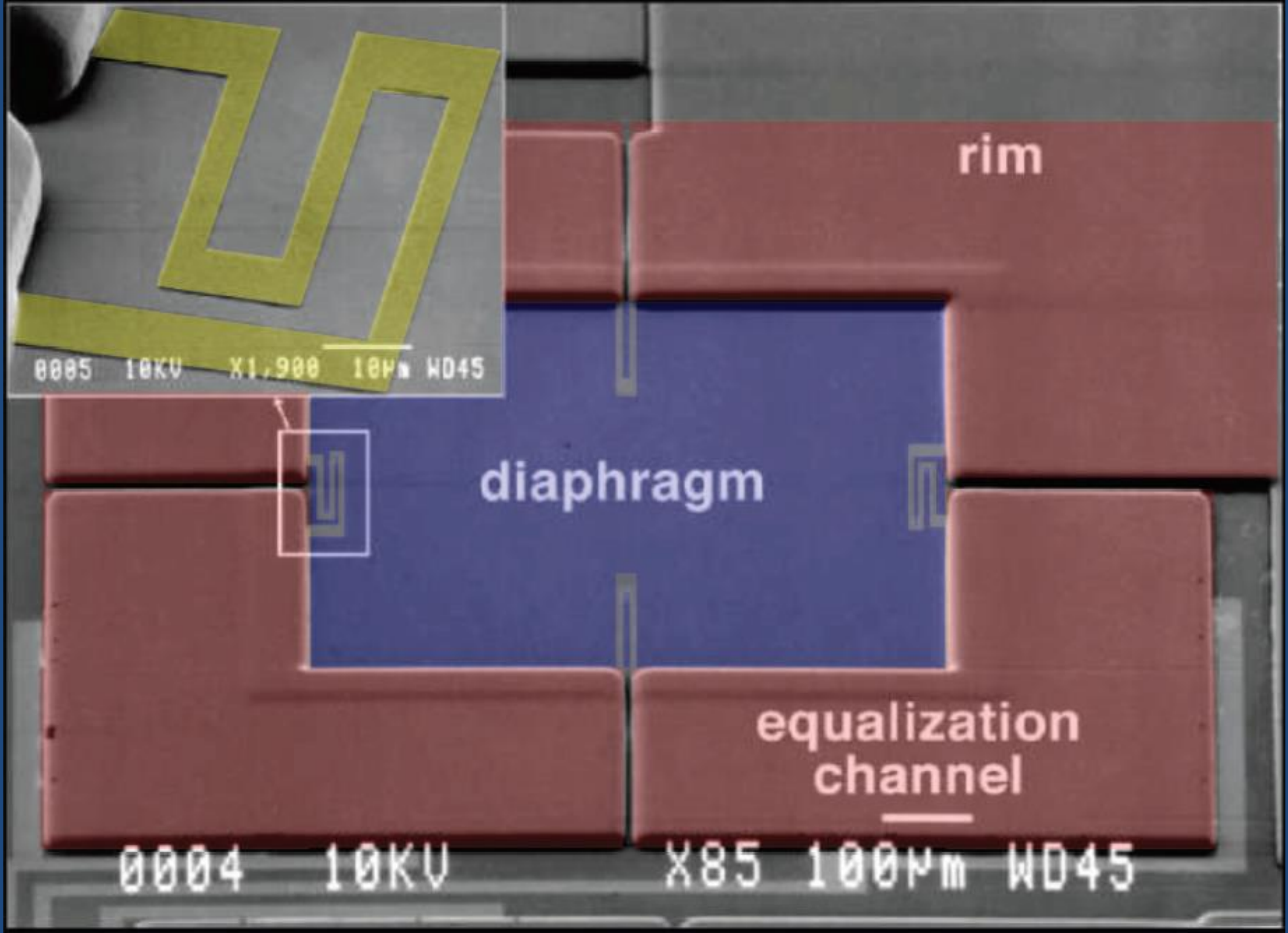
Piezoresistive sensor



LEC 2. RESISTIVE CIRCUITS

ECEN203

Analogue Circuits and Systems

Agenda

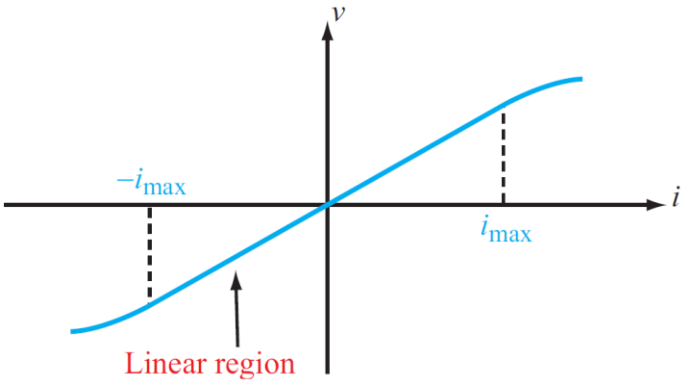
- Apply Ohm’s law and explain the basic properties of resistive elements,
- State Kirchhoff’s current and voltage laws; apply them to resistive circuits,
- Define circuit equivalency, combine resistors in series and in parallel, apply voltage and current division,
- Apply source transformation between voltage and current sources.

Ohm's Law

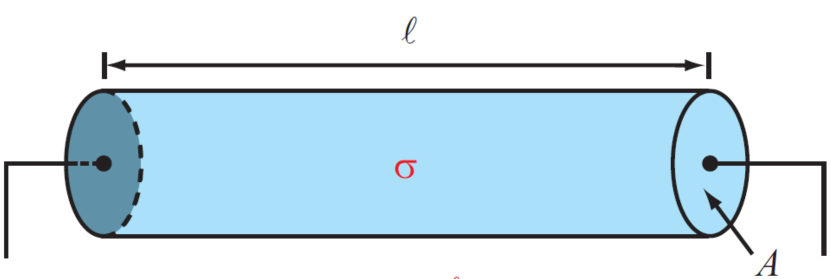
Voltage across resistor is proportional to current

$$\upsilon = iR$$

$$R = \frac{\upsilon}{i}$$



Resistance: ability to resist flow of electric current



$$R = \frac{\ell}{\sigma A}$$

$$R = \frac{\ell}{\sigma A} = \rho \frac{\ell}{A} \qquad (\Omega),$$

ρ = **resistivity (Ω-m)**

▶ The ***conductivity*** σ of a material is a measure of how easily electrons can drift through the material when an external voltage is applied across it. ***Resistivity*** (ρ) is the inverse $(1/\sigma)$ of conductivity. ◀

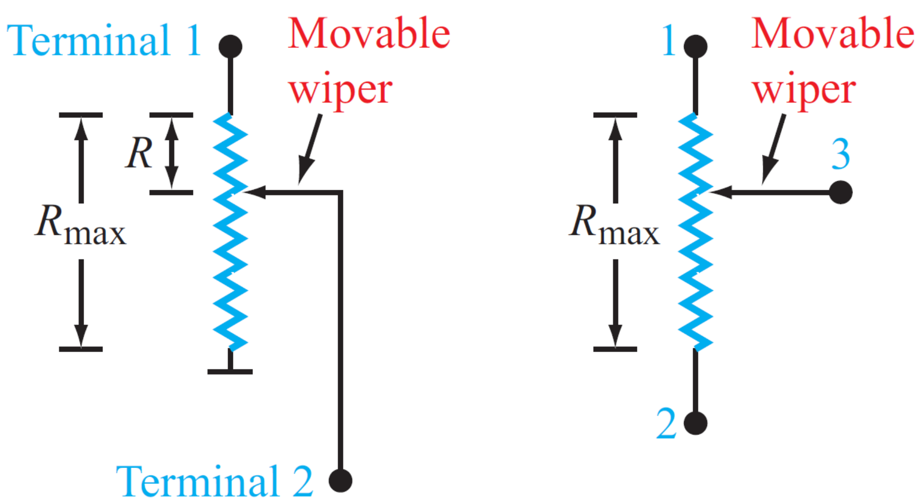
Conductivity

Table 2-1: Conductivity and resistivity of some common materials at 20 °C.

Material	Conductivity σ (S/m)	Resistivity ρ (Ω-m)
Conductors		
Silver	6.17×10^{7}	1.62×10^{-8}
Copper	5.81×10^{7}	1.72×10^{-8}
Gold	4.10×10^{7}	2.44×10^{-8}
Aluminum	3.82×10^{7}	2.62×10^{-8}
Iron	1.03×10^{7}	9.71×10^{-8}
Mercury (liquid)	1.04×10^{6}	9.58×10^{-7}
Semiconductors		
Carbon (graphite)	7.14×10^{4}	1.40×10^{-5}
Pure germanium	2.13	0.47
Pure silicon	4.35×10^{-4}	2.30×10^{3}
Insulators		
Paper	$\sim 10^{-10}$	$\sim 10^{10}$
Glass	$\sim 10^{-12}$	$\sim 10^{12}$
Teflon	$\sim 3.3 \times 10^{-13}$	$\sim 3 \times 10^{12}$
Porcelain	$\sim 10^{-14}$	$\sim 10^{14}$
Mica	$\sim 10^{-15}$	$\sim 10^{15}$
Polystyrene	$\sim 10^{-16}$	$\sim 10^{16}$
Fused quartz	$\sim 10^{-17}$	$\sim 10^{17}$
Common materials		
Distilled water	5.5×10^{-6}	1.8×10^{5}
Drinking water	$\sim 5 \times 10^{-3}$	~ 200
Sea water	4.8	0.2
Graphite	1.4×10^{-5}	71.4×10^{3}
Rubber	1×10^{-13}	1×10^{13}
Biological tissues		
Blood	~ 1.5	~ 0.67
Muscle	~ 1.5	~ 0.67
Fat	~ 0.1	10

Table 2-3: Common resistor terminology.

Thermistor	R sensitive to temperature
Piezoresistor	R sensitive to pressure
Rheostat	2-terminal variable resistor
Potentiometer	3-terminal variable resistor



(a) Rheostat (b) Potentiometer

Resistor Codes

(a) 4-Band color code: $\boldsymbol{b_1}$ $\boldsymbol{b_2}$ $\boldsymbol{b_4}$ $\boldsymbol{b_5}$

Note that a wider spacing exists between b_3 and b_5 than between the earlier bands. The resistor value is given by

$$R = (b_1 b_2) \times 10^{b_4} \pm b_5,$$

with the values of b_1, b_2, b_4, and b_5 specified by the color code shown in Fig. 2-2. For example,

$$= 25 \times 10^0 \pm 10\% = 25 \pm 10\%\ \Omega.$$

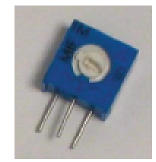
(b) 5-Band color code: $\boldsymbol{b_1}$ $\boldsymbol{b_2}$ $\boldsymbol{b_3}$ $\boldsymbol{b_4}$ $\boldsymbol{b_5}$

In this case

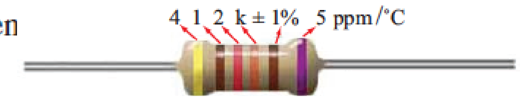
$$R = (b_1 b_2 b_3) \times 10^{b_4} \pm b_5.$$

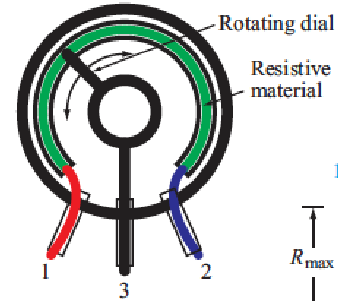
(c) 6-Band color code: $\boldsymbol{b_1}$ $\boldsymbol{b_2}$ $\boldsymbol{b_3}$ $\boldsymbol{b_4}$ $\boldsymbol{b_5}$ $\boldsymbol{b_6}$

Rotatable-shaft potentiometer

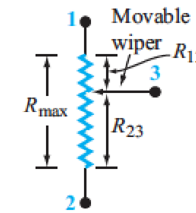


Screw-top potentiometer

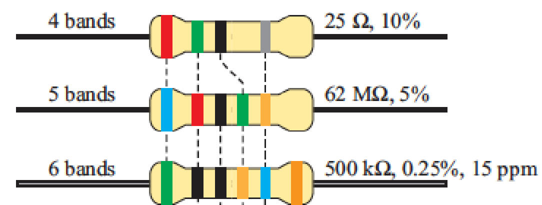




Potentiometer resistor



Potentiometer

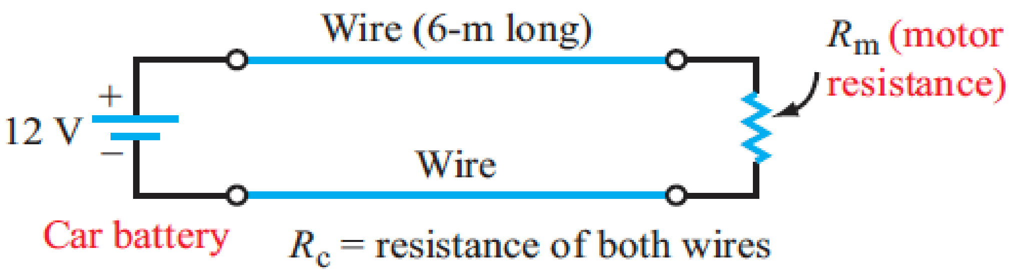


	b_1	b_2	b_3	b_4	b_5	b_6
Silver				0.01	10%	
Gold				0.1	5%	
Black	0	0	0	1		
Brown	1	1	1	10	1%	100 ppm
Red	2	2	2	100	2%	50 ppm
Orange	3	3	3	1K		15 ppm
Yellow	4	4	4	10K		25 ppm
Green	5	5	5	100K	0.5%	
Blue	6	6	6	1M	0.25%	10 ppm
Purple	7	7	7	10M	0.1%	5 ppm
Gray	8	8	8			
White	9	9	9			
	1st digit	2nd digit	3rd digit	Multiplier $\times 10^{b_4}$ # of zeros	Tolerance	Temperature coefficient ppm/°C

4-, 5-, and 6-band color code system

Example 2-1: dc Motor

What fraction of power supplied by the battery is dissipated in the motor?



Solution:

$$R_c = \rho \frac{\ell}{A}$$

$$= 1.72 \times 10^{-8} \times \frac{12}{\pi (1.3 \times 10^{-3})^2}$$

$$= 0.04\ \Omega.$$

$$R = R_c + R_m$$

$$= 0.04 + 2$$

$$= 2.04\ \Omega.$$

$$I = \frac{V}{R}$$

$$= \frac{12}{2.04}$$

$$= 5.88\ \text{A}$$

$$P = IV$$

$$= 5.88 \times 12$$

$$= 70.56\ \text{W}$$

$$P_m = I^2 R_m$$

$$= (5.88)^2 \times 2$$

$$= 69.15\ \text{W},$$

$$= 98\%\ \text{of}\ P$$

Problem 2.6 A light bulb has a filament whose resistance is characterized by a temperature coefficient $\alpha = 6 \times 10^{-3}\ ^\circ\mathrm{C}^{-1}$ (see resistance model given in Problem 2.5). The bulb is connected to a 100-V household voltage source via switch. After turning on the switch, the temperature of the filament increases rapidly from the initial room temperature of 20°C to an operating temperature of 1800°C. When it reaches its operating temperature, it consumes 80 W of power.

(a) Determine the filament resistance at 1800°C.

(b) Determine the filament resistance at room temperature.

(c) Determine the current that the filament draws at room temperature and also at 1800°C.

(d) If the filament deteriorates when the current through it approaches 10 A, is the damage done to the filament greater when it is first turned on or later on when it arrives at its operating temperature?

Solution:

(a) R = resistance at 1800°C.

$$p = \frac{V^2}{R}; \quad R = \frac{V^2}{p} = \frac{(100)^2}{80} = 125\ \Omega.$$

(b) $R = R_0(1 + \alpha T)$.

$$R_0 = \frac{R}{1 + \alpha T} = \frac{125}{1 + 6 \times 10^{-3} \times 1800} = 10.6\ \Omega \text{ @ } 0^\circ\mathrm{C}.$$

At $T = 20^\circ$C,

$$R = R_0(1 + \alpha T) = 10.6(1 + 6 \times 10^{-3} \times 20) = 11.9\ \Omega \text{ @ } 20^\circ\mathrm{C}.$$

(c) At $T = 20^\circ$C,

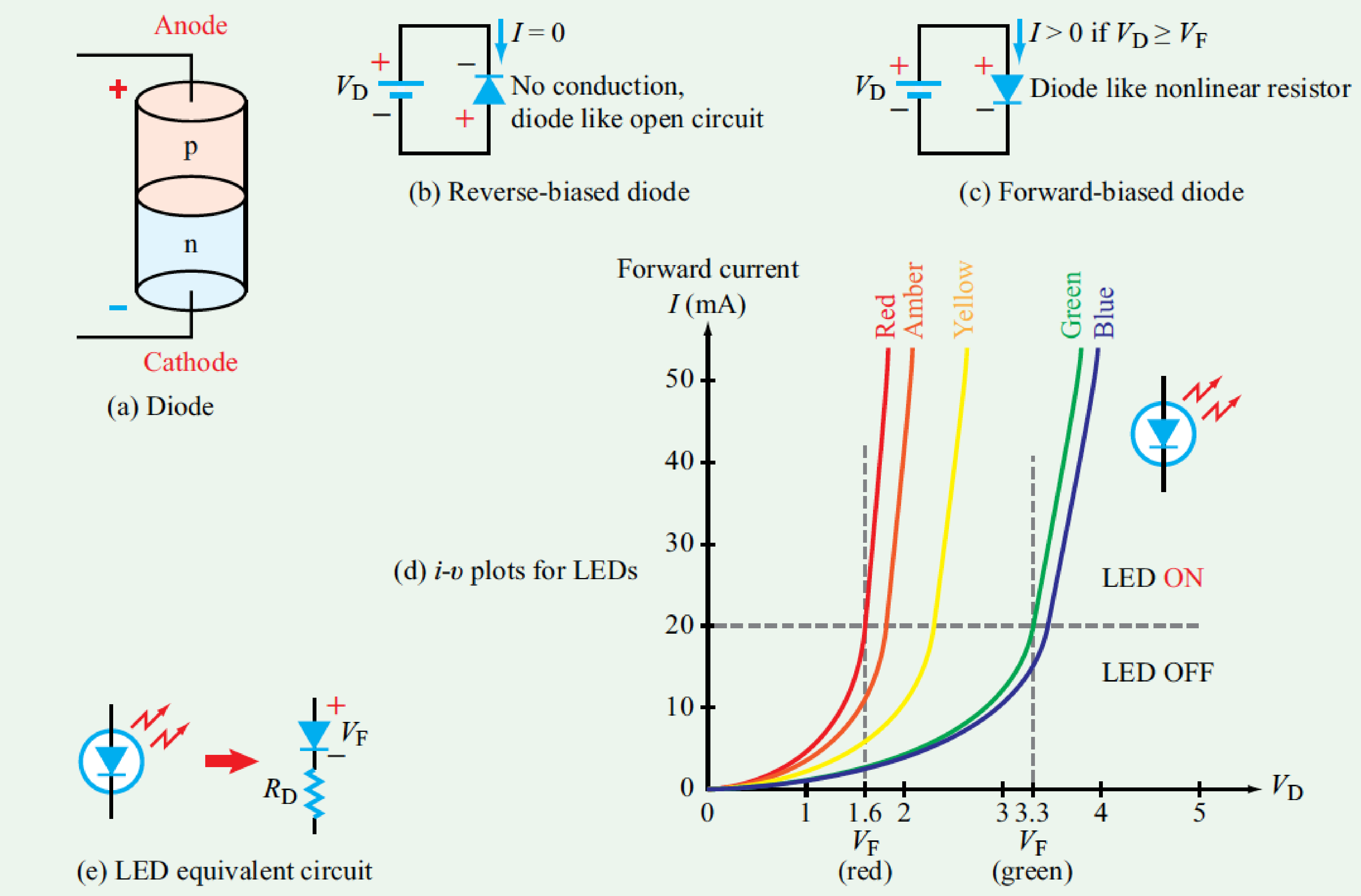
$$I = \frac{V}{R(20^\circ\mathrm{C})} = \frac{100}{11.9} = 8.43\ \mathrm{A}.$$

At $T = 1800^\circ$C,

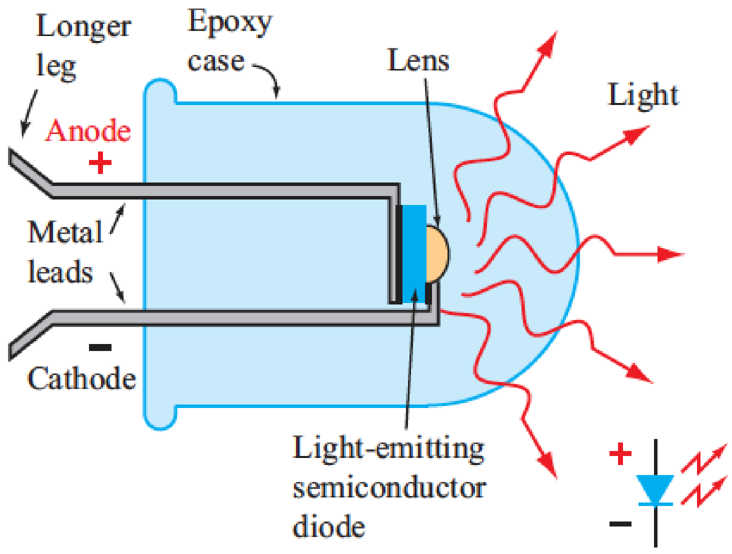
$$I = \frac{p}{V} = \frac{80}{100} = 0.8\ \mathrm{A}.$$

(d) The damage is greater at room temperature because the current is much closer to 10 A.

Light Emitting Diode



LEDs



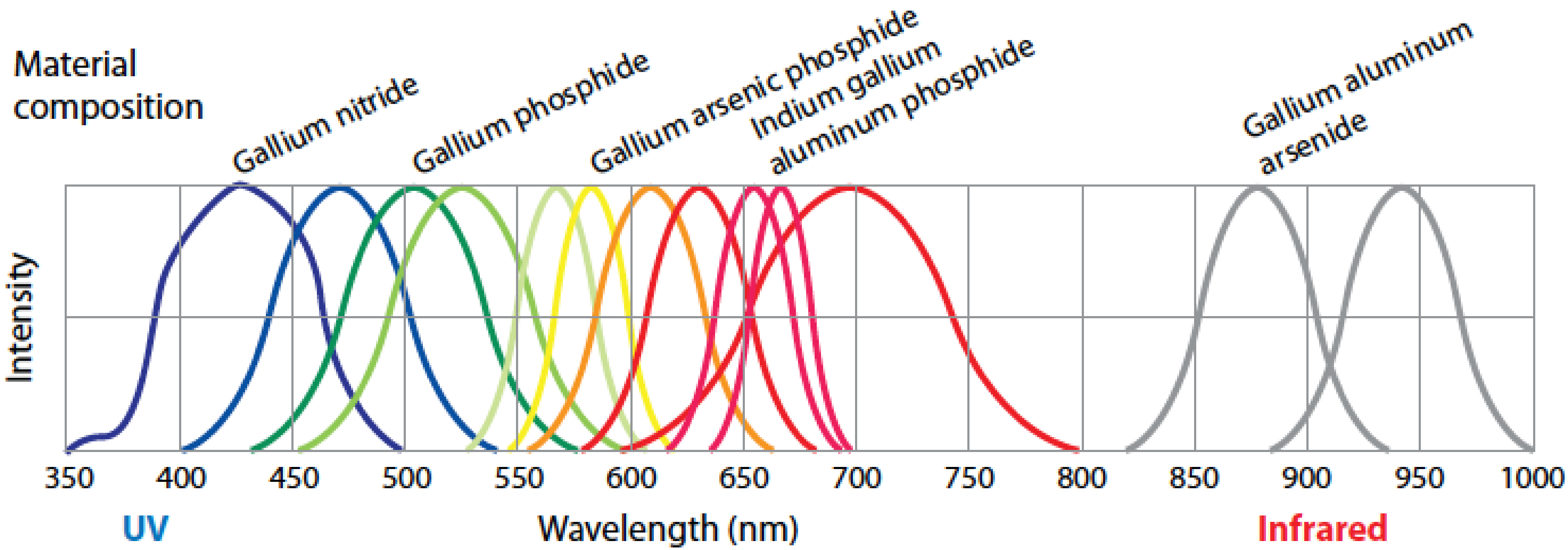


Figure TF5-1: Emission spectra of LEDs made of different material composites.

LED Examples

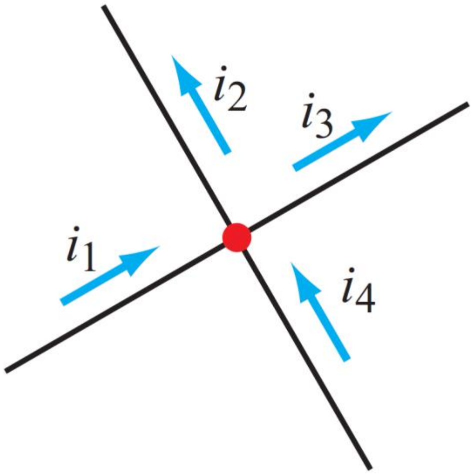
Figure TF5-4: LED eyelashes can be worn in many colors, and can be made to turn on or off with a tip of the head. (Credit: Soomi Park.)

Figure TF5-5: Pulse oximeter used to measure blood oxygen content.

Kirchhoff's Current Law

Sum of currents entering a node is zero

Also holds for closed boundary



$$\sum_{n=1}^{N} i_n = 0 \qquad \text{(KCL)},$$

► A common convention is to assign a positive "+" sign to a current if it is entering the node and a negative "−" sign if it is leaving it. ◄

$$i_1 - i_2 - i_3 + i_4 = 0$$

► The total current entering a node must be equal to the total current leaving it. ◄

$$i_1 + i_4 = i_2 + i_3$$

Example 2-4

If V_4, the voltage across the 4 Ω resistor in **Fig. 2-11**, is 8 V, determine I_1 and I_2.

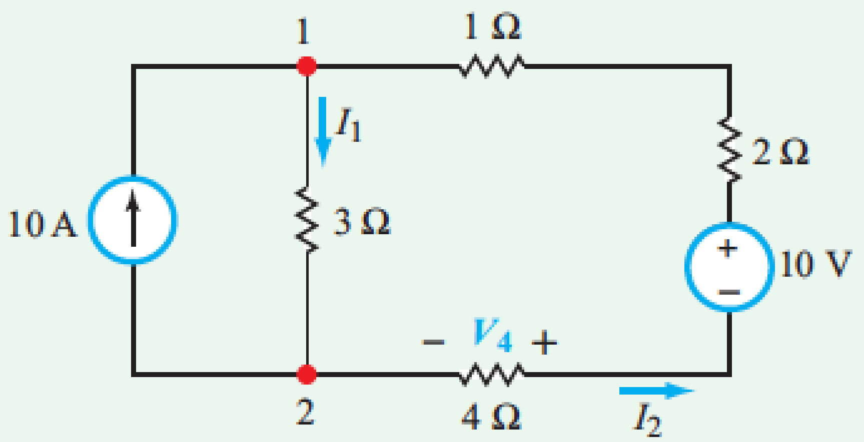


Figure 2-11: Circuit for Example 2-4.

Solution: The designated direction of I_2 is such that it enters the negative (−) terminal of V_4, whereas according to Ohm's law, the current should enter through the positive (+) terminal of the voltage across a resistor. Hence, in the present case, we should include a negative sign in the relationship between I_2 and V_4, namely

$$I_2 = -\frac{V_4}{4} = -\frac{8}{4} = -2 \text{ A}.$$

Thus, the true direction of the current flowing through the 4 Ω resistor is opposite of that of I_2.

Using the KCL convention that defines a current as positive if it is leaving a node and negative if it is entering it, at node 2:

$$10 - I_1 + I_2 = 0,$$

which leads to

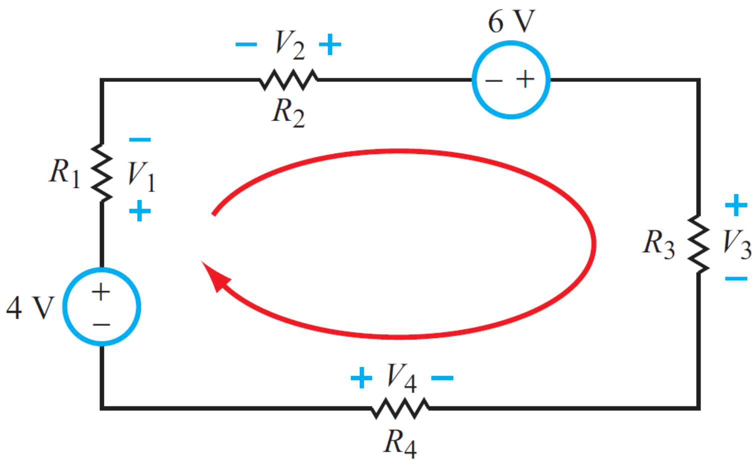
$$I_1 = 10 + I_2 = 10 - 2 = 8 \text{ A}.$$

Kirchhoff's Voltage Law (KVL)

Sum of voltages around a closed path is zero

Sum of voltage drops = sum of voltage rises

$$\sum_{n=1}^{N} v_n = 0 \qquad \text{(KVL)},$$



Sign Convention

- Add up the voltages in a systematic clockwise movement around the loop.
- Assign a positive sign to the voltage across an element if the (+) side of that voltage is encountered first, and assign a negative sign if the (−) side is encountered first.

$$-4 + V_1 - V_2 - 6 + V_3 - V_4 = 0$$

KCL/KVL Rules

Table 2-4: Equally valid, multiple statements of Kirchhoff's Current Law (KCL) and Kirchhoff's Voltage Law (KVL).

KCL

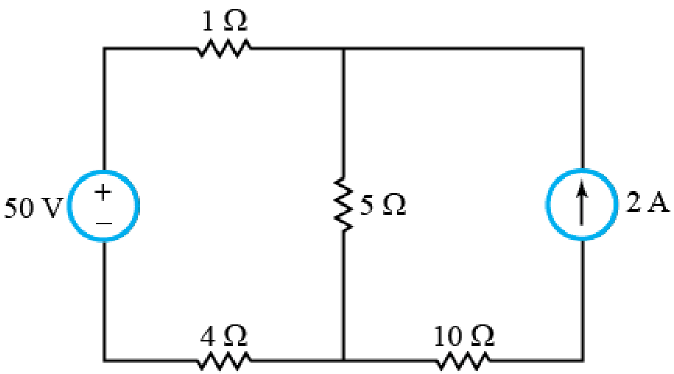
- Sum of all currents entering a node $= 0$
  [$i =$ "+" if entering; $i =$ "−" if leaving]
- Sum of all currents leaving a node $= 0$
  [$i =$ "+" if leaving; $i =$ "−" if entering]
- Total of currents entering $=$ Total of currents leaving

KVL

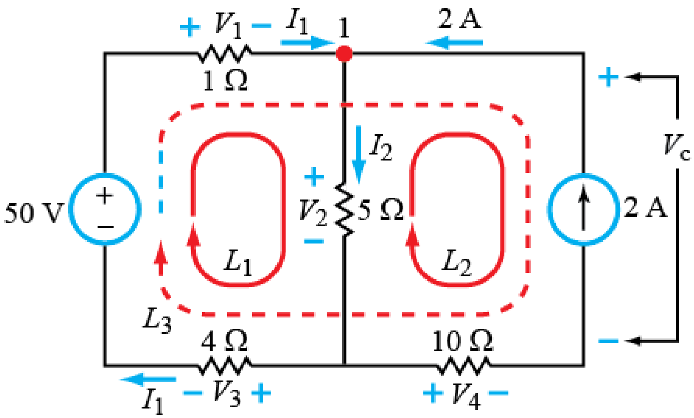
- Sum of voltages around closed loop $= 0$
  [$\upsilon =$ "+" if $+$ side encountered first in clockwise direction]
- Total voltage rise $=$ Total voltage drop

► An alternative statement of KVL is that *the total voltage rise around a closed loop must equal the total voltage drop around the loop.* ◄

Example : KCL/KVL



(a)



(b)

Determine all currents & voltages

Solution:

$$-50 + I_1 + 5I_2 + 4I_1 = 0, \qquad \text{Loop 1}$$

$$-5I_2 + V_c - (10 \times 2) = 0. \qquad \text{Loop 2}$$

Next, we apply KCL at node 1 which gives

$$I_1 - I_2 + 2 = 0.$$

Three equations w/three unknowns:

$$I_1 = 4 \text{ A},$$

$$I_2 = 6 \text{ A},$$

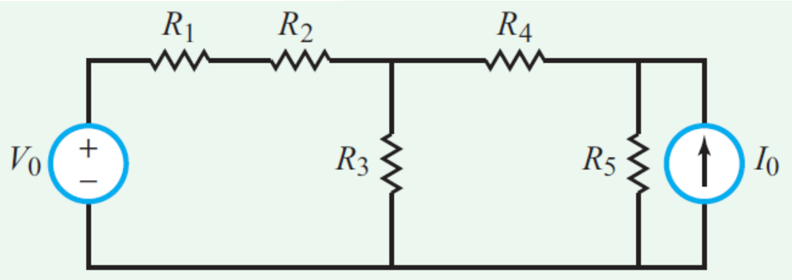
$$V_c = 50 \text{ V}.$$

KCL/KVL Solution Recipe

- Use KCL, KVL and Ohm's law to develop as many independent equations as the number of unknowns (N).
- Write as many KVL loop equations as you can, picking up at least one additional circuit element for each loop. Let M be the number of such loop equations. Exclude loops that go through current sources.
- Write (N-M) KCL equations, making sure each node picking up an additional current.
- Write the equations in standard form.
- Cast the standard-form equations in matrix form.
- Apply matrix inversion to compute the values of the circuit unknowns.

Example: Matrix Inversion of KVL/KCL Equations

For the circuit in **Fig. 2-14(a)**: (a) identify all N unknown branch currents and assign them preliminary directions, (b) develop M KVL loop equations through all possible elements (while excluding loops containing current sources), (c) develop $(N - M)$ KCL node equations, (d) arrange the equations in matrix form, (e) solve by matrix reduction to find the unknown branch currents, (f) determine the power dissipated in R_5, and (g) find the voltages of all extraordinary nodes relative to the negative terminal of the voltage source. The element values are: $V_0 = 10$ V, $I_0 = 0.8$ A, $R_1 = 2\ \Omega$, $R_2 = 3\ \Omega$, $R_3 = 5\ \Omega$, $R_4 = 10\ \Omega$, and $R_5 = 2.5\ \Omega$.

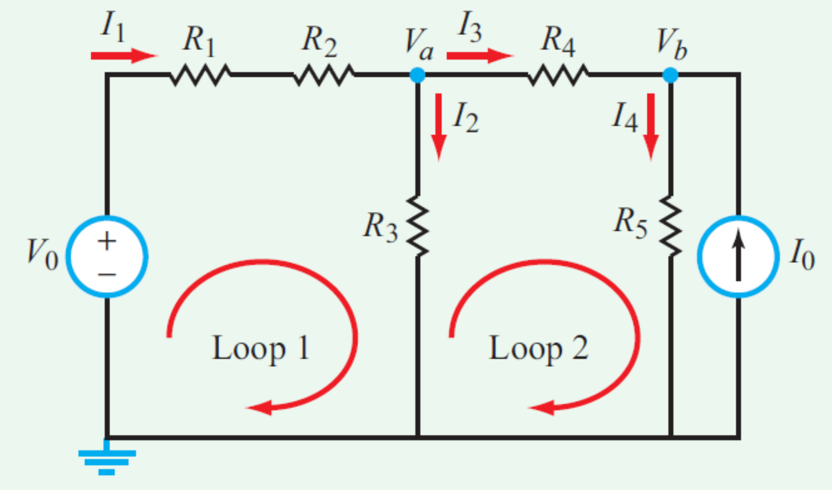


(a) Original circuit

Solution:

(a) Identify unknown currents

Excluding the branch containing I_0 (since we know that the current in that branch is $I_0 = 0.8$ A), we have 4 unknown branch currents, which we denote I_1 to I_4 in **Fig. 2-14(b)**. Also, with the negative terminal of the voltage source denoted as the voltage reference (ground), we have two extraordinary nodes, with designated voltages V_a and V_b.



(b)

(b) KVL equations

The circuit contains two independent loops that do not contain the current source I_0. The associated KVL equations are:

$$-V_0 + I_1 R_1 + I_1 R_2 + R_3 I_2 = 0 \qquad \textbf{(Loop 1)},$$
$$-I_2 R_3 + I_3 R_4 + I_4 R_5 = 0 \qquad \textbf{(Loop 2)}.$$

(c) KCL equations

We have two extraordinary nodes (in addition to the ground node). We designate their voltages as shown in **Fig. 2-14(b)**. With current defined as positive when entering a node, their KCL equations are

$$I_1 - I_2 - I_3 = 0 \qquad \textbf{(Node } a\textbf{)},$$
$$I_3 - I_4 + I_0 = 0 \qquad \textbf{(Node } b\textbf{)}.$$

(d) Arrange equations in matrix form

$$\underbrace{\begin{bmatrix} (R_1 + R_2) & R_3 & 0 & 0 \\ 0 & -R_3 & R_4 & R_5 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}}_{\mathbf{I}} = \underbrace{\begin{bmatrix} V_0 \\ 0 \\ 0 \\ -I_0 \end{bmatrix}}_{\mathbf{B}}.$$

This is in the form

$$\mathbf{AI} = \mathbf{B}.$$

(e) Matrix inversion

- After replacing sources and resistors with their values, can use Matlab or Cramer's Rule (see appendix later), to find the answers:

$$I_1 = 1.1 \text{ A}, \qquad I_2 = 0.9 \text{ A},$$
$$I_3 = 0.2 \text{ A}, \qquad I_4 = 1 \text{ A}, .$$

(f) Power in R_5

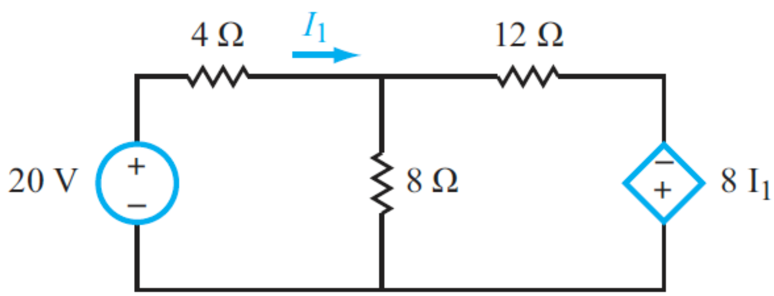
$$P = I_4^2 R_5 = 1^2 \times 2.5 = 2.5 \text{ W}.$$

(g) Node voltages

$$V_a = I_2 R_3 = 0.9 \times 5 = 4.5 \text{ V},$$
$$V_b = I_4 R_5 = 1 \times 2.5 = 2.5 \text{ V}.$$

Example 2-8 Circuit with Dependent Source

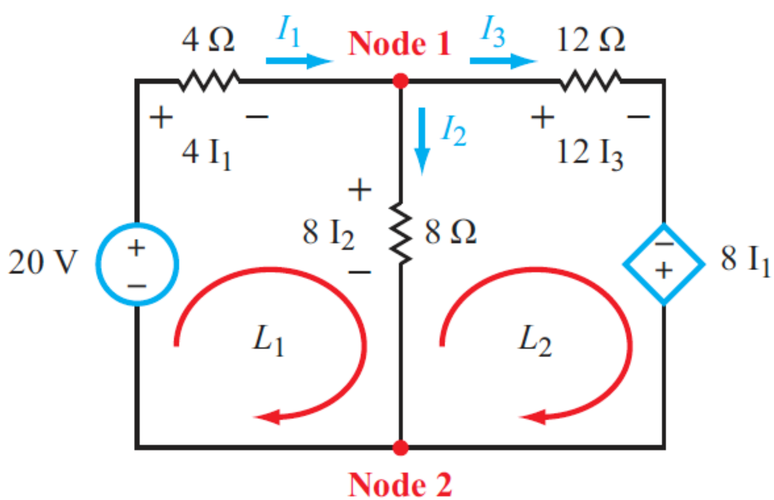
The circuit in Fig. 2-13 includes a current-dependent voltage source. Apply KVL and KCL to determine the amount of power consumed by the 12-Ω resistor.



(a) Given circuit

$$-20 + 4I_1 + 8I_2 = 0, \qquad \text{(KVL for Loop 1)}$$

$$-8I_2 + 12I_3 - 8I_1 = 0. \qquad \text{(KVL for Loop 2)}$$



(b) After assigning currents at node 1

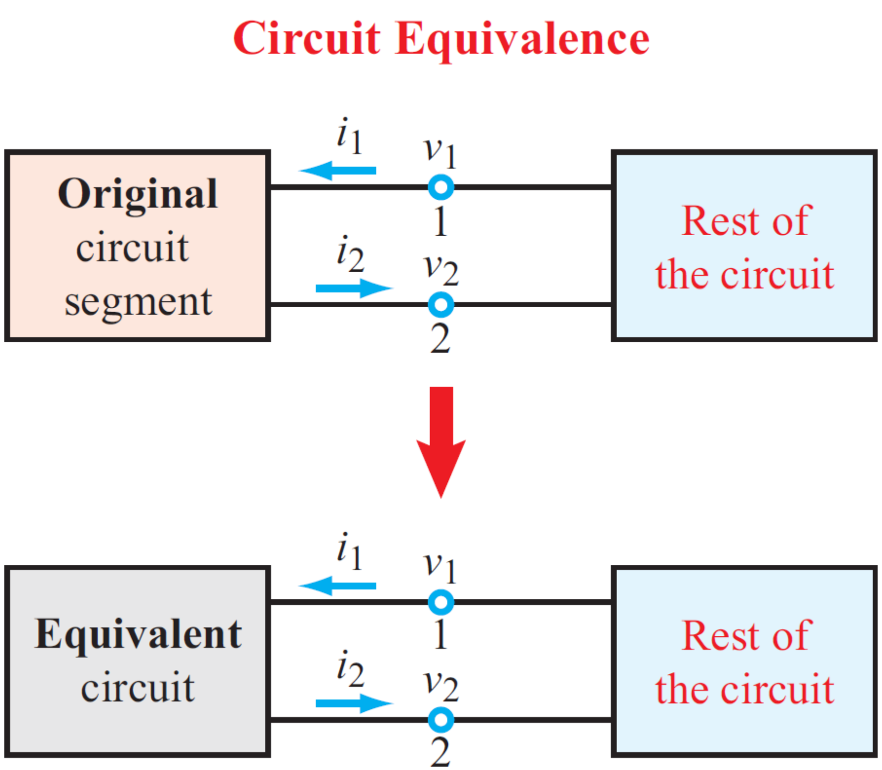
$$I_1 = I_2 + I_3. \qquad \text{KCL}$$

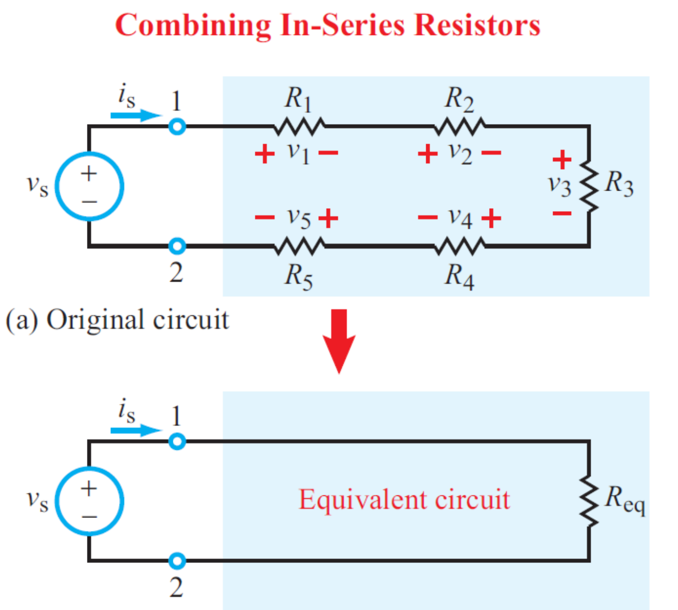
$$I_1 = \frac{25}{7}\text{ A}, \qquad I_2 = \frac{5}{7}\text{ A}, \qquad I_3 = \frac{20}{7}\text{ A}.$$

$$P = I_3^2 R = \left(\frac{20}{7}\right)^2 \times 12 = 97.96\text{ W}.$$

Equivalent Circuits

- If the current and voltage characteristics at nodes are identical, the circuits are considered "equivalent"
- Identifying equivalent circuits simplifies analysis

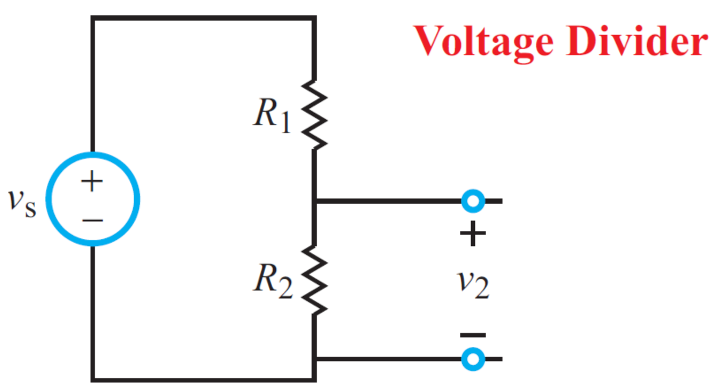




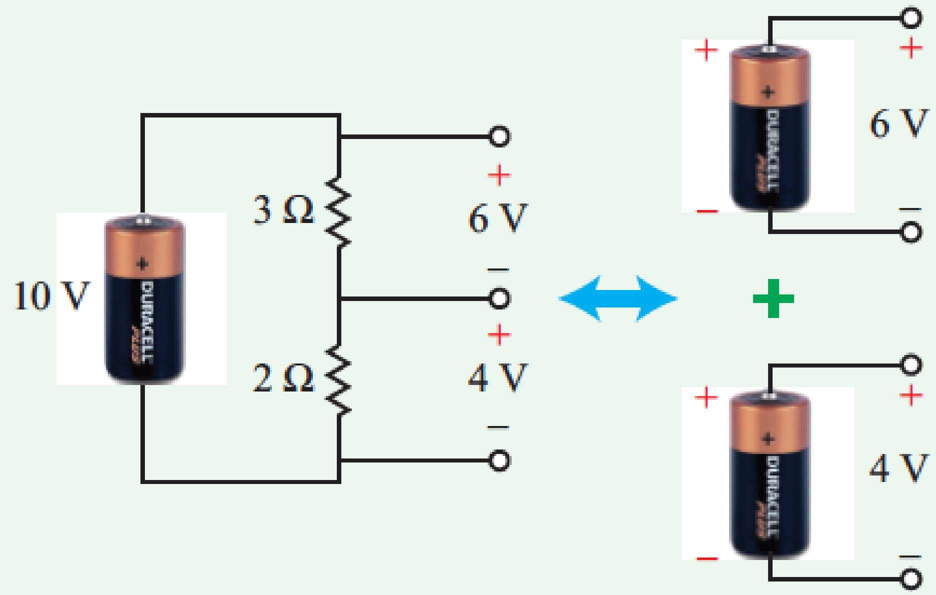
$$
\begin{aligned}
v_s &= R_1 i_s + R_2 i_s + R_3 i_s + R_4 i_s + R_5 i_s \\
&= (R_1 + R_2 + R_3 + R_4 + R_5) i_s \\
&= R_{eq} i_s,
\end{aligned}
$$

Resistors in Series

Equivalent resistance (series) is sum of resistances



$$v_2 = \left(\frac{R_2}{R_1 + R_2}\right) v_s$$



(b) Voltage divider is equivalent to subdividing a battery into two separate batteries

Voltage divided over resistors (voltage divider)

Adding Sources In Series

Unrealizable Circuit

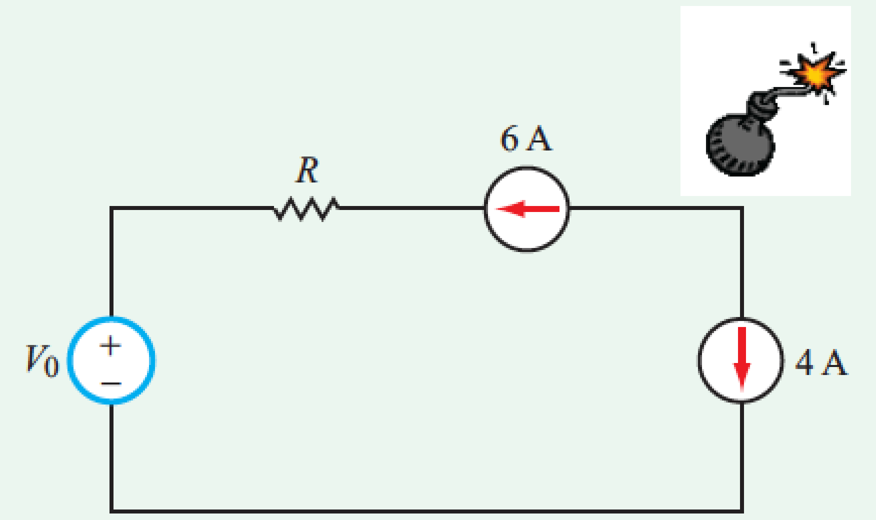
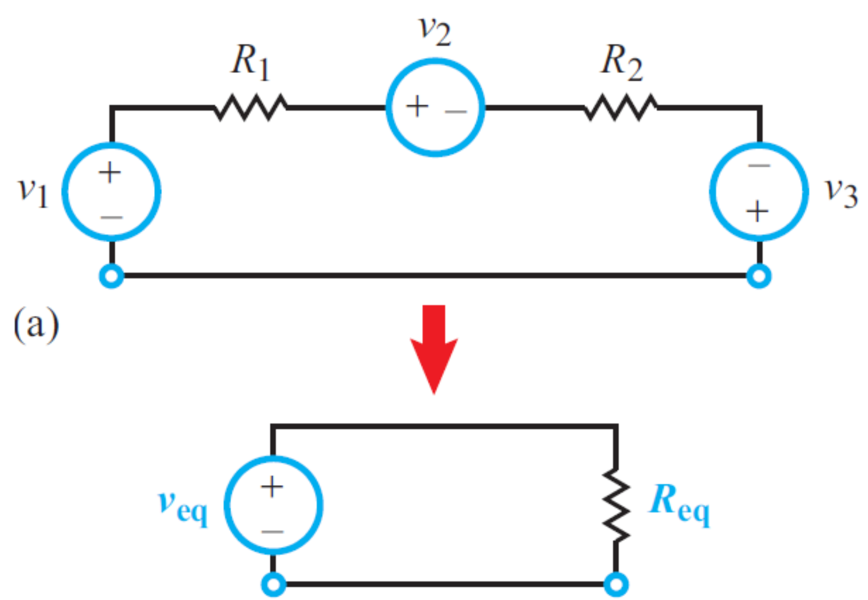


Figure 2-20: Unrealizable circuit; two current sources with different magnitudes or directions cannot be connected in series.

▶ Ideal current sources cannot be added in series. ◀

Combining voltage sources

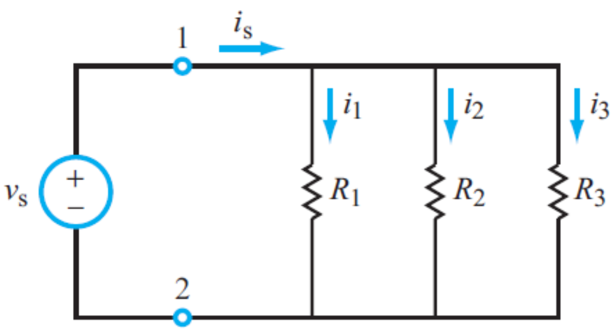


(b) $v_{eq} = v_1 - v_2 + v_3$ $\quad R_{eq} = R_1 + R_2$

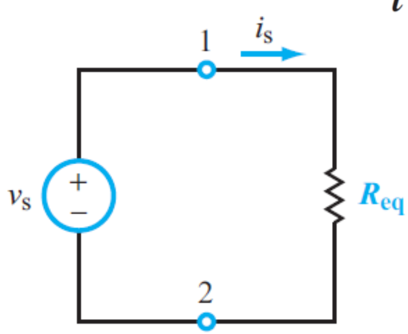
▶ Multiple voltage sources connected in series can be combined into an equivalent voltage source whose voltage is equal to the algebraic sum of the voltages of the individual sources. ◀

Resistors in Parallel

Combining In-Parallel Resistors



(a) Original circuit

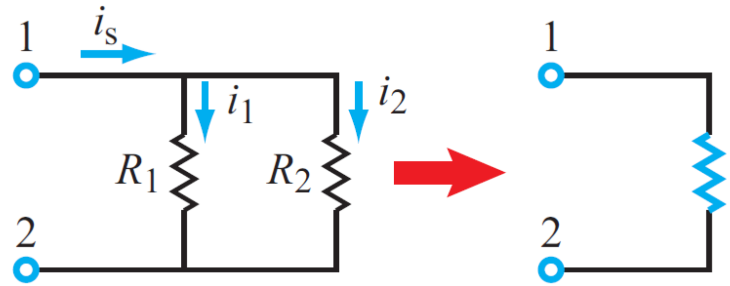


(b) Equivalent circuit

$$i_s = \frac{v_s}{R_1} + \frac{v_s}{R_2} + \frac{v_s}{R_3} = \frac{v_s}{R_{eq}}.$$

$$R_{eq} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1} \qquad i_2 = \left(\frac{R_{eq}}{R_2}\right) i_s$$

Current Division



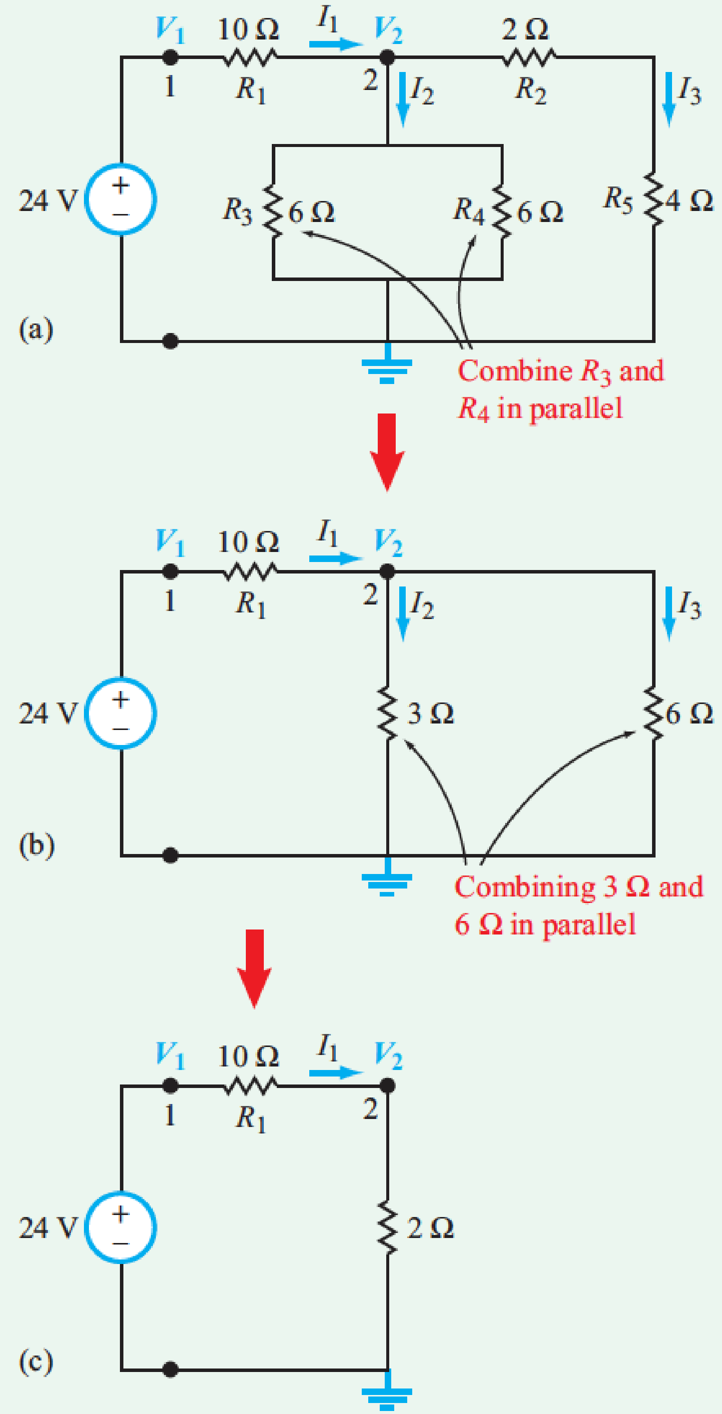
$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

$$i_1 = \left(\frac{R_2}{R_1 + R_2}\right) i_s \qquad i_2 = \left(\frac{R_1}{R_1 + R_2}\right) i_s$$

$$\frac{1}{R_{eq}} = \sum_{i=1}^{N} \frac{1}{R_i} \qquad \text{(resistors in parallel).}$$

Example 2-12: Equivalent-Circuit Solution

Use the equivalent-resistance approach to determine V_2, I_1, I_2, and I_3 in the circuit of Fig. 2-28(a).



$$I_1 = \frac{24}{10 + 2} = 2 \text{ A}$$

and

$$V_2 = 2I_1 = 2 \times 2 = 4 \text{ V}.$$

Returning to Fig. 2-28(b), we apply Ohm's law to find I_2 and I_3.

$$I_2 = \frac{V_2}{3} = \frac{4}{3} = 1.33 \text{ A},$$

and

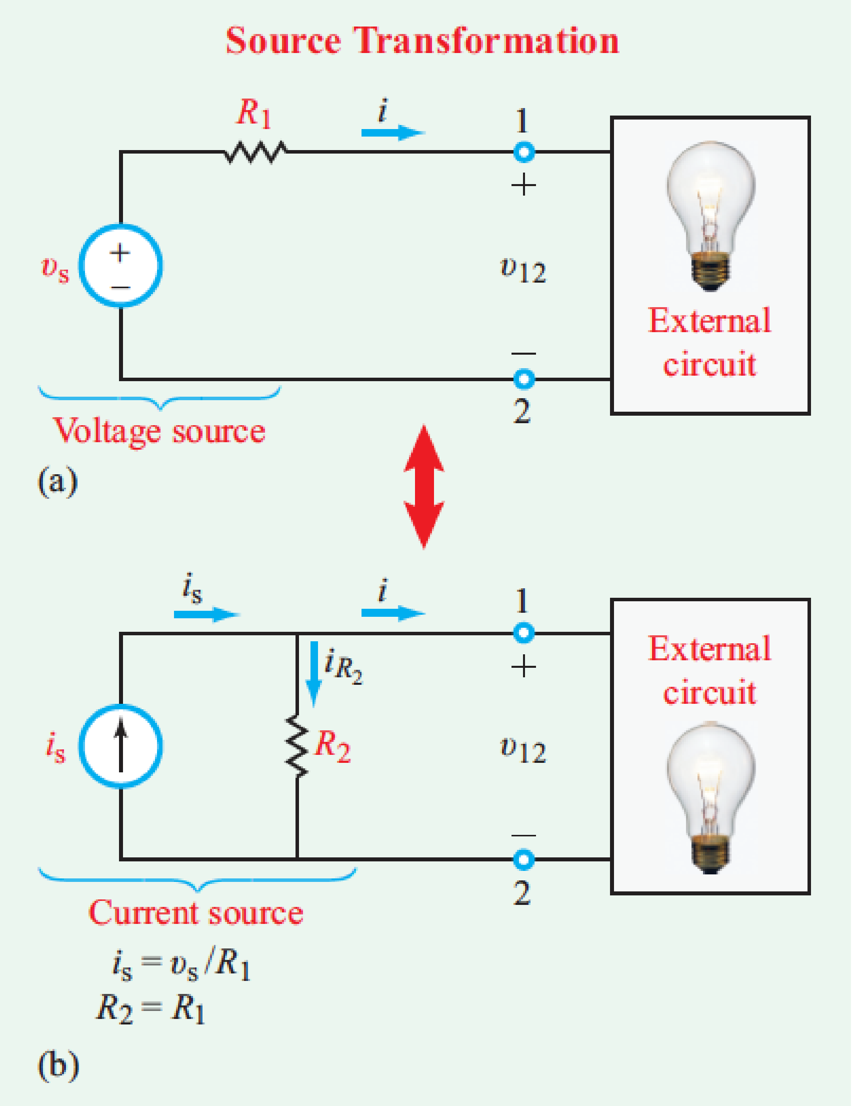
$$I_3 = \frac{V_2}{6} = \frac{4}{6} = 0.67 \text{ A}.$$

Source Transformation

$$-v_s + iR_1 + v_{12} = 0$$

Hence, $$i = \frac{v_s}{R_1} - \frac{v_{12}}{R_1}.$$

$$i = i_s - i_{R_2}$$
$$= i_s - \frac{v_{12}}{R_2}$$



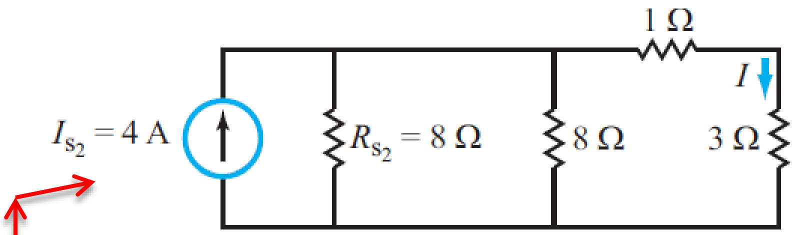
For the two circuits to be equivalent :

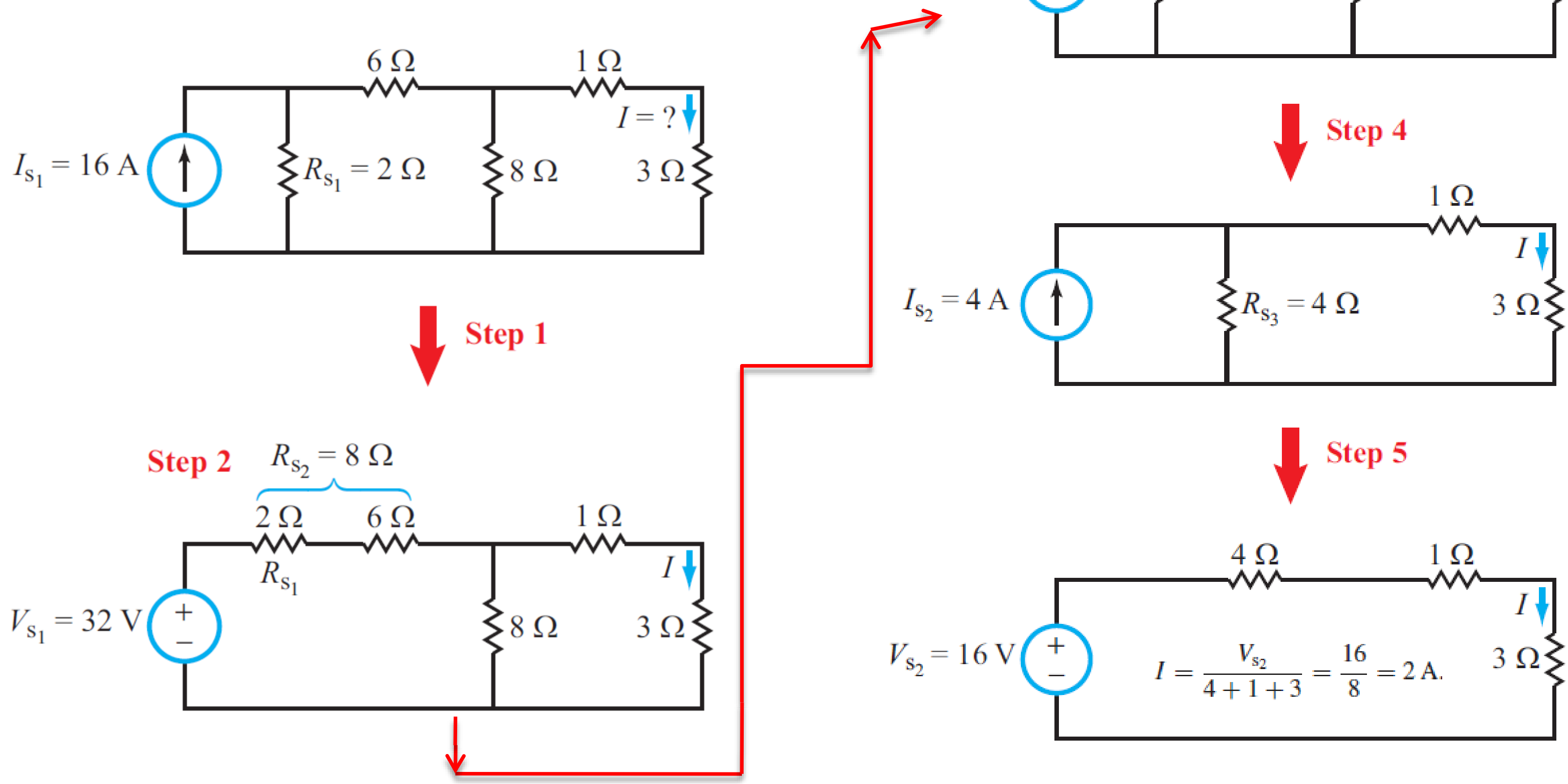
$$R_1 = R_2$$

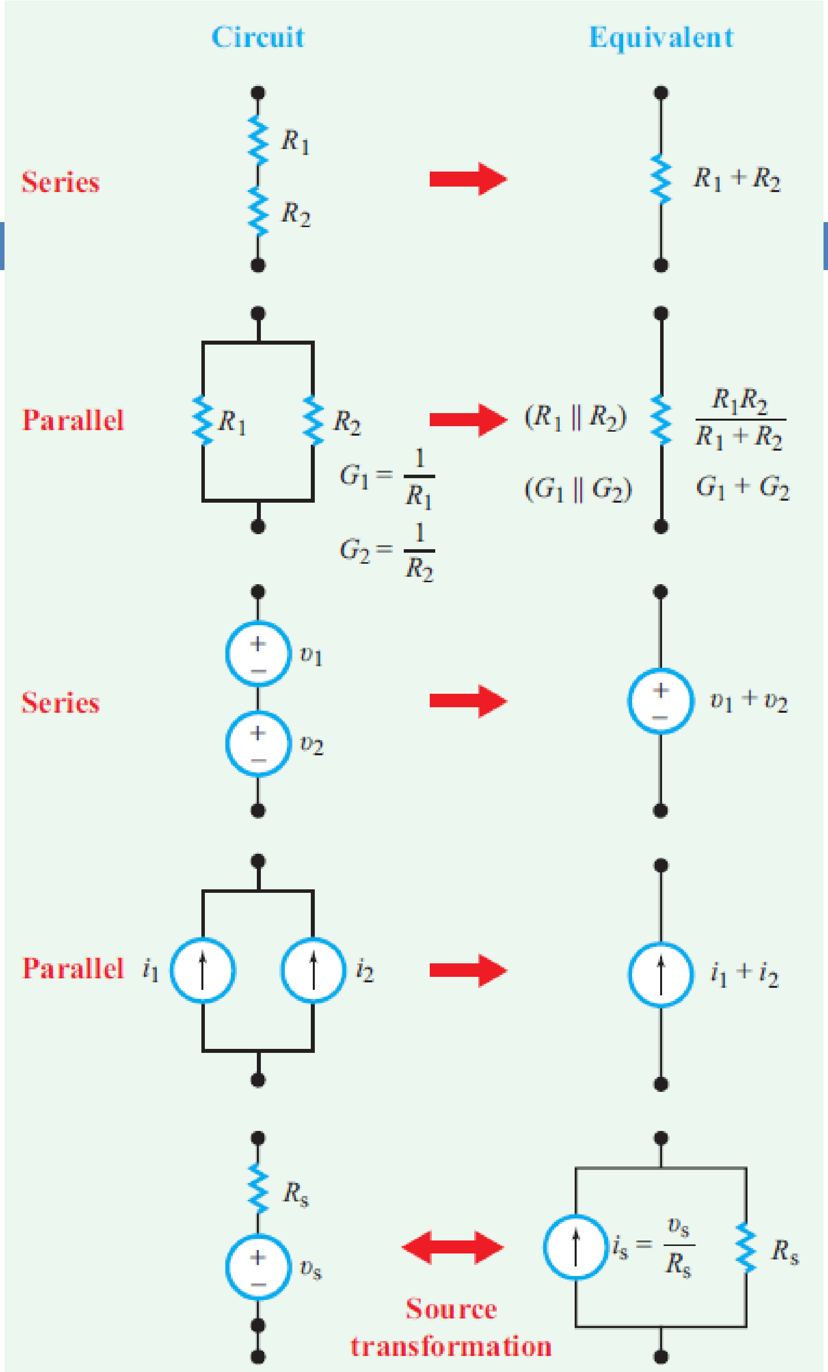
$$i_s = \frac{v_s}{R_1}.$$

Example 2-13: Source Transformation

Determine the current I in the circuit of Fig. 2-26(a).







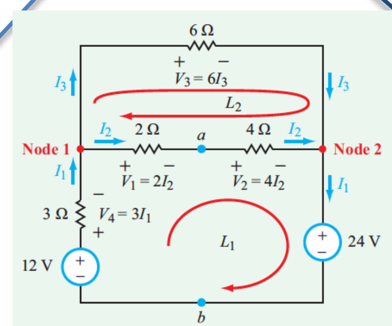
Circuit
Equivalent
Series
R_1
R_2
$R_1 + R_2$
Parallel
R_1
R_2
$G_1 = \frac{1}{R_1}$
$G_2 = \frac{1}{R_2}$
$(R_1 \parallel R_2)$
$\frac{R_1 R_2}{R_1 + R_2}$
$(G_1 \parallel G_2)$
$G_1 + G_2$
Series
υ_1
υ_2
$\upsilon_1 + \upsilon_2$
Parallel
i_1
i_2
$i_1 + i_2$
R_s
υ_s
$i_s = \frac{\upsilon_s}{R_s}$
R_s
Source transformation

Complimentary Approaches To Learning Circuits and Electronics

Analytical

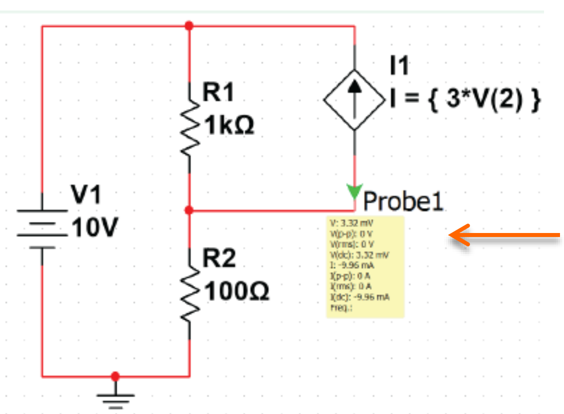
(By-hand)

$$-20 + 4I_1 + 8I_2 = 0, \qquad \text{(KVL for Loop 1)}$$
$$-8I_2 + 12I_3 - 8I_1 = 0. \qquad \text{(KVL for Loop 2)}$$



Computer Simulation

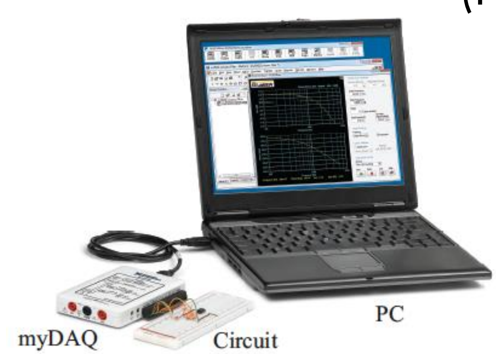
(LTSpice)



Labs

Measurements

(Physical Lab)



Summary

Mathematical and Physical Models

Linear resistor

$$R = \rho \ell / A$$
$$p = i^2 R$$

Kirchhoff current law (KCL)

$$\sum_{n=1}^{N} i_n = 0$$

i_n = current entering node n

Kirchhoff voltage law (KVL)

$$\sum_{n=1}^{N} \upsilon_n = 0$$

υ_n = voltage across branch n

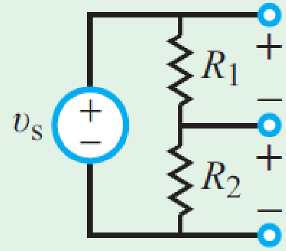
Resistor combinations

In series $\quad R_{eq} = \sum_{i=1}^{N} R_i$

In parallel $\quad \dfrac{1}{R_{eq}} = \sum_{i=1}^{N} \dfrac{1}{R_i}$

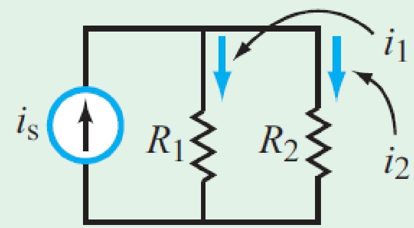
or $\quad G_{eq} = \sum_{i=1}^{N} G_i$

Voltage division



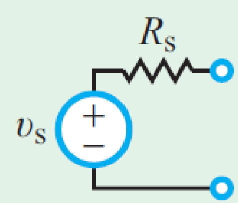
$$\upsilon_1 = \left(\frac{R_1}{R_1 + R_2}\right)\upsilon_s$$
$$\upsilon_2 = \left(\frac{R_2}{R_1 + R_2}\right)\upsilon_s$$

Current division

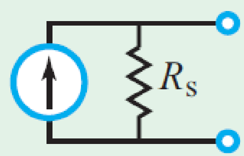


$$i_1 = \left(\frac{R_2}{R_1 + R_2}\right) i_s = \frac{G_1}{G_{eq}} i_s$$
$$i_2 = \left(\frac{R_1}{R_1 + R_2}\right) i_s = \frac{G_2}{G_{eq}} i_s$$

Source transformation



$$i_s = \frac{\upsilon_s}{R_s}$$



Appendix: Cramer's Rule

Step 1: Cast Equations in Standard Form

Example, given these equation:

$$\begin{aligned} i_1 + 5i_2 - 4i_3 &= 10, \\ -i_1 - 3i_2 + 6i_3 &= 0, \\ i_1 - \; i_2 \qquad\quad &= 5. \end{aligned}$$

Generate a matrix equation of form **AI=B**

$$\begin{bmatrix} 1 & 5 & -4 \\ -1 & -3 & 6 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 5 \end{bmatrix}$$

Cramer's Rule

Step 2: General Solution

The solutions to currents are solved by

$$i_1 = \frac{\Delta_1}{\Delta},$$

$$i_2 = \frac{\Delta_2}{\Delta},$$

$$i_3 = \frac{\Delta_3}{\Delta},$$

where Δ is the value of the ***characteristic determinant***

and Δ_1 to Δ_3 are the ***affiliated determinants*** for variables i_1 to i_3.

Note: Δ cannot be zero.

Cramer's Rule

Step 3: Evaluating the Characteristic Determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

$$\Delta = \begin{vmatrix} 1 & 5 & -4 \\ -1 & -3 & 6 \\ 1 & -1 & 0 \end{vmatrix}.$$

We expand the top row, and this gives:

$$\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = C_{11} + 5C_{12} - 4C_{13},$$

where C_{11}, C_{12}, and C_{13} are the ***cofactors*** of elements a_{11}, a_{12}, and a_{13}, respectively.

$$C_{jk} = (-1)^{j+k} M_{jk},$$

Cramer's Rule

and the minor determinant M_{jk} is obtained by deleting from the parent determinant all elements contained in row j and column k. Hence, M_{11} is given by Δ after removal of the top row and the left column,

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} -3 & 6 \\ -1 & 0 \end{vmatrix}.$$

$$M_{11} = a_{22}M_{22} - a_{23}M_{23} = a_{22}a_{33} - a_{23}a_{32},$$

$$M_{11} = (-3) \times 0 - 6 \times (-1) = 6.$$

Similarly, M_{12} is obtained by removing from Δ the elements in row 1 and the elements in column 2,

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31} \quad = (-1) \times 0 - 6 \times 1 = -6.$$

Cramer's Rule

$$M_{13} = \begin{vmatrix} -1 & -3 \\ 1 & -1 \end{vmatrix} = (-1) \times (-1) - (-3) \times 1 = 4.$$

$$\begin{aligned} \Delta &= C_{11} + 5C_{12} - 4C_{13} \\ &= M_{11} - 5M_{12} - 4M_{13} \\ &= 6 - 5 \times (-6) - 4 \times 4 = 20. \end{aligned}$$

Cramer's Rule

Step 4: Evaluating the Affiliated Determinants

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 10 & 5 & -4 \\ 0 & -3 & 6 \\ 5 & -1 & 0 \end{vmatrix}.$$

Evaluation of Δ_1 follows the same rules of expansion discussed earlier in Step 3 in connection with the evaluation of Δ. Hence

$$\Delta_1 = 10 \begin{vmatrix} -3 & 6 \\ -1 & 0 \end{vmatrix} - 5 \begin{vmatrix} 0 & 6 \\ 5 & 0 \end{vmatrix} - 4 \begin{vmatrix} 0 & -3 \\ 5 & -1 \end{vmatrix}$$
$$= 10 \times 6 - 5 \times (-30) - 4 \times 15 = 150.$$

And so,

$$i_1 = \frac{\Delta_1}{\Delta} = \frac{150}{20} = 7.5.$$

Similarly, Δ_2 is obtained from Δ upon replacing column 2 with the b-column, and Δ_3 is obtained from Δ upon replacing column 3 with the b-column. The procedure leads to $\Delta_2 = 50$, $\Delta_3 = 50$, $i_2 = \Delta_2/\Delta = 50/20 = 2.5$, and $i_3 = \Delta_3/\Delta = 2.5$.