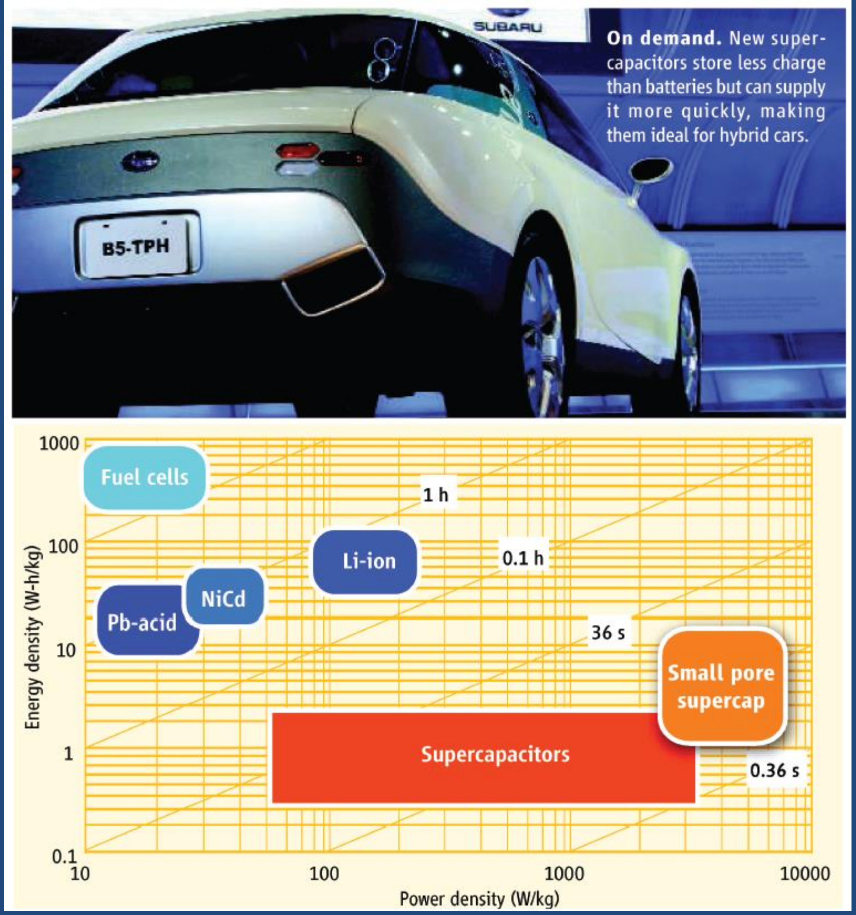


LEC 5. RC AND RL FIRST-ORDER CIRCUITS

ECEN203

Analogue Electronics and Systems

RC and RL 1st Order Circuit

- Use mathematical functions to describe several types of nonperiodic waveforms.
- Define the electrical properties of a capacitor, including its i-v relationship and energy equation.
- Combine multiple capacitors when connected in series or parallel.
- Define the electrical properties of an inductor, including its *i-v* relationship and energy equation.
- Combine multiple capacitors when connected in series or parallel.
- Analyse the transient responses of RC and RL circuits.
- Design RC op-amp circuits to perform differentiation and integration and related operations.

Revision: Resistive circuit

- Governed by Ohm's law => a linear relationship between *i*-*v*. This is static and does not change with time.
- Due to this linearity, even if current becomes time-varying, voltage will also change linearly.
- We can still solve resistive circuit using *static* formulas.
- Resistive circuits *consume* electrical energy by converting it into *heat*.

Things are different with C and L

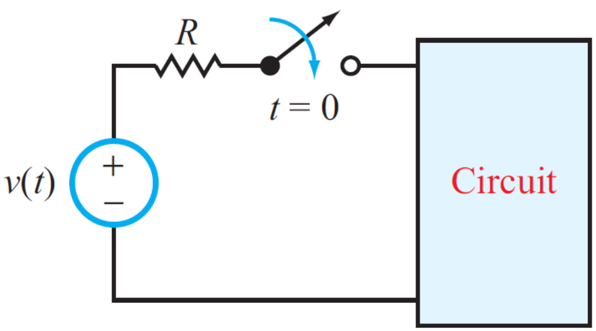
- With C and L, circuit becomes dynamic.

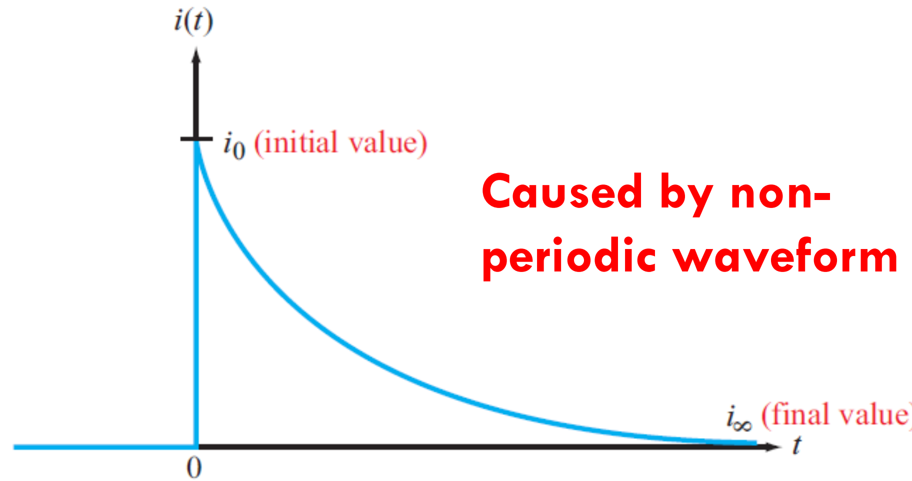
> ► Capacitors and inductors represent a contrasting (yet complimentary) class of electrical devices. Not only is time t (or more precisely d/dt) at the heart of how capacitors and inductors function, but they also differ from resistors in that they do not dissipate energy. They can store energy and then release it—but not consume it. ◄

- The dynamic response of circuit to a certain voltage or current source depends on both the architecture of the circuit and the waveform characterising the time variation of that source.
- This usually consists of a transient response and a steady-state response.

Transient Response

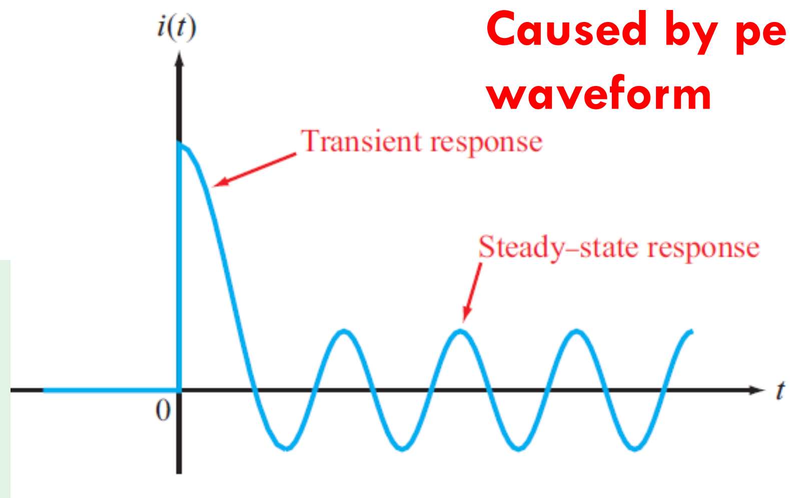
The transient response represents the initial reaction immediately after a sudden change, such as closing or opening a switch to connect a source to the circuit.





(a) dc transient response

Caused by non-periodic waveform



(b) Combined response to ac excitation

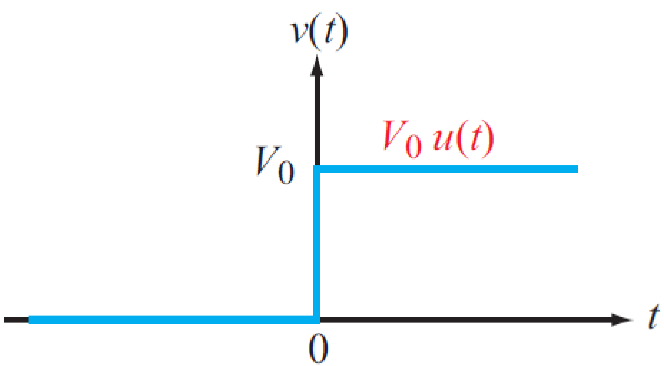
Caused by periodic waveform

Note:

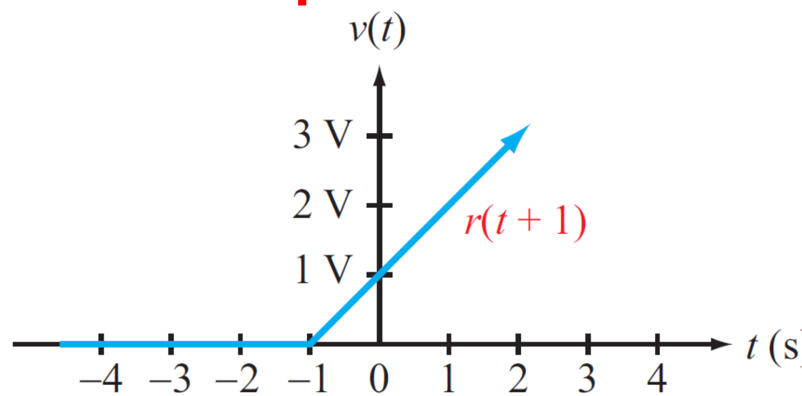
► First-order circuits include ***RC circuits***—composed of sources, resistors, and a single capacitor (or multiple capacitors that can be combined into a single equivalent capacitor)—and ***RL circuits***, but not circuits containing capacitors and inductors simultaneously. ◄

Non-Periodic Waveforms

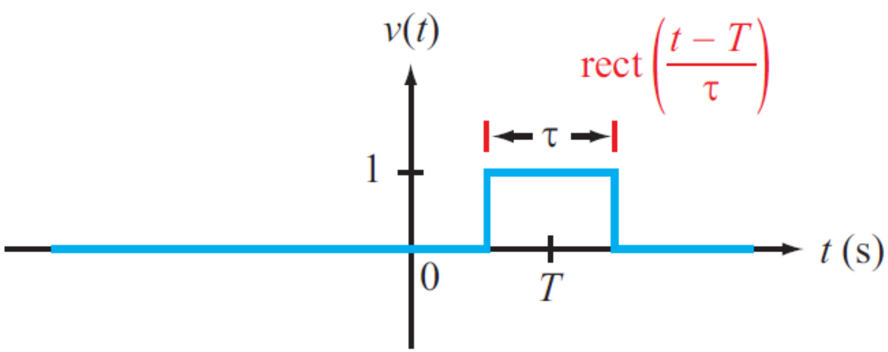
Step Function



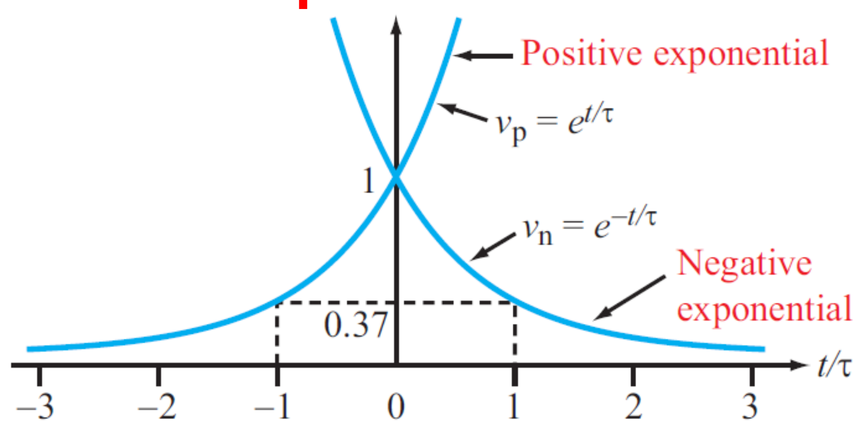
Ramp Function



Square Pulse



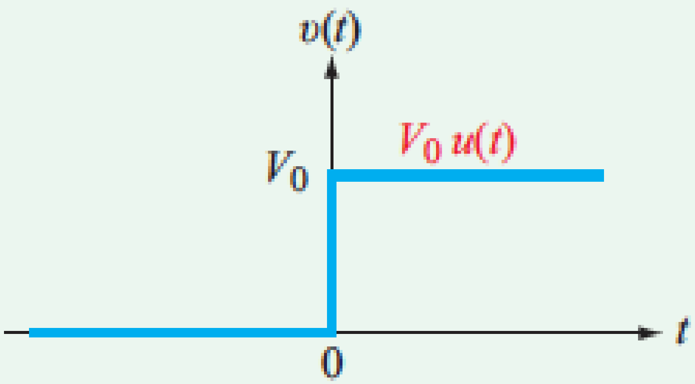
Exponential



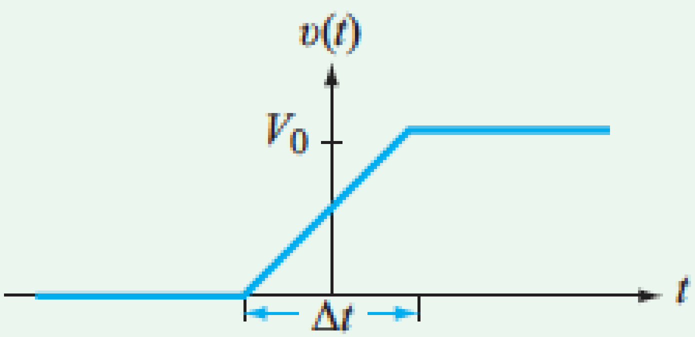
Non-Periodic Waveforms: Step Function

$$v(t) = V_0\, u(t - T) = \begin{cases} 0 & \text{for } t < T, \\ V_0 & \text{for } t > T. \end{cases}$$

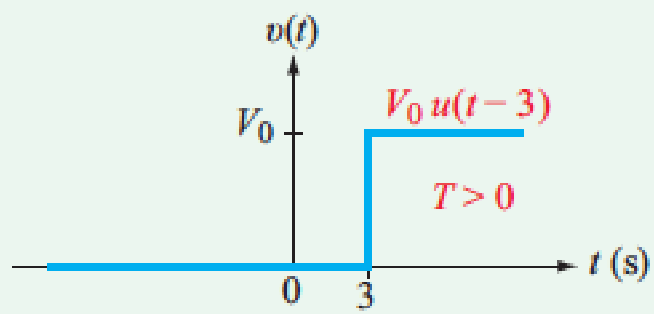
Step Functions



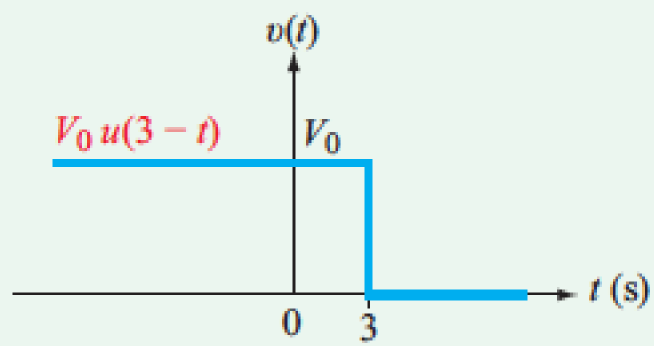
(a) Ideal step function



(b) Realistic step function



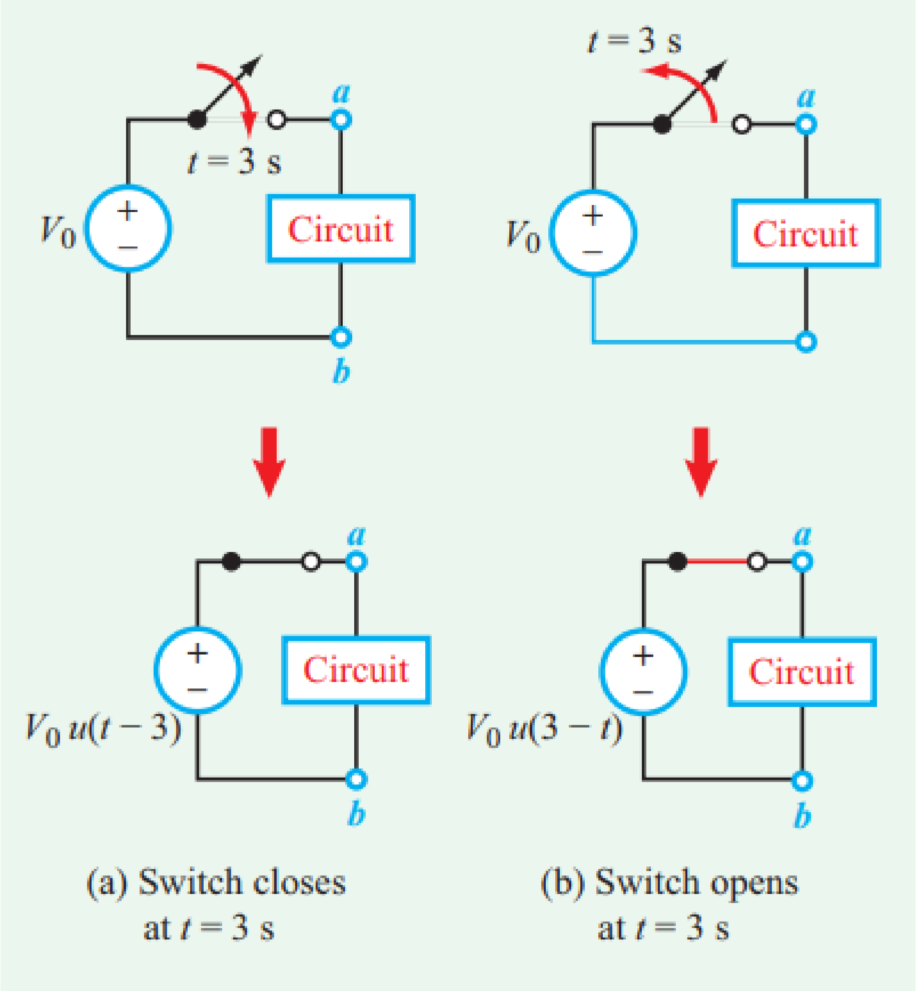
(c) Time-shifted step $u(t - T)$ with $T = 3$s



(d) Time-shifted step $u(t - T)$ with $T = 3$s

Switch on and off: step function

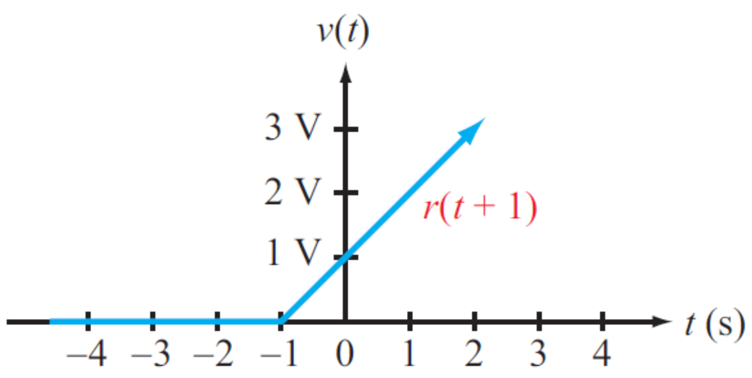
- Connecting/disconnecting a voltage source to/from a circuit via a switch can be represented mathematically by a step function.



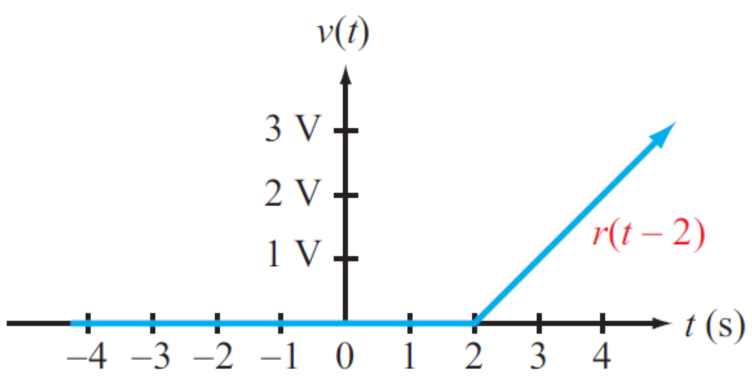
(a) Switch closes at $t = 3$ s
(b) Switch opens at $t = 3$ s

Non-Periodic Waveforms: Ramp Function

Ramp Functions



(a)



(b)

$$r(t-T) = \begin{cases} 0 & \text{for } t \le T, \\ (t-T) & \text{for } t \ge T. \end{cases}$$

A unit ramp function is related to the unit step function by

$$r(t) = \int_{-\infty}^{t} u(t)\, dt = t\, u(t), \tag{5.6}$$

Example 5-1: Realistic Step Waveform

Waveform synthesis as sum of two ramp functions

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) \\ &= 3r(t + 2\text{ ms}) - 3r(t - 2\text{ ms}) \qquad \text{V}. \end{aligned}$$

In view of Eq. (5.7), $v(t)$ also can be expressed in terms of time-shifted step functions as

$$\begin{aligned} v(t) = {} & 3(t + 2\text{ ms})\, u(t + 2\text{ ms}) \\ & - 3(t - 2\text{ ms})\, u(t - 2\text{ ms}) \qquad \text{V}. \end{aligned}$$

Note:

$$r(t - T) = \int_{-\infty}^{t} u(t - T)\, dt = (t - T)\, u(t - T). \qquad (5.7)$$

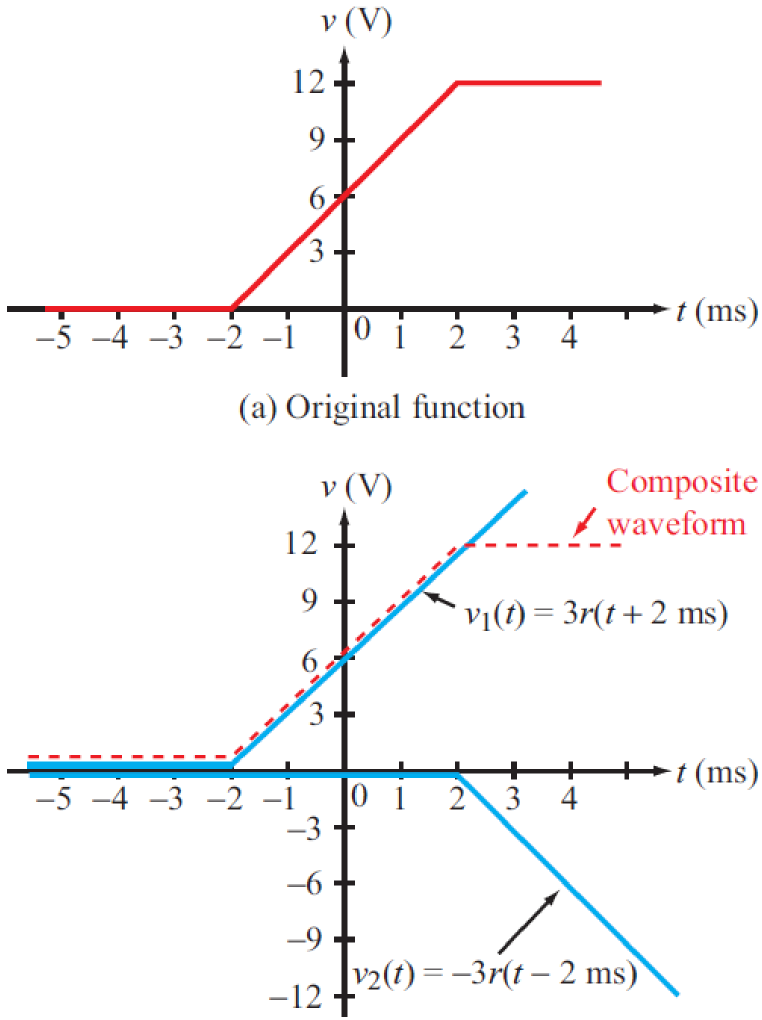
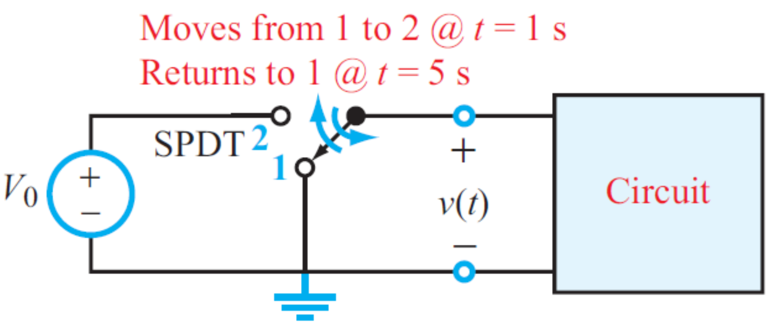


Figure 5-5: Step waveform of Example 5-1.

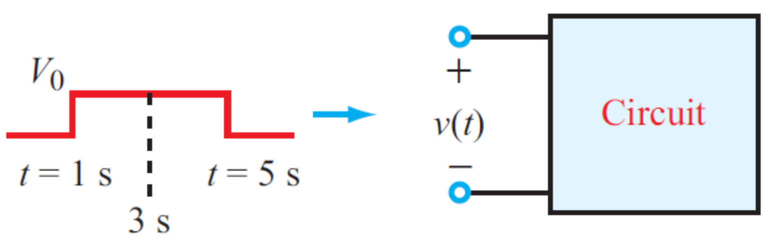
Non-Periodic Waveforms: Pulses

$$\mathrm{rect}\left(\frac{t-T}{\tau}\right) = \begin{cases} 0 & \text{for } t < (T-\tau/2), \\ 1 & \text{for } (T-\tau/2) \le t \le (T+\tau/2), \\ 0 & \text{for } t > (T+\tau/2). \end{cases}$$

=>



(a) Circuit with input switch



(b) Equivalent input pulse

$\mathrm{rect}\left(\frac{t-3}{4}\right)$

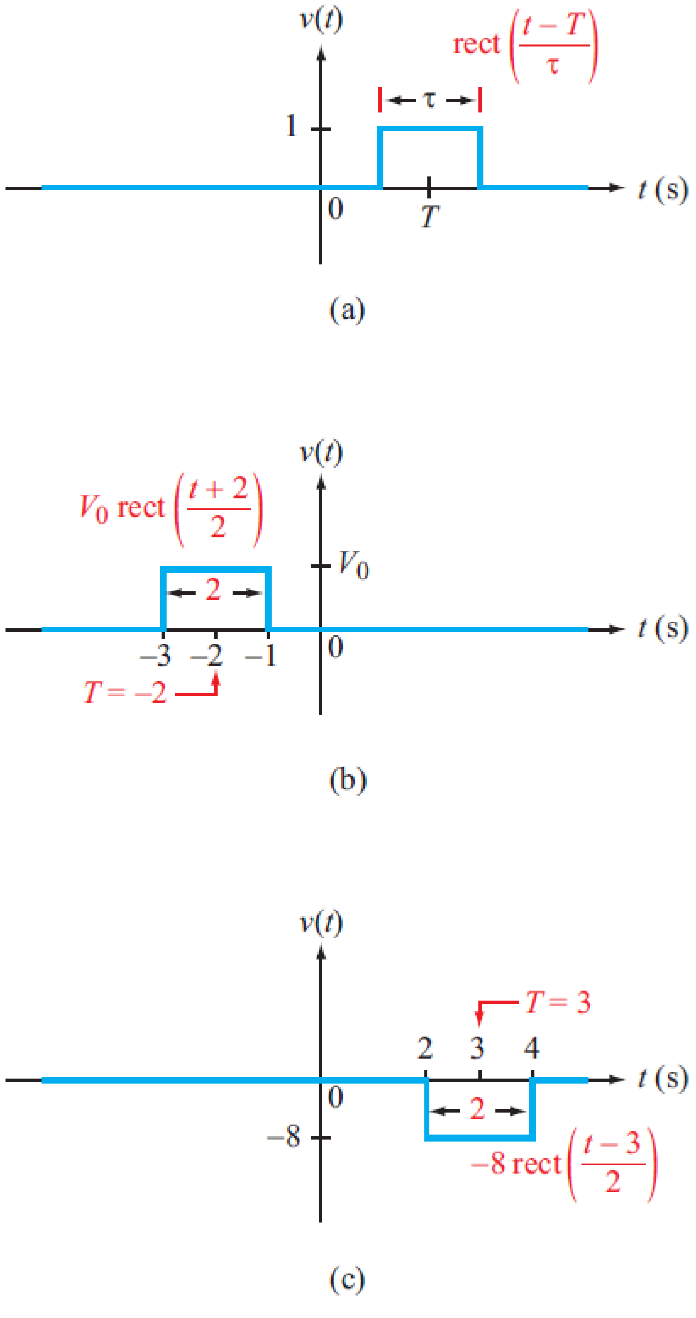
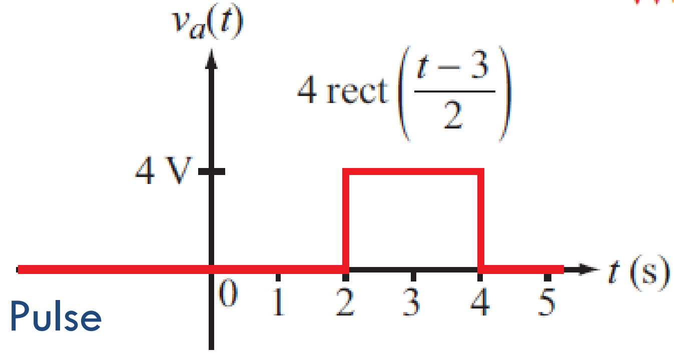


Figure 5-7: Rectangular pulses.

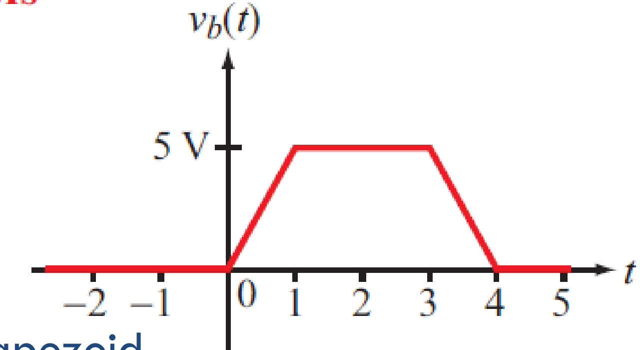
Waveform Synthesis

1. Pulse

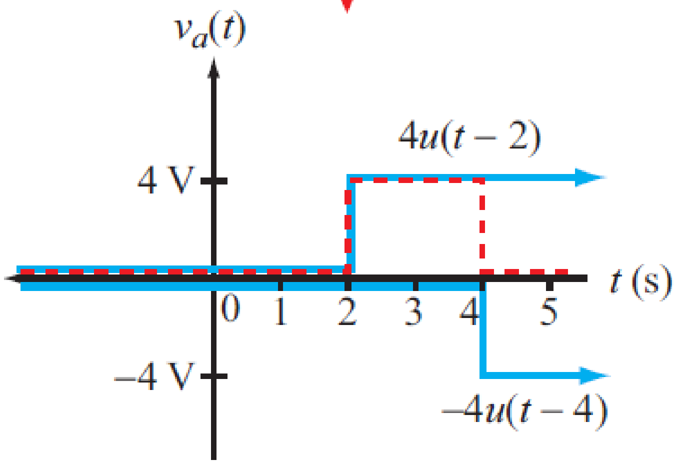


(a) Rectangular pulse

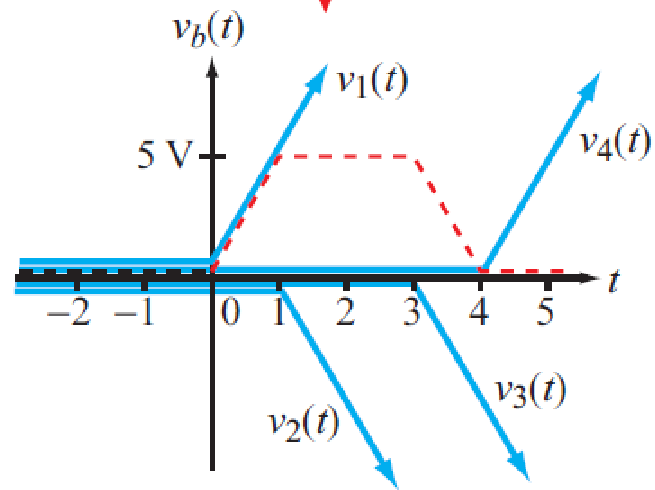
2. Trapezoid



(b) Trapezoidal pulse



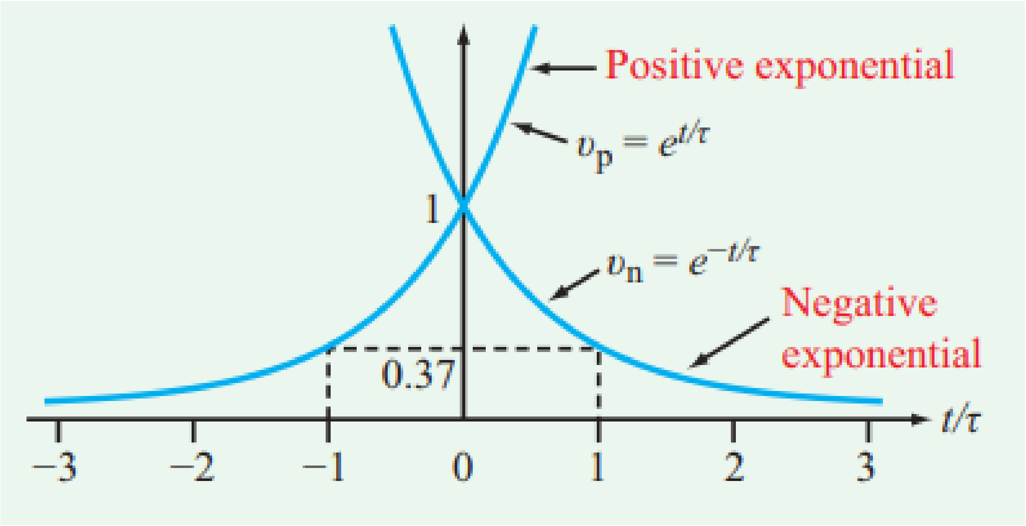
(c) $v_a(t) = 4u(t-2) - 4u(t-4)$



(d) $v_b(t) = v_1(t) + v_2(t) + v_3(t) + v_4(t)$

$$\upsilon_b(t) = 5[t\,u(t) - (t-1)\,u(t-1) - (t-3)\,u(t-3) + (t-4)\,u(t-4)] \quad \text{V.}$$

Non-Periodic Waveforms: Exponentials



When $t = \tau$, $\upsilon_n = e^{-1} = 0.37$. Thus, if a certain quantity (such as a voltage or current) is said to decay exponentially with time, it means that after τ seconds its amplitude decreases to $1/e$ or 37 percent of its initial value. Symmetrically, $\upsilon_p = e^{-1} = 0.37$ when $t = -\tau$.

Non-Periodic Waveforms: Exponentials

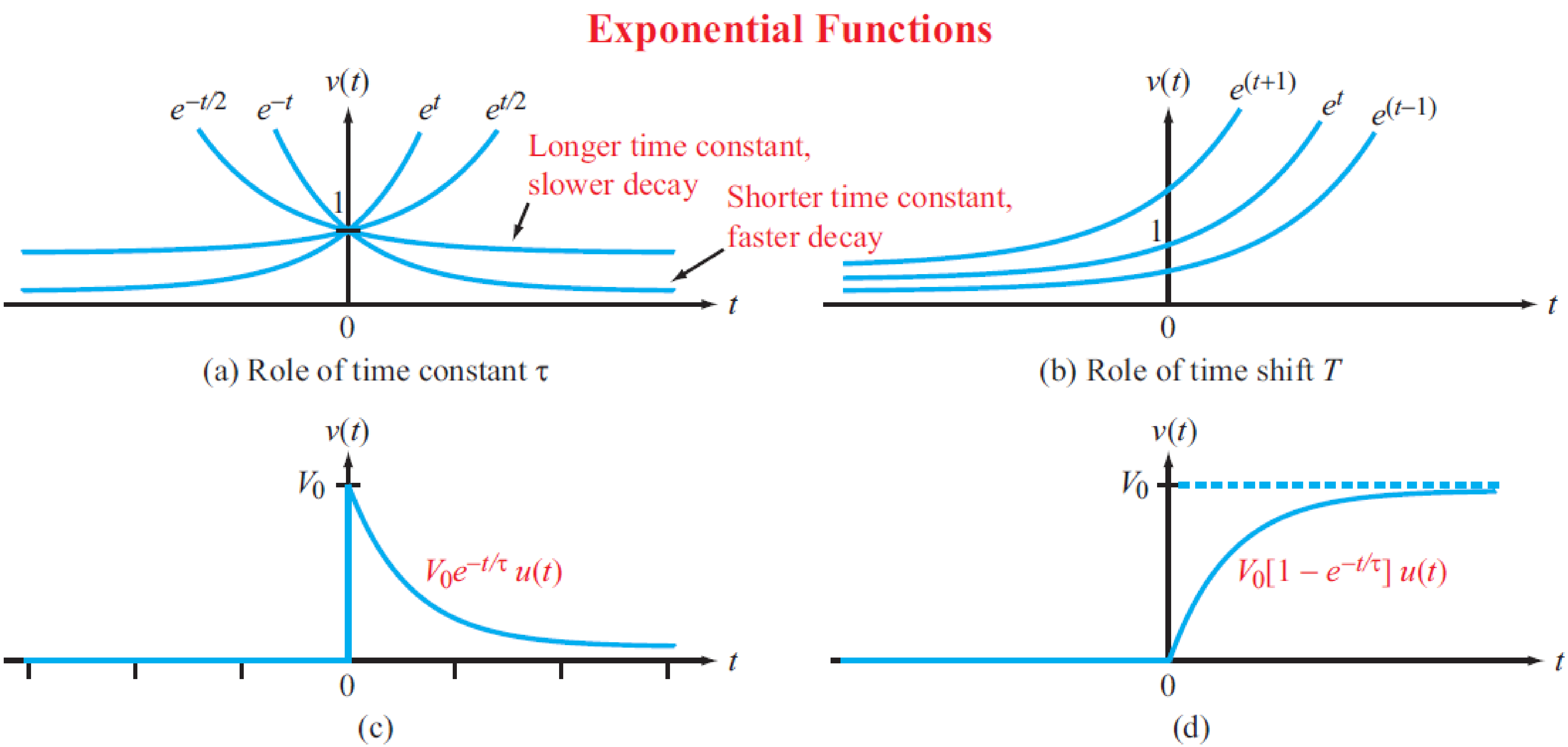
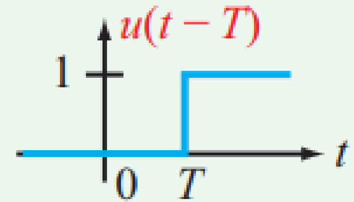
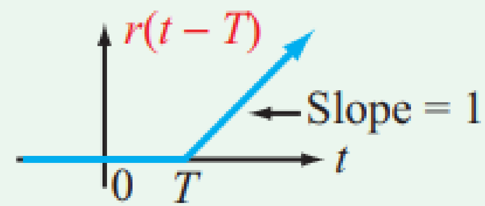
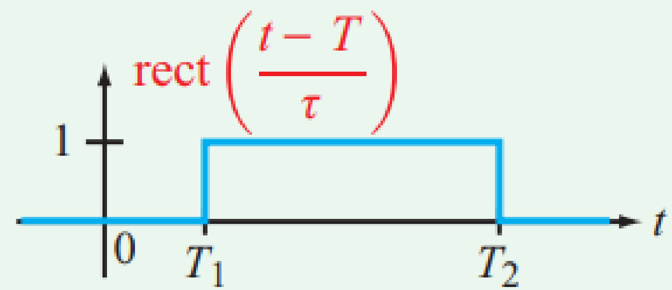
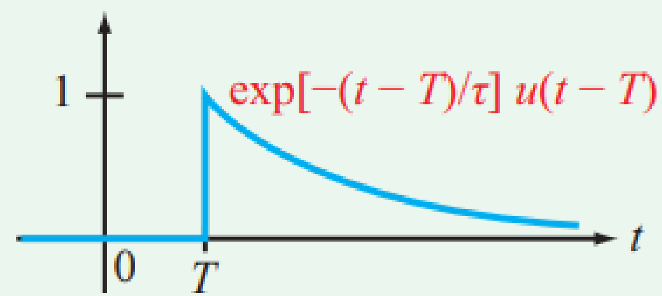


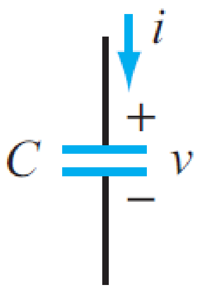
Figure 5-10: Properties of the exponential function.

Common non-periodic waveforms

waveform	Expression	General Shape
Step	$u(t-T)=\begin{cases}0 & \text{for } t<T\\ 1 & \text{for } t>T\end{cases}$	
Ramp	$r(t-T)=(t-T)\,u(t-T)$	
Rectangle	$\operatorname{rect}\left(\frac{t-T}{\tau}\right)=u(t-T_1)-u(t-T_2)$ $T_1=T-\frac{\tau}{2};\quad T_2=T+\frac{\tau}{2}$	
Exponential	$\exp[-(t-T)/\tau]\,u(t-T)$	

Capacitors

Passive element that stores energy in electric field



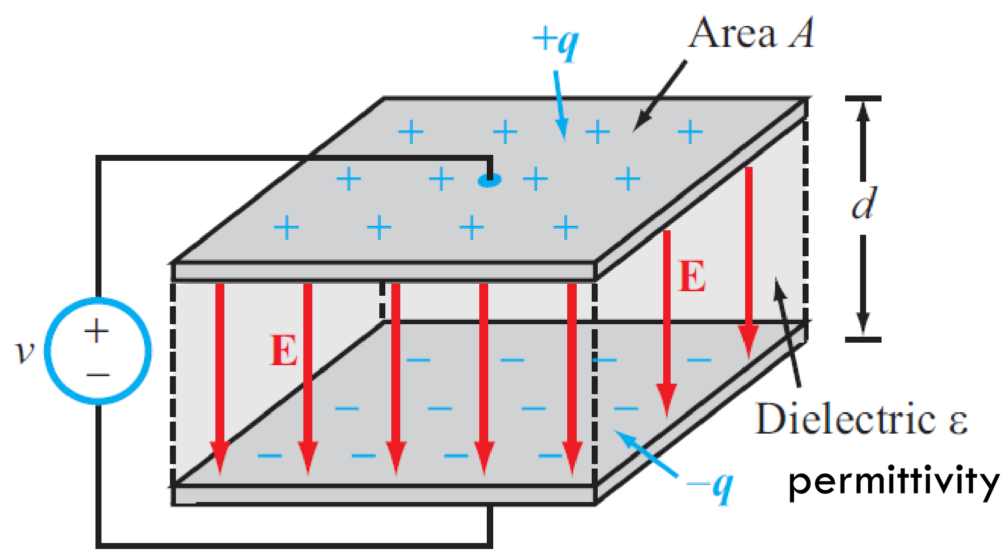
$$i = C\frac{dv}{dt} \qquad q = Cv$$

$$\upsilon = \frac{1}{C}\int_{t_0}^{t} i\,dt + \upsilon(t_0)$$

- For DC, capacitor looks like open circuit
- Voltage on capacitor must be continuous (no abrupt change)

Parallel plate capacitor

$$C = \frac{\varepsilon A}{d}$$



Various types of capacitors

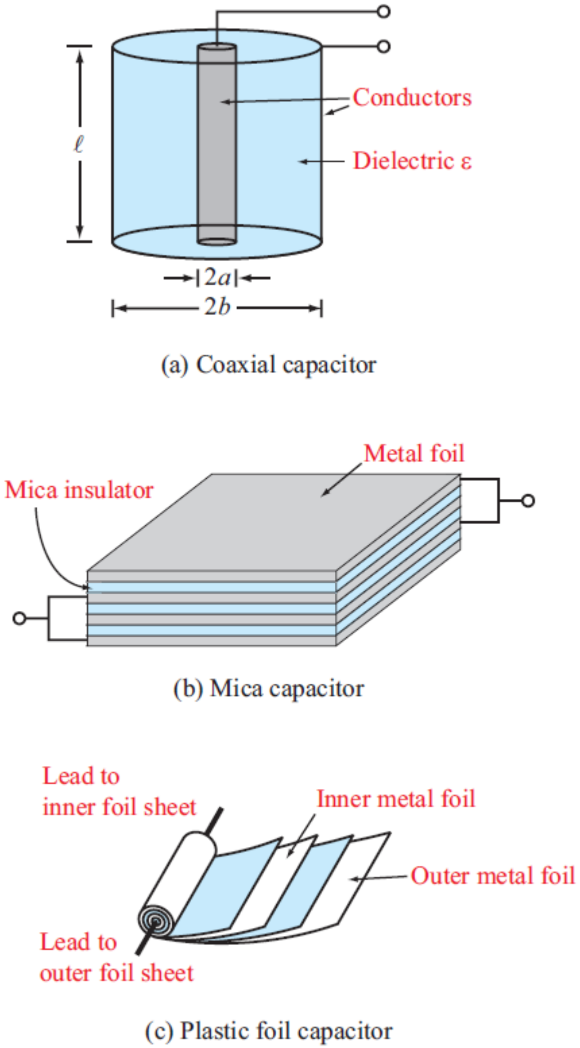


Figure 5-12: Various types of capacitors.

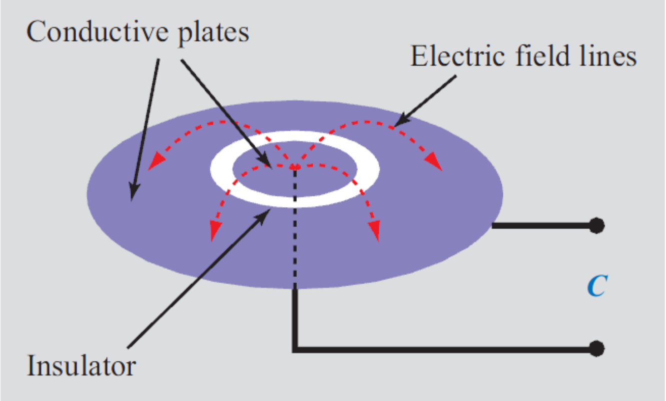


Figure TF9-4: Concentric-plate capacitor.

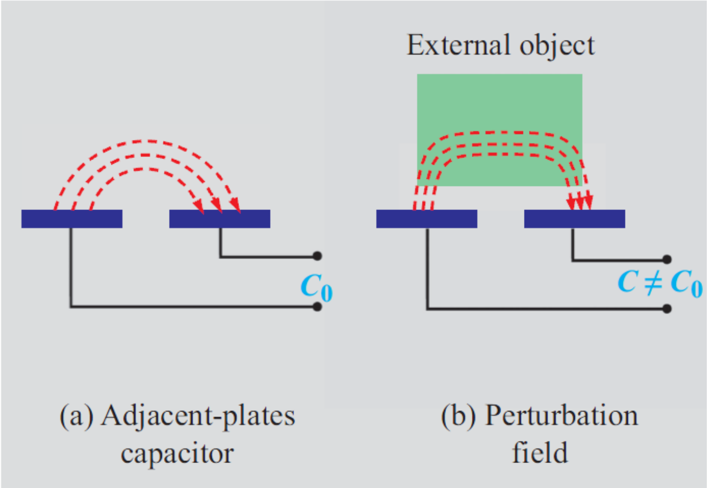
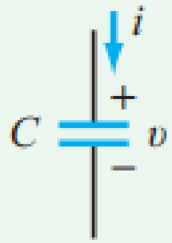


Figure TF9-5: (a) Adjacent-plates capacitor; (b) perturbation field.

Electrical Properties of Capacitors

- Given q = Cv, then from formula of current, we have
- $$i = \frac{dq}{dt} = C\,\frac{dv}{dt},$$ =>



$$i = C\,\frac{dv}{dt}$$

- This means 2 things:

► The voltage across a capacitor cannot change instantaneously, but the current can. ◄

► Under dc conditions, a capacitor behaves like an open circuit. ◄

- Integrating *i*, we get voltage =>

$$v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i\,dt'.$$

- Assume no charge for $t < 0$,

$$v(t) = \frac{1}{C}\int_{0}^{t} i\,dt'$$

(capacitor uncharged before $t = 0$).

Energy Stored in Capacitors

The instantaneous power $p(t)$ transferring into or out of a capacitor is given by

$$p(t) = vi$$

$$= Cv\,\frac{dv}{dt} \qquad \text{(W)}. \tag{5.27}$$

If the magnitude of $p(t)$ is positive, then by the passive sign convention, the capacitor is receiving power, and if $p(t)$ is negative, it is delivering power.

Energy is the integral of the product of power and time. Hence, the amount of energy stored in the capacitor at any time t is equal to the time integral of $p(t)$ from $-\infty$ (at which time the capacitor was uncharged) to t and is given by

$$w = \int_{-\infty}^{t} p\,dt = C\int_{-\infty}^{t} \left(v\,\frac{dv}{dt}\right) dt$$

$$= C\int_{-\infty}^{t} \left[\frac{d}{dt}\left(\frac{1}{2}v^2\right)\right] dt, \tag{5.28}$$

which yields

$$w = \frac{1}{2}Cv^2 \qquad \text{(J)}. \tag{5.29}$$

Q: What is the unit for capacitance?

A: Farad (F)

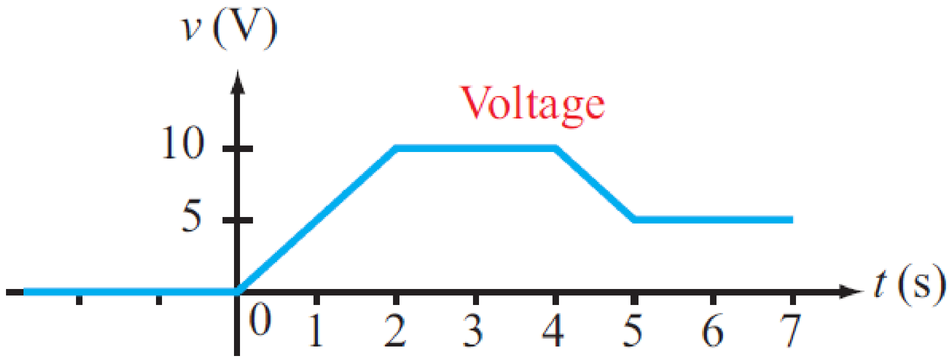
Energy Storage Comparison

Table TT12-1: Comparison of a conventional capacitor, supercapacitor, and lithium battery size and mass required to hold ~ 1 megajoule (MJ) of energy (300 watt-hours). 1 MJ of energy will power a laptop with an average consumption of 50 W for 6 hours. Note from the first column that a lithium ion battery might hold 1000 times more energy than a conventional capacitor for reasonable voltages (< 50 V).

Sample device	Specific Energy [Watt hours/ kg]	Specific Energy [MJ / kg]	Energy Density [MJ / liter]	Volume required to hold 1 MJ [liter]	Weight required to hold 1 MJ [kg]
Conventional capacitor	0.01 – 0.1	4×10^{-5}-4×10^{-4}	6×10^{-5}-6×10^{-4}	17000-1700	25000 - 2500
Supercapacitor	1 - 10	0.004 – 0.04	0.006 - 0.06	166 – 16	250 – 25
Lithium ion battery	100 - 250	0.36 - 0.9	1 - 2	1 – 0.5	2.8 – 1.1

Example 5-3: Capacitor Response to Voltage Waveform

Given *v*(*t*) below, determine *i*(*t*), *p*(*t*), and *w*(*t*); C = $0.6\text{-}\mu F$



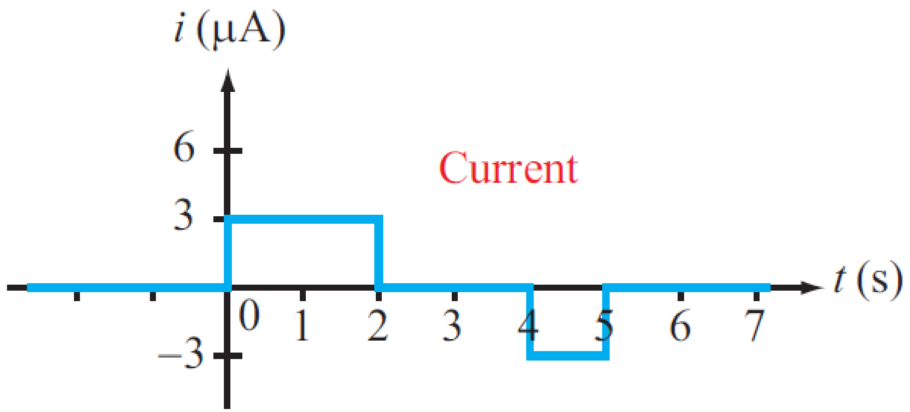
- Start by expressing voltage waveform in term of ramp functions => starts at t=0 and has a slope of 10/2 = 5 V/s, so $\upsilon(t) = 5r(t) - 5r(t-2) - 5r(t-4) + 5r(t-5) \quad \text{V}.$
- Recall that $r(t-T) = (t-T)\, u(t-T),$
- the expression for v(t) is then given by

$$\upsilon(t) = \begin{cases} 0 & \text{for } t \le 0, \\ 5t \text{ V} & \text{for } 0 \le t \le 2 \text{ s}, \\ 10 \text{ V} & \text{for } 2 \text{ s} \le t \le 4 \text{ s}, \\ (-5t + 30) \text{ V} & \text{for } 4 \text{ s} \le t \le 5 \text{ s}, \\ 5 \text{ V} & \text{for } t \ge 5 \text{ s}. \end{cases}$$

Example 5-3: Capacitor Response to Voltage Waveform

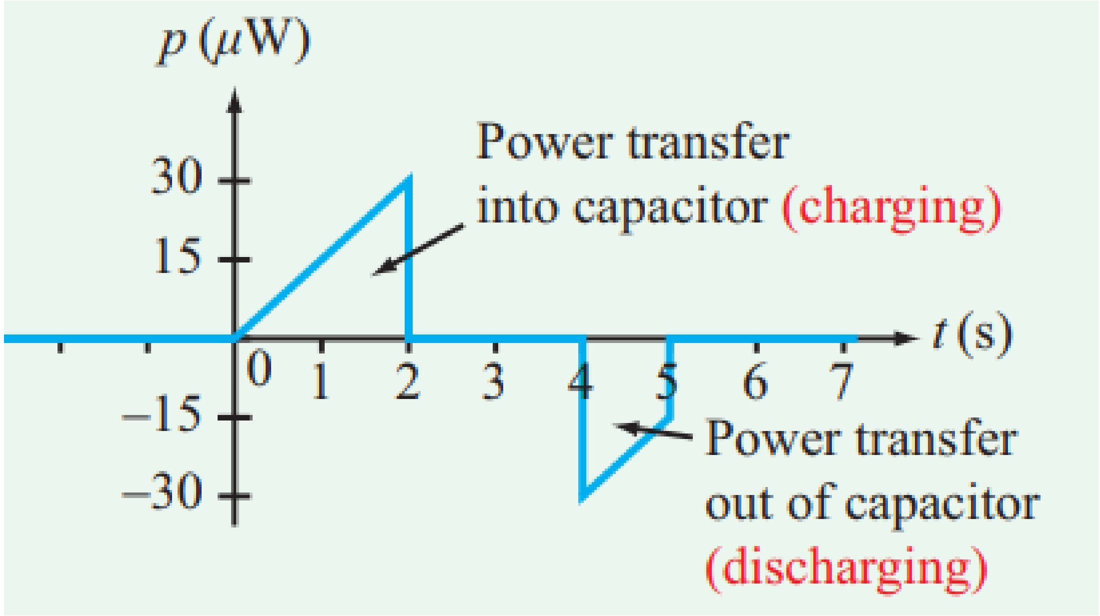
- We know that $i(t) = C\,\dfrac{dv}{dt}$, and the derivative is the same as the slope ot the line, so this gives

$$i(t) = C\,\frac{dv}{dt} = \begin{cases} 0 & \text{for } t \le 0, \\ 3\ \mu\text{A} & \text{for } 0 \le t \le 2\ \text{s}, \\ 0 & \text{for } 2\ \text{s} \le t \le 4\ \text{s}, \\ -3\ \mu\text{A} & \text{for } 4\ \text{s} \le t \le 5\ \text{s}, \\ 0 & \text{for } t \ge 5\ \text{s}. \end{cases}$$



Example 5-3: Capacitor Response to Voltage Waveform

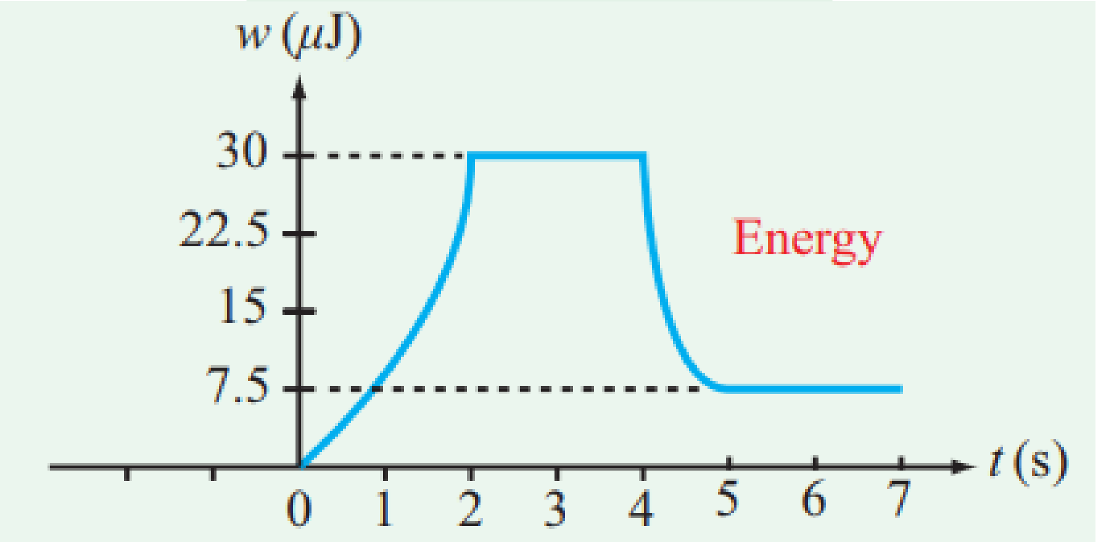
- The power, p(t), is the product of v(t) and i(t) given above, shown here:



Example 5-3: Capacitor Response to Voltage Waveform

- To find stored energy, we can integrate p(t) – which is to find area under the curve, or by using equation:

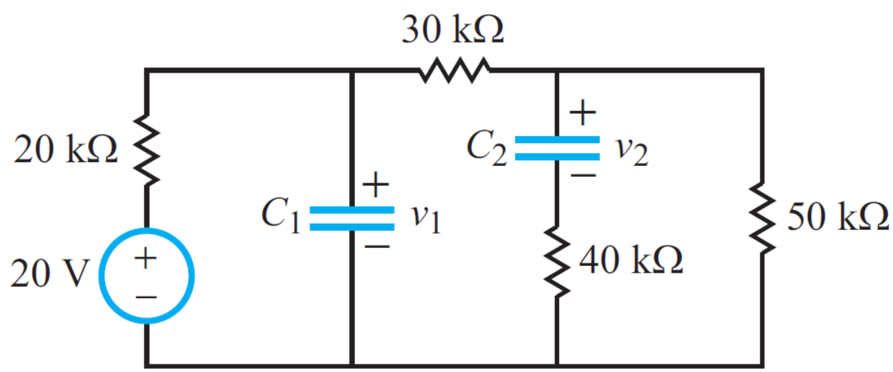
$$w(t) = \frac{1}{2} C\, v^2(t) \qquad \text{(J).}$$



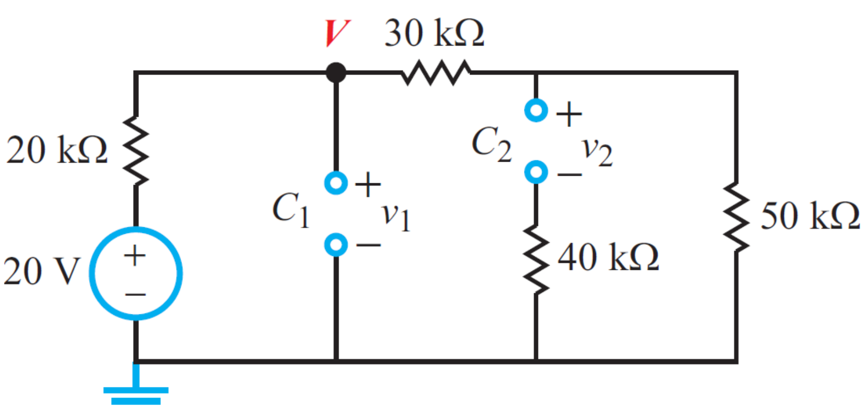
- Now, can you explain the energy transfer process from the standpoint of the current and voltage?

Example 5-4: RC Circuits at dc

Find v_1 and v_2. Assume circuit has been in present (charged) condition for a long time.



- At dc no currents flow through capacitors: open circuits.
- At node V, find currents:



$$\frac{V-20}{20\times10^3}+\frac{V}{(30+50)\times10^3}=0,$$

which gives $V=16$ V. Hence,

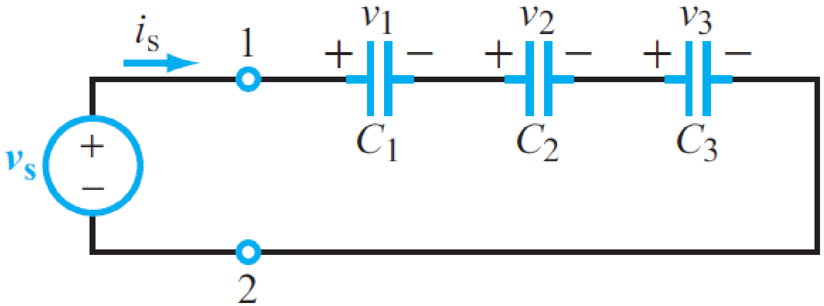
$$v_1 = V = 16\text{ V}.$$

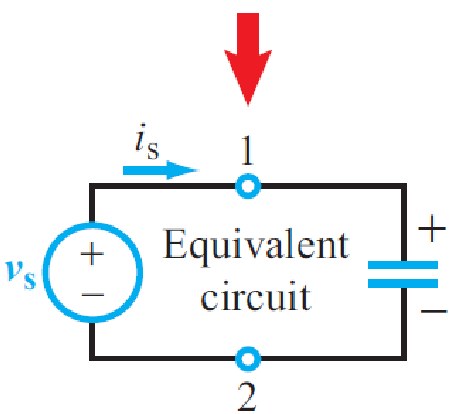
Through voltage division, v_2 across the 50-kΩ resistor is given by

$$v_2=\frac{V\times 50\text{k}}{(30+50)\text{k}}=\frac{16\times50}{80}=10\text{ V}.$$

Capacitors in Series

Combining In-Series Capacitors





$$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)^{-1}$$

Use KVL, same current through each capacitor

$$i_s = C_1 \frac{dv_1}{dt} = C_2 \frac{dv_2}{dt} = C_3 \frac{dv_3}{dt}.$$

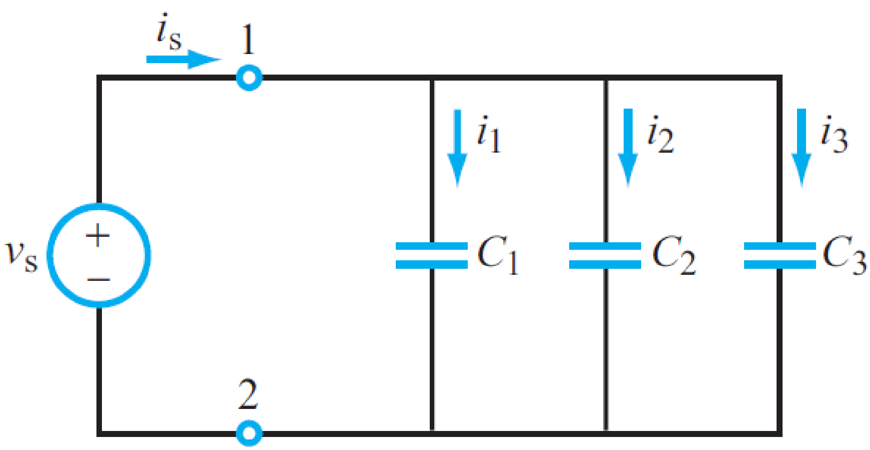
Also,

$$\begin{aligned} i_s &= C_{eq} \frac{dv_s}{dt} \\ &= C_{eq}\left(\frac{dv_1}{dt} + \frac{dv_2}{dt} + \frac{dv_3}{dt}\right) \\ &= C_{eq}\left(\frac{i_s}{C_1} + \frac{i_s}{C_2} + \frac{i_s}{C_3}\right), \end{aligned}$$

which leads to

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

Capacitors in Parallel

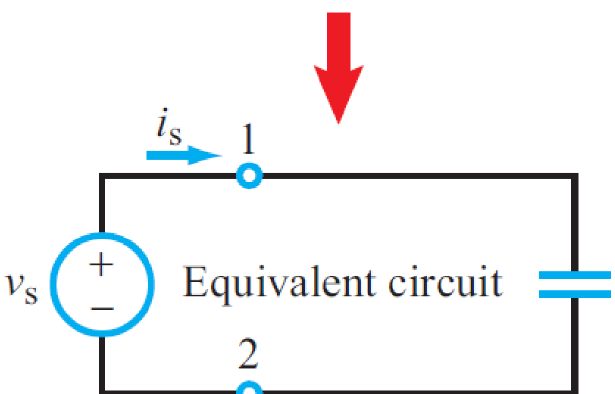


Use KCL, same voltage across each capacitor

$$i_s = i_1 + i_2 + i_3$$

$$= C_1 \frac{dv_s}{dt} + C_2 \frac{dv_s}{dt} + C_3 \frac{dv_s}{dt}$$

$$i_s = C_{eq} \frac{dv_s}{dt}$$

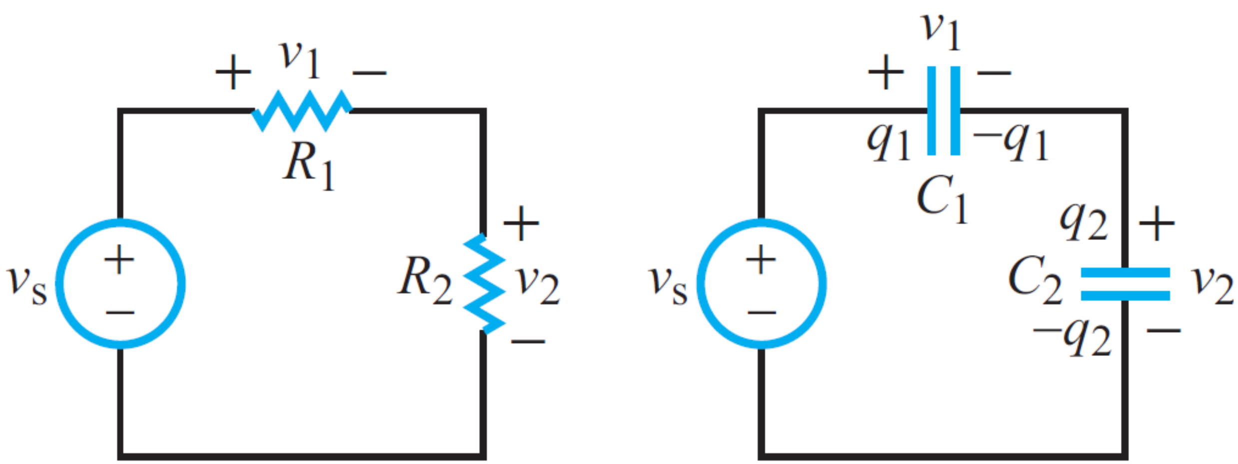


$C_{eq} = C_1 + C_2 + C_3$

$$\boxed{C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N}$$

Voltage Division

Voltage Division



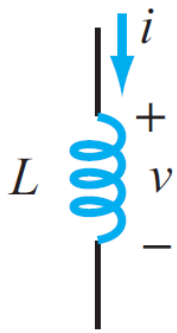
$$\text{(a)}\quad v_1 = \left(\frac{R_1}{R_1 + R_2}\right) v_s \qquad \text{(b)}\quad v_1 = \left(\frac{C_2}{C_1 + C_2}\right) v_s$$

$$v_2 = \left(\frac{R_2}{R_1 + R_2}\right) v_s \qquad v_2 = \left(\frac{C_1}{C_1 + C_2}\right) v_s$$

Question: can you derive the formula for (b)?

Inductors

Passive element that stores energy in magnetic field



$$v = L\frac{di}{dt}$$

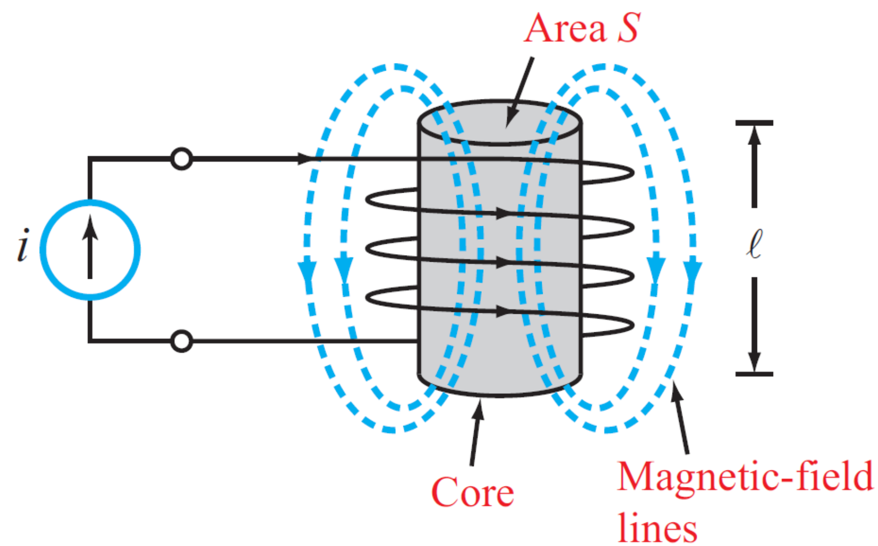
$$i = \frac{1}{L}\int_{t_0}^{t} v(t)\,dt + i(t_0)$$

- At dc, inductor looks like a short circuit
- Current through inductor must be continuous (no abrupt change)

$$w = \frac{1}{2}Li^2 \qquad \text{(J)},$$

A typical inductor:

Solenoid Wound Inductor



$$L = \frac{N^2 \mu A}{l}$$

μ = Magnetic permeability

Various Styles of Inductors

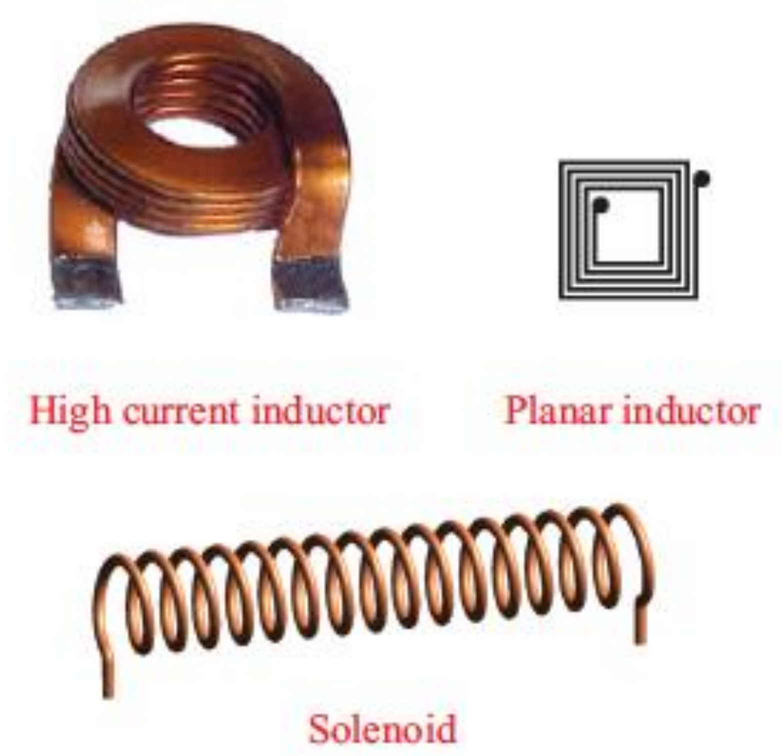


Figure 5-21: Various types of inductors.

Q: What is the unit for inductance?

A: Henry (H).

Example: Inductor Response to $i(t) = 10e^{-0.8t}\sin(\pi t/2)$ A, $\quad$ (for $t \geq 0$)

(a) Plot waveform for *i(t)* versus *t*, and determine the locations of its first maximum, first minimum, and their corresponding amplitudes.

(b) Given that L = 50mH, obtain an expression for v(t) across the inductor and plot its waveform.

(c) Generate a plot of the power p(t) delivered to the inductor.

Example: Inductor Response to $i(t) = 10e^{-0.8t} \sin(\pi t/2)$ A, (for $t \geq 0$)

- To determine the locations for maxima and minima, we take the derivative of i(t) and equate it to zero. This gives:

$$-0.8 \times 10e^{-0.8t} \sin(\pi t/2) + \left(\frac{\pi}{2}\right) \times 10e^{-0.8t} \cos\left(\frac{\pi t}{2}\right) = 0,$$

which in turn simplifies to

$$\tan\left(\frac{\pi t}{2}\right) = \frac{\pi}{1.6}.$$

Its solution is

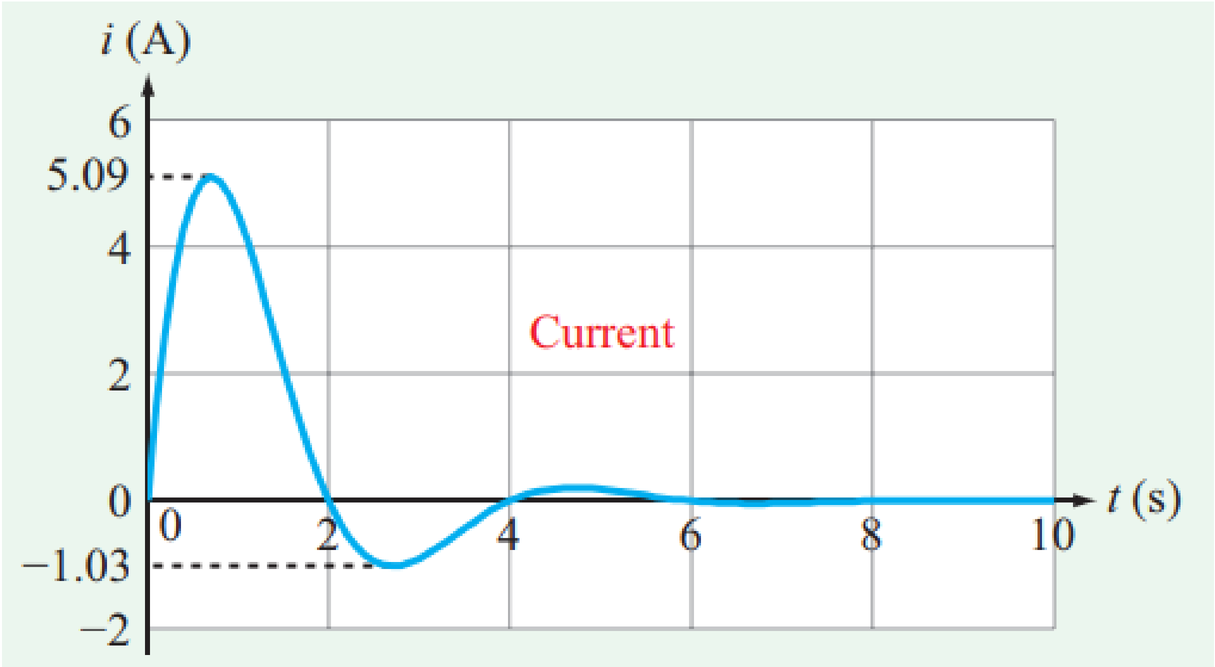
$$\frac{\pi t}{2} = 1.1 + n\pi \qquad (\text{for } n = 0, 1, 2, \ldots).$$

For $n = 0$, $t = 0.7$ s, which is the location in time of the first maximum of $i(t)$. The next solution, corresponding to $n = 1$, gives the location of the first minimum of $i(t)$ at 2.7 s. The amplitudes of $i(t)$ at these locations are

$$i_{\max} = i(t = 0.7 \text{ s}) = 10e^{-0.8 \times 0.7} \sin(\pi \times 0.7/2) = 5.09 \text{ A}$$

and

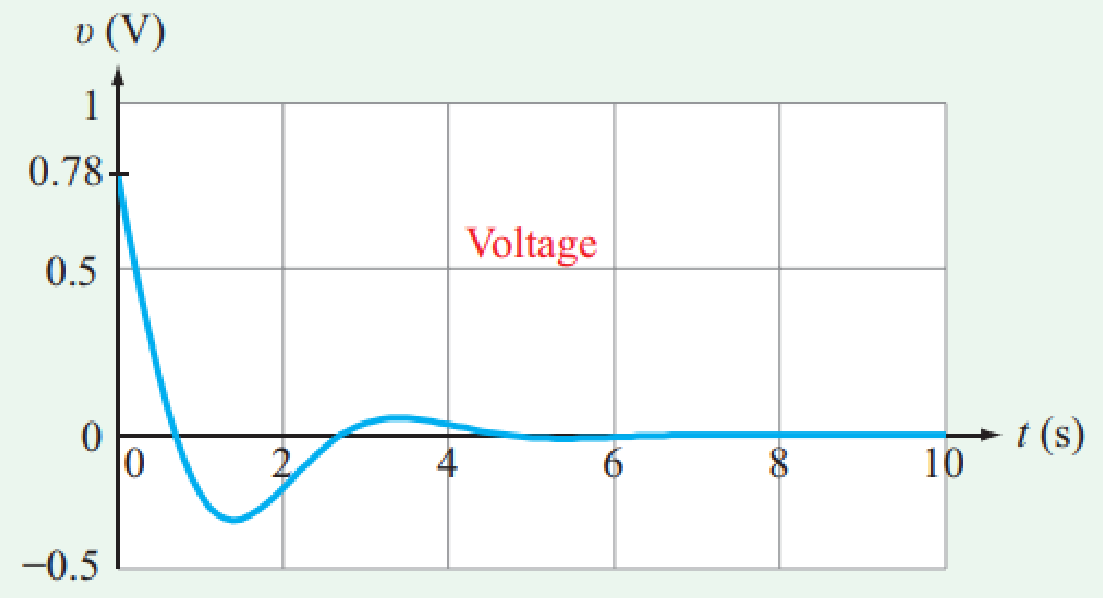
$$i_{\min} = i(t = 2.7\text{ s}) = 10e^{-0.8\times 2.7}\sin(\pi \times 2.7/2) = -1.03\text{ A}.$$



Example: Inductor

Response to $i(t) = 10e^{-0.8t}\sin(\pi t/2)$ A, (for $t \geq 0$)

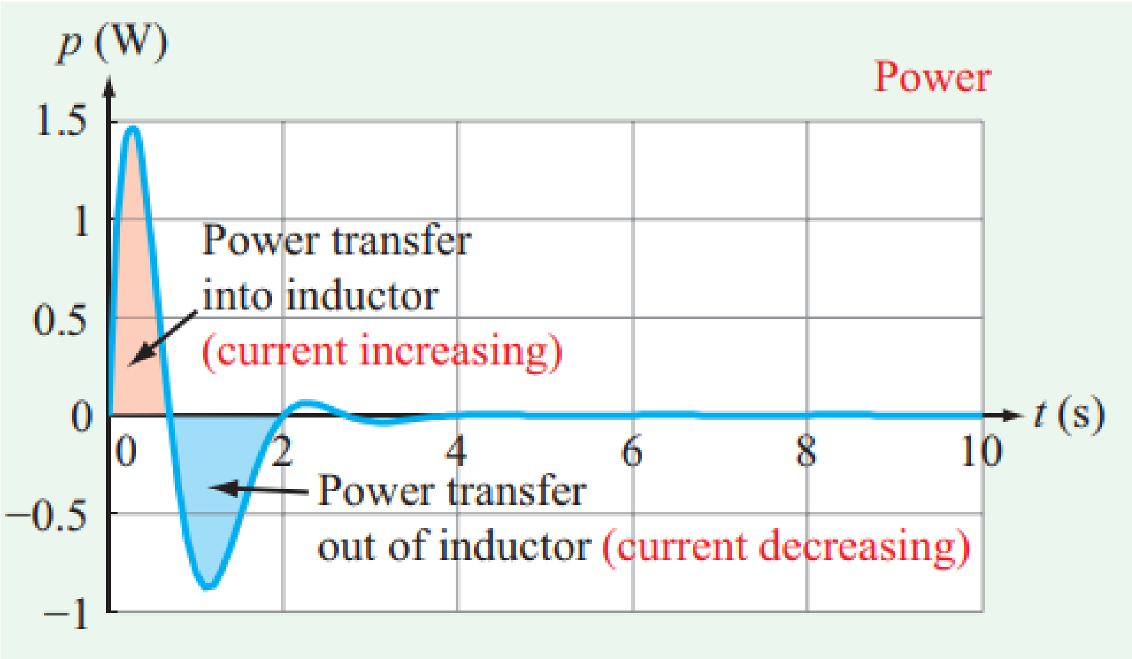
$$\begin{aligned} v(t) &= L\,\frac{di}{dt} \\ &= L\,\frac{d}{dt}[10e^{-0.8t}\sin(\pi t/2)] \\ &= 50\times 10^{-3}\cdot[-8e^{-0.8t}\sin(\pi t/2) + 5\pi e^{-0.8t}\cos(\pi t/2)] \\ &= [-0.4\sin(\pi t/2) + 0.25\pi\cos(\pi t/2)]e^{-0.8t}\ \text{V}. \end{aligned}$$



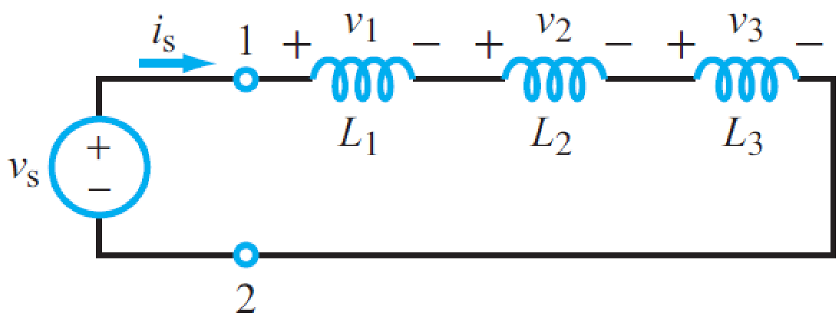
Example: Inductor

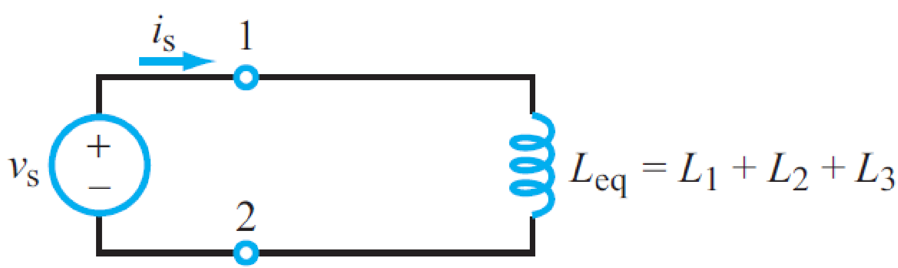
Response to $i(t) = 10e^{-0.8t}\sin(\pi t/2)$ A, (for $t \geq 0$)

$$
\begin{aligned}
p(t) &= \upsilon(t)\, i(t) \\
&= [-0.4\sin(\pi t/2) + 0.25\pi\cos(\pi t/2)]e^{-0.8t} \\
&\quad \times 10e^{-0.8t}\sin(\pi t/2) \\
&= [-4\sin^2(\pi t/2) + 2.5\pi\cos(\pi t/2)\sin(\pi t/2)] \\
&\quad \times e^{-1.6t}\ \text{W}.
\end{aligned}
$$



Inductors in Series





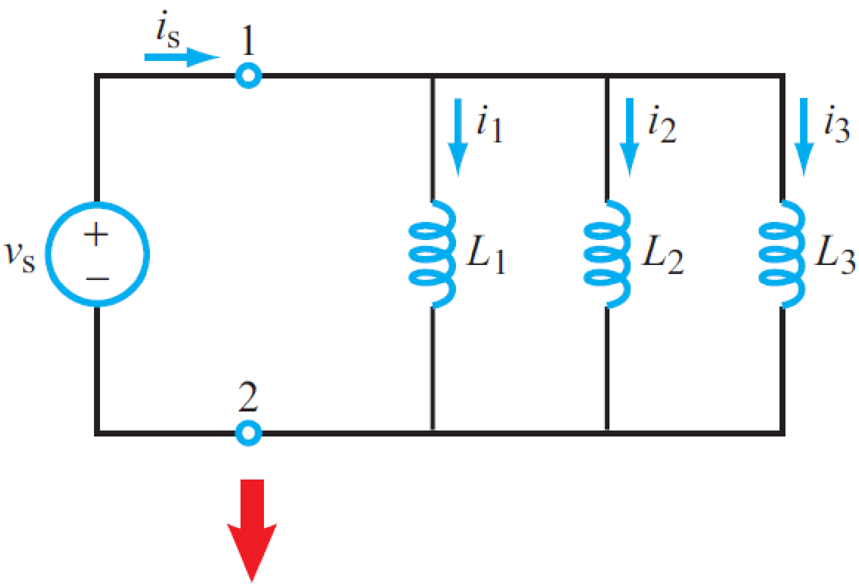
Use KVL, current is same through all inductors

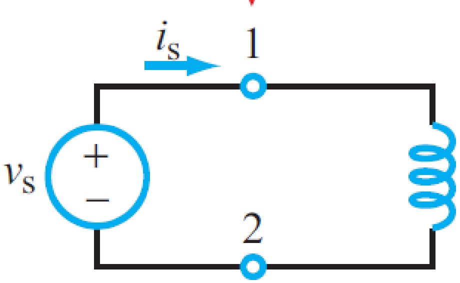
$$v_s = v_1 + v_2 + v_3$$
$$= L_1 \frac{di_s}{dt} + L_2 \frac{di_s}{dt} + L_3 \frac{di_s}{dt}$$
$$= (L_1 + L_2 + L_3) \frac{di_s}{dt},$$
$$v_s = L_{eq} \frac{di_s}{dt}$$

$$L_{eq} = L_1 + L_2 + L_3$$

Inductors in Parallel

Combining In-Parallel Inductors





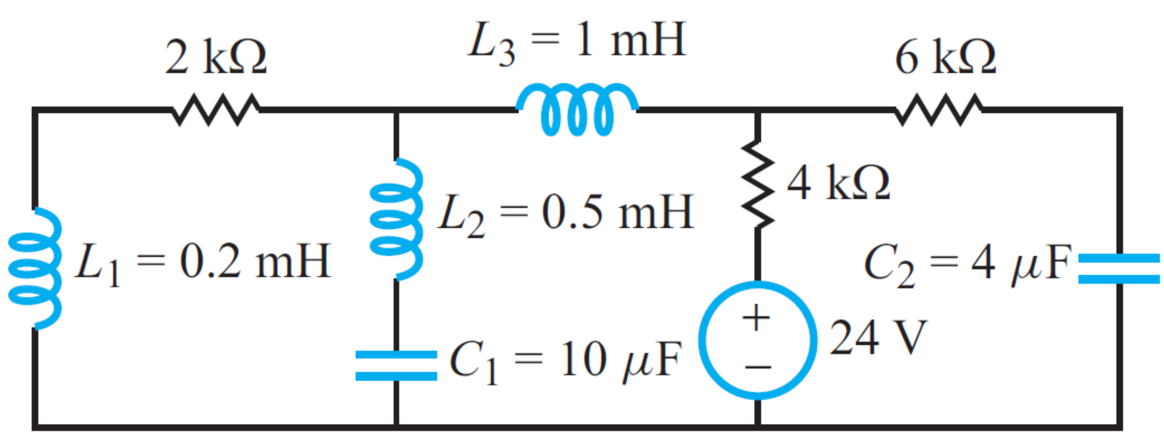
$$L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)^{-1}$$

Voltage is same across all inductors

Inductors add together in the same way resistors do

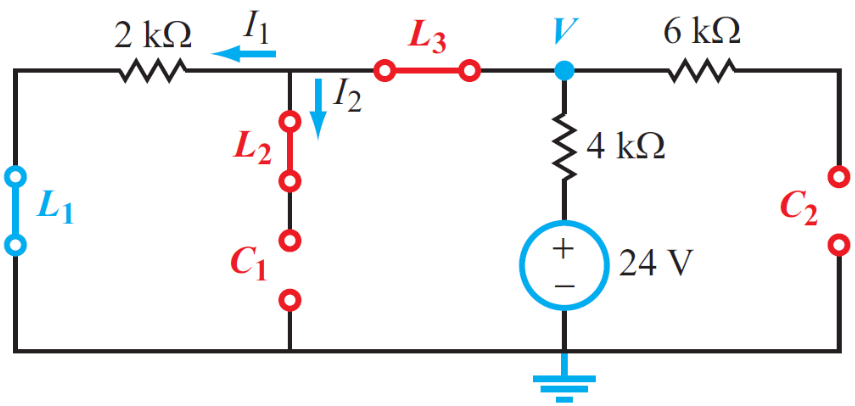
Example: RL Circuits at dc

Given circuit below, determine the amount of energy stored in the capacitors ad inductors. Assume that the circuit has been in the present state for a long time.



Example: RL Circuits at dc

- At dc no voltage across inductors: short circuit
- Capacitors: open circuit. Equivalent circuit:



$$I_1 = \frac{24}{(2+4)\mathrm{k}} = 4 \text{ mA},$$

and node voltage V is

$$V = 24 - (4 \times 10^{-3} \times 4 \times 10^{3}) = 8 \text{ V}.$$

Hence, the amounts of energy stored in C_1, C_2, L_1, L_2, and L_3 are

$$C_1: \quad W = \frac{1}{2}C_1V^2 = \frac{1}{2} \times 10^{-5} \times 64 = 0.32 \text{ mJ},$$

$$C_2: \quad W = \frac{1}{2}C_2V^2 = \frac{1}{2} \times 4 \times 10^{-6} \times 64 = 0.128 \text{ mJ},$$

and…

Example: RL Circuits at dc

$$L_1: \quad W = \frac{1}{2} L_1 I_1^2$$

$$= \frac{1}{2} \times 0.2 \times 10^{-3} \times (4 \times 10^{-3})^2 = 1.6 \text{ nJ},$$

$$L_2: \quad W = \frac{1}{2} L_2 I_2^2 = \frac{1}{2} \times 0.5 \times 10^{-3} \times (0) = 0,$$

and

$$L_3: \quad W = \frac{1}{2} L_3 I_1^2 = \frac{1}{2} \times 10^{-3} \times (4 \times 10^{-3})^2 = 8 \text{ nJ}.$$

Basic Properties of *R*, *L* and C

Property	*R*	*L*	*C*
i–v relation	$i = \frac{v}{R}$	$i = \frac{1}{L}\int_{t_0}^{t} v\, dt' + i(t_0)$	$i = C\frac{dv}{dt}$
v-i relation	$v = iR$	$v = L\frac{di}{dt}$	$v = \frac{1}{C}\int_{t_0}^{t} i\, dt' + v(t_0)$
p (power transfer in)	$p = i^2 R$	$p = Li\frac{di}{dt}$	$p = Cv\frac{dv}{dt}$
w (stored energy)	0	$w = \frac{1}{2}Li^2$	$w = \frac{1}{2}Cv^2$
Series combination	$R_{eq} = R_1 + R_2$	$L_{eq} = L_1 + L_2$	$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$
Parallel combination	$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$	$\frac{1}{L_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$	$C_{eq} = C_1 + C_2$
dc behavior	no change	short circuit	open circuit
Can v change instantaneously?	yes	yes	no
Can i change instantaneously?	yes	no	yes

Response Terminology

Source dependence

Natural response – response in absence of sources

Forced response – response due to external source

Complete response = Natural + Forced

Time dependence

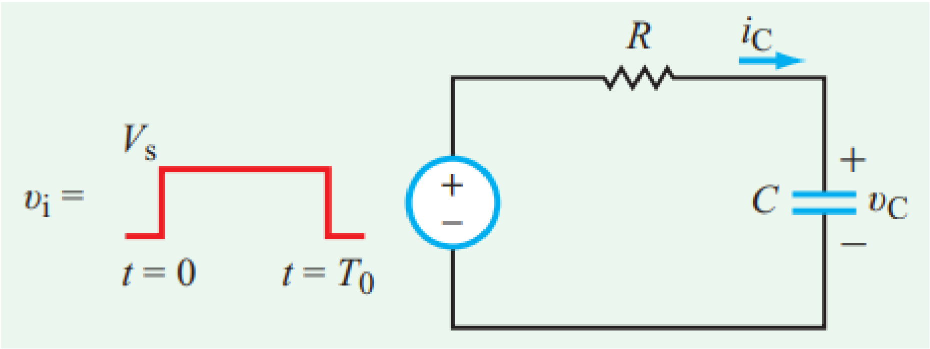
Transient response – time-varying response (temporary)

Steady state response – time-independent or periodic (permanent)

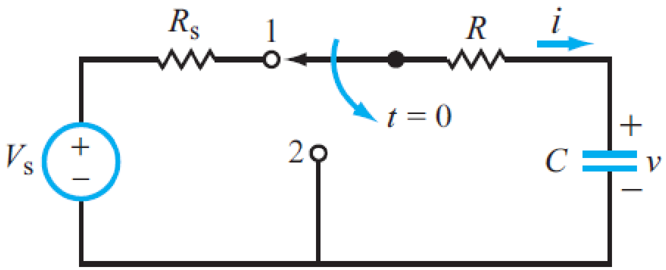
Complete response = Transient + Steady State

First order response

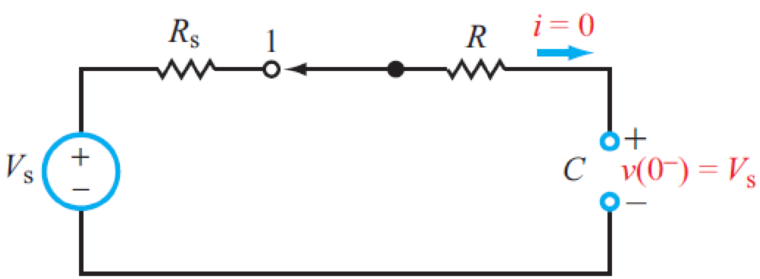
- Response of RC or RL circuits due to sudden changes – closing or opening of switch, or both sequentially.
- We start with just dc voltage and current sources.
- In later lectures, will look at RLC circuits with dc source, and ac source.
- Example, RC circuit: to be solved using 1st-order differential equation



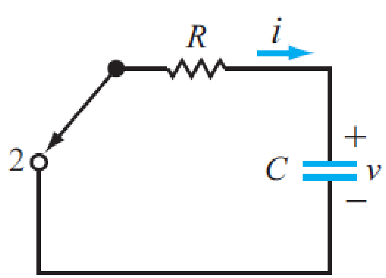
Natural Response of Charged Capacitor (from charged to discharged)



(a) RC circuit



(b) At $t = 0^-$



(c) At $t \geq 0$

(a) $t = 0^-$ is the instant just before the switch is moved from terminal 1 to terminal 2

(b) $t = 0$ is the instant just after it was moved; $t = 0$ is synonymous with $t = 0^+$

since the voltage across the capacitor cannot change instantaneously, it follows that

$$v(0) = v(0^-) = V_s.$$

For $t \geq 0$, application of KVL to the loop in Fig. 5-28(c) gives

$$Ri + v = 0 \qquad (\text{for } t \geq 0), \tag{5.68}$$

where i is the current through and v is the voltage across the capacitor. Since $i = C\,dv/dt$,

$$RC\,\frac{dv}{dt} + v = 0. \tag{5.69}$$

Upon dividing both terms by RC, Eq. (5.69) takes the form

$$\boxed{\frac{dv}{dt} + av = 0 \qquad (\text{source-free}),} \tag{5.70}$$

where

$$a = \frac{1}{RC}. \tag{5.71}$$

Solution of First-Order Diff. Equations

The standard procedure for solving Eq. (5.70) starts by multiplying both sides by e^{at},

$$\frac{dv}{dt}\,e^{at} + ave^{at} = 0. \qquad (5.72)$$

Next, we recognize that the sum of the two terms on the left-hand side is equal to the expansion of the differential of (ve^{at}),

$$\frac{d}{dt}(ve^{at}) = \frac{dv}{dt}\,e^{at} + ave^{at}. \qquad (5.73)$$

Hence, Eq. (5.72) becomes

$$\frac{d}{dt}(ve^{at}) = 0. \qquad (5.74)$$

Integrating both sides, we have

$$\int_0^t \frac{d}{dt}(ve^{at})\,dt = 0, \qquad (5.75)$$

▶ The solution of the source-free equation is called the ***natural response*** (discharging condition) of the circuit. ◀

Performing the integration gives

$$ve^{at}\Big|_0^t = 0$$

or

$$v(t)\,e^{at} - v(0) = 0. \qquad (5.76)$$

Solving for $v(t)$, we have

$$\begin{aligned} v(t) &= v(0)\,e^{-at}, \\ &= v(0)\,e^{-t/RC} \qquad (\text{for } t \geq 0), \end{aligned} \qquad (5.77)$$

$$v_C(t) = v_C(0)\,e^{-t/\tau}, \qquad (5.78)$$

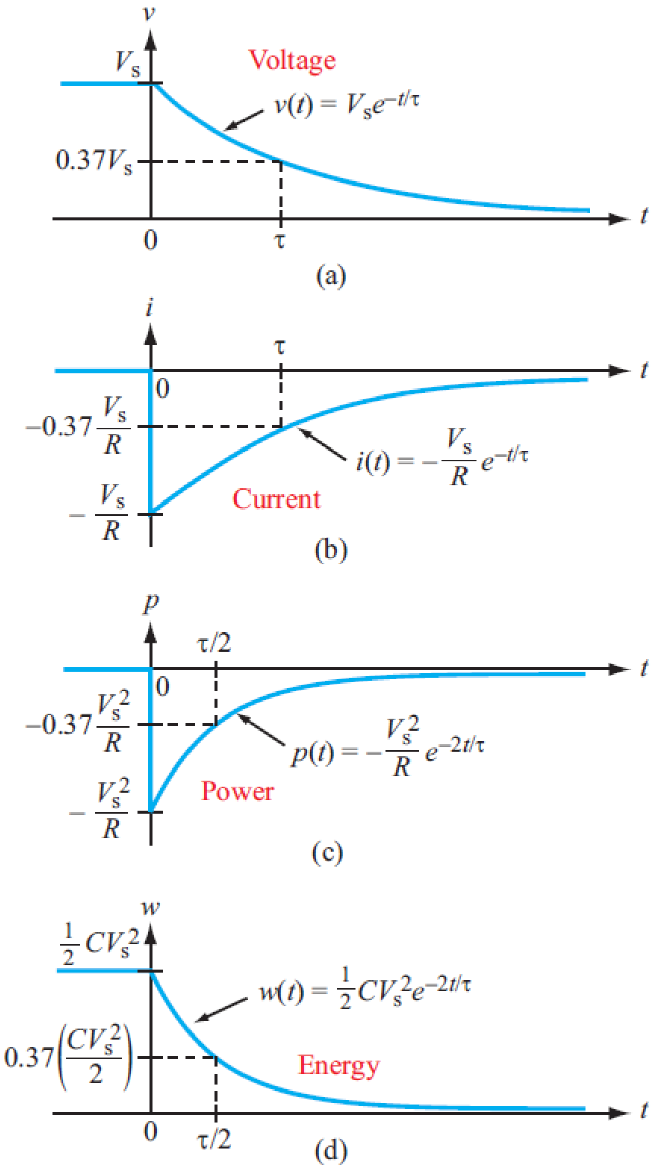
(natural response discharging),

with

$$\tau = RC \qquad (\mathrm{s}), \qquad (5.79)$$

*τ is called the **time constant** of the circuit.*

Natural Response of Charged Capacitor



Current flowing through capacitor is:

$$i(t) = C\,\frac{dv}{dt} = C\,\frac{d}{dt}(V_s e^{-t/\tau})$$

$$= -C\,\frac{V_s}{\tau}\,e^{-t/\tau} \qquad (\text{for } t \geq 0),$$

which simplifies to

$$i(t) = -\frac{V_s}{R}\,e^{-t/\tau}\,u(t) \qquad (\text{for } t \geq 0)$$

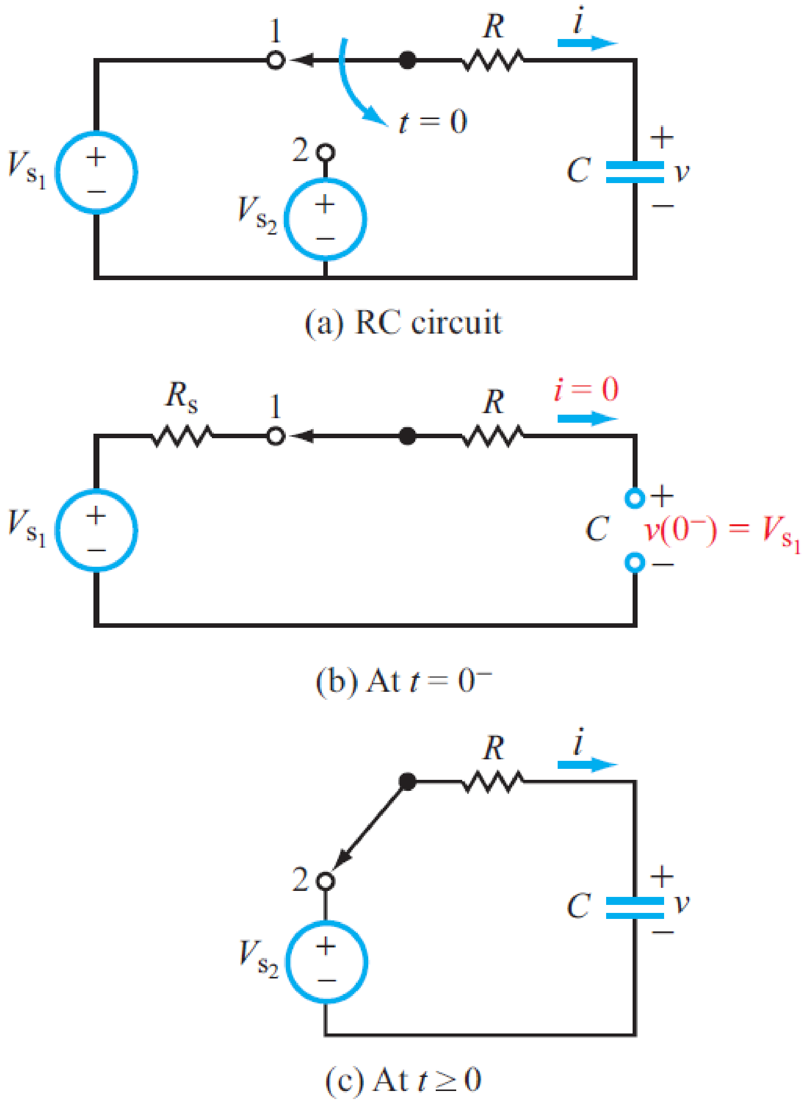
(natural response).

$$p(t) = iv = -\frac{V_s}{R}\,e^{-t/\tau} \times V_s e^{-t/\tau}$$

$$= -\frac{V_s^2}{R}\,e^{-2t/\tau} \qquad (\text{for } t \geq 0).$$

► The decay rate for $p_C(t)$ is $2/\tau$, which is twice as fast as that for $\upsilon_C(t)$ or $i_C(t)$. ◄

General Response of RC Circuit
(from V_{s1} changed to V_{s2})

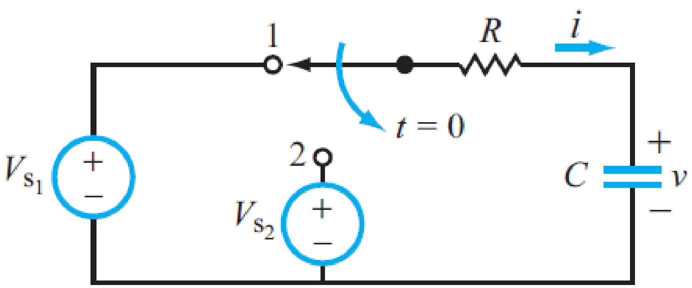


- Possible scenarios:
  - ***Step response*** (due to V_{s_2}) of an ***uncharged capacitor*** (if $V_{s_1} = 0$)
  - ***Step response*** (due to V_{s_2}) of a ***charged capacitor*** (if $V_{s_1} \neq 0$)
  - ***Natural response*** (if $V_{s_2} = 0$) of a ***charged capacitor*** ($V_{s_1} \neq 0$)

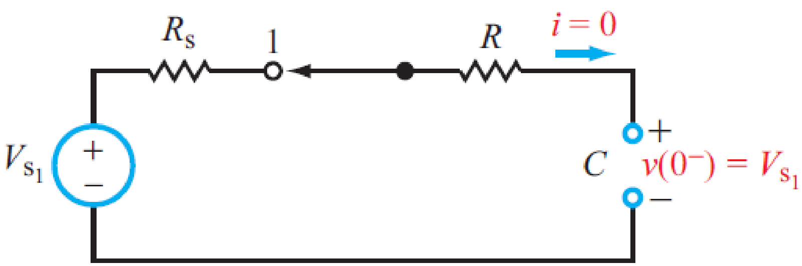
We will look at case #2 (when both sources are non-zero).

General Response of RC Circuit

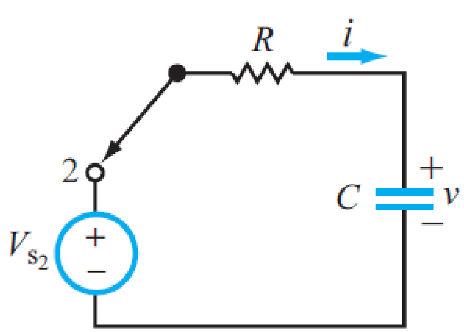
(from V_{s1} changed to V_{s2})



(a) RC circuit



(b) At $t = 0^-$



(c) At $t \geq 0$

Looking at (b),

$$v(0) = v(0^-) = V_{s_1}. \tag{5.86}$$

For $t \geq 0$, the voltage equation for the loop in Fig. 5-30(c) is

$$-V_{s_2} + iR + v = 0. \tag{5.87}$$

Upon using $i = C\, dv/dt$ and rearranging its terms, Eq. (5.87) can be written in the differential-equation form

$$\boxed{\frac{dv}{dt} + av = b,} \tag{5.88}$$

where

$$a = \frac{1}{RC} \quad \text{and} \quad b = \frac{V_{s_2}}{RC}. \tag{5.89}$$

Solution of $\frac{dv}{dt} + av = b,$

$$\frac{d}{dt}(ve^{at}) = be^{at}.$$

Integrating both sides,

$$\int_0^t \frac{d}{dt}(ve^{at})\, dt = \int_0^t be^{at}$$

gives

$$ve^{at}|_0^t = \frac{b}{a}\, e^{at}\Big|_0^t .$$

Upon evaluating the functions at the two limits, we have

$$v(t)\, e^{at} - v(0) = \frac{b}{a}\, e^{at} - \frac{b}{a},$$

and then solving for $v(t)$, we have

$$v(t) = v(0)\, e^{-at} + \frac{b}{a}\,(1 - e^{-at}).$$

As $t \to \infty$, $v(t)$ reduces to

$$v(\infty) = \frac{b}{a} = V_{s_2}.$$

By reintroducing the time constant $\tau = RC = 1/a$ and replacing b/a with $v(\infty)$, we can rewrite Eq. (5.94) in the general form:

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \qquad \text{(for } t \geq 0\text{)}$$
$$\text{(switch action at } t = 0\text{)}.$$

If the switch action causing the change in voltage across the capacitor occurs at time T_0 instead of at $t = 0$, Eq. (5.96) assumes the form

$$v(t) = v(\infty) + [v(T_0) - v(\infty)]e^{-(t-T_0)/\tau} \qquad \text{(for } t \geq T_0\text{)}$$
$$\text{(switch action at } t = T_0\text{)},$$

Series RC circuit solution

Series RC Circuit Solution

1: If switch action is at $t = 0$, analyze circuit at $t = 0^-$ to determine initial conditions $\upsilon_C(0^-)$ and $i_C(0^-)$. Use this information to determine $\upsilon_C(0)$ and $i_C(0)$, at t immediately after the switch action. Remember that the voltage across a capacitor cannot change instantaneously (between $t = 0^-$ and $t = 0$), but the current can.

2: Analyze the circuit to determine $\upsilon_C(\infty)$, the voltage across the capacitor long after the switch action.

3: Determine the time constant $\tau = RC$.

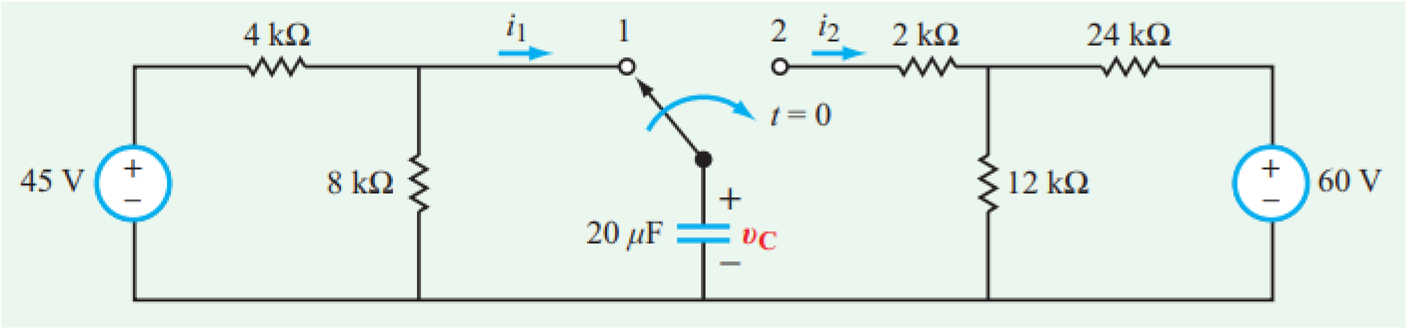
4: Incorporate the information obtained in the previous three steps in Eq. (5.96):

$$\upsilon_C(t) = \left\{\upsilon_C(\infty) + [\upsilon_C(0) - \upsilon_C(\infty)]e^{-t/\tau}\right\} u(t).$$

5: If the switch action is at $t = T_0$ instead of $t = 0$, replace 0 with T_0 and use Eq. (5.98):

$$\upsilon_C(t) = \left\{\upsilon_C(\infty) + [\upsilon_C(T_0) - \upsilon_C(\infty)] \cdot e^{-(t-T_0)/\tau}\right\} \cdot u(t - T_0).$$

Example 5-10: Determine Capacitor Voltage



In circuit above, switch is moved from position 1 to 2 after it Ha been in position 1 for a long time. Determine the voltage $v_c(t)$ for $t \geq 0$ if the switch is moved at (a) $t = 0$, and (b) $t = 3$ s.

Example 5-10: Determine Capacitor Voltage

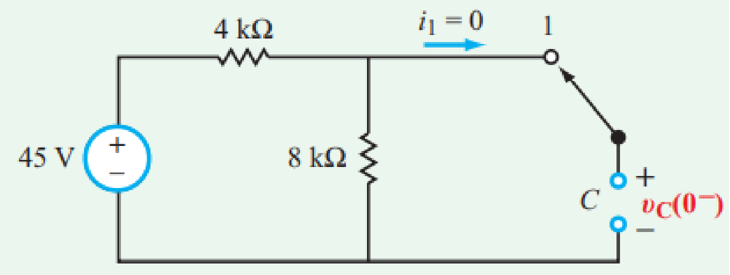
- For $T_0 = 0$ and $t \geq 0$, the complete solution of $v_c(t)$ is given by $$v_C(t) = \{v_C(\infty) + [v_C(0) - v_C(\infty)]e^{-t/\tau}\}\ u(t).$$
- There are 3 unknowns that need to be determined:

$v_c(0)$, the final voltage $v_c(\infty)$, and the time constant τ.

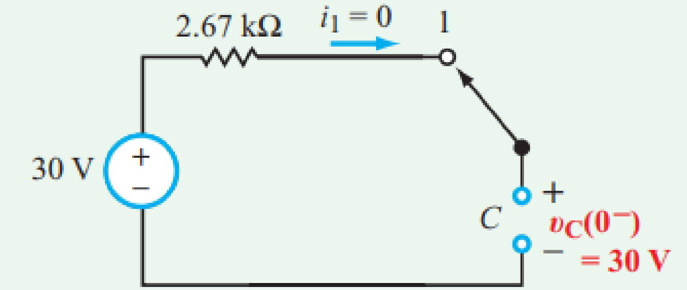
- For $v_c(0)$, we have =>
- By voltage division,

$$v_C(0^-) = \left(\frac{8\text{k}}{4\text{k} + 8\text{k}}\right) \times 45 = 30 \text{ V}.$$

and so $v_C(0) = v_C(0^-) = 30 \text{ V}.$



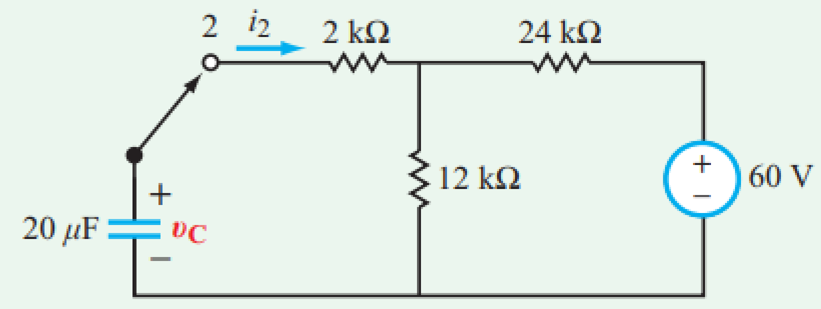
Circuit



Thévenin equivalent

Example 5-10: Determine Capacitor Voltage

- For $v_c(\infty)$, this is when switch is in position 2 for a long time. So we have =>



Circuit

- When circuit reaches final state, capacitor is open circuit, means $i_2 = 0$ at $t = \infty$.

Voltage division gives

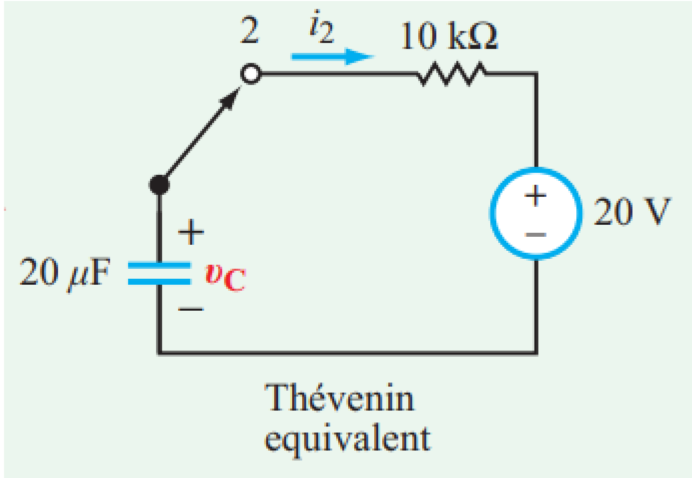
$$v_C(\infty) = \left(\frac{12k}{12k + 24k}\right) \times 60 = 20 \text{ V}.$$

Example 5-10: Determine Capacitor Voltage

The time constant of the circuit to the right of terminal 2 is given by $\tau = RC$, with R being the Thévenin resistance of that circuit. After suppressing (short-circuiting) the 60 V source, we get

$$R = R_{\mathrm{Th}} = 2\ \mathrm{k\Omega} + 12\ \mathrm{k\Omega} \parallel 24\ \mathrm{k\Omega}$$

$$= 2\ \mathrm{k\Omega} + \frac{12\mathrm{k} \times 24\mathrm{k}}{12\mathrm{k} + 24\mathrm{k}} = 10\ \mathrm{k\Omega}.$$



Hence,

$$\tau = RC = 10 \times 10^3 \times 20 \times 10^{-6} = 0.2\ \mathrm{s}.$$

- These give $\upsilon_C(t) = [(20 + 10e^{-5t})\ u(t)]\ \mathrm{V}.$

Example 5-10: Determine Capacitor Voltage

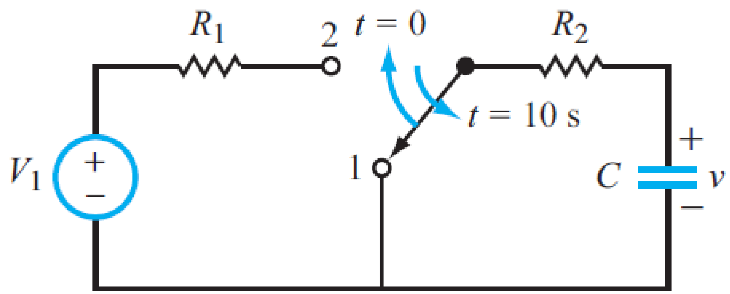
(b) For t = 3, this is equivalent to when $T_0 = 3$ s. The expression is then given by

$$\upsilon_C(t) = \left\{ \upsilon_C(\infty) + [\upsilon_C(3) - \upsilon_C(\infty)]e^{-(t-3)/\tau} \right\} u(t-3).$$

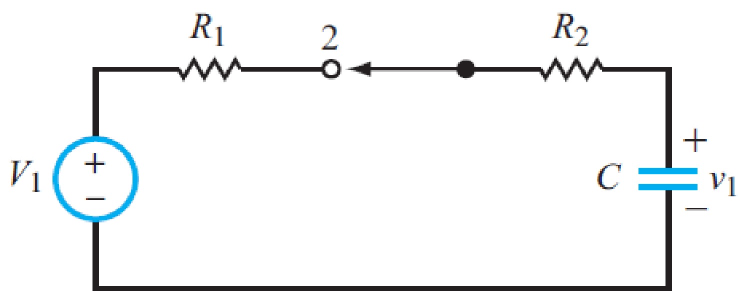
Of course, $\upsilon_C(t) = 30$ V before $t = 3$ s. Hence, for the specified time duration $t \geq 0$,

$$\upsilon_C(t) = \begin{cases} 30 \text{ V} & \text{for } 0 \leq t \leq 3 \text{ s}, \\ [20 + 10e^{-5(t-3)}] \text{ V} & \text{for } t \geq 3 \text{ s}. \end{cases}$$

Example 5-11: Charge/Discharge Action

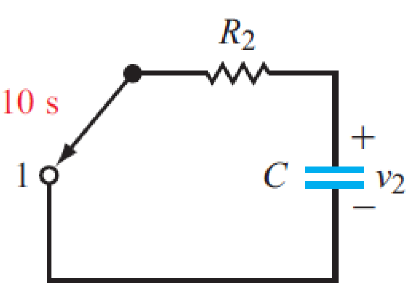


(a) Actual circuit



(b) Circuit during $0 \le t \le 10$ s

Circuit after $t = 10$ s



(c)

Given that the switch in Fig. 5-32 was moved to position 2 at $t = 0$ (after it had been in position 1 for a long time) and then returned to position 1 at $t = 10$ s, determine the voltage response $v(t)$ for $t \ge 0$ and evaluate it for $V_1 = 20$ V, $R_1 = 80$ kΩ, $R_2 = 20$ kΩ, and $C = 0.25$ mF.

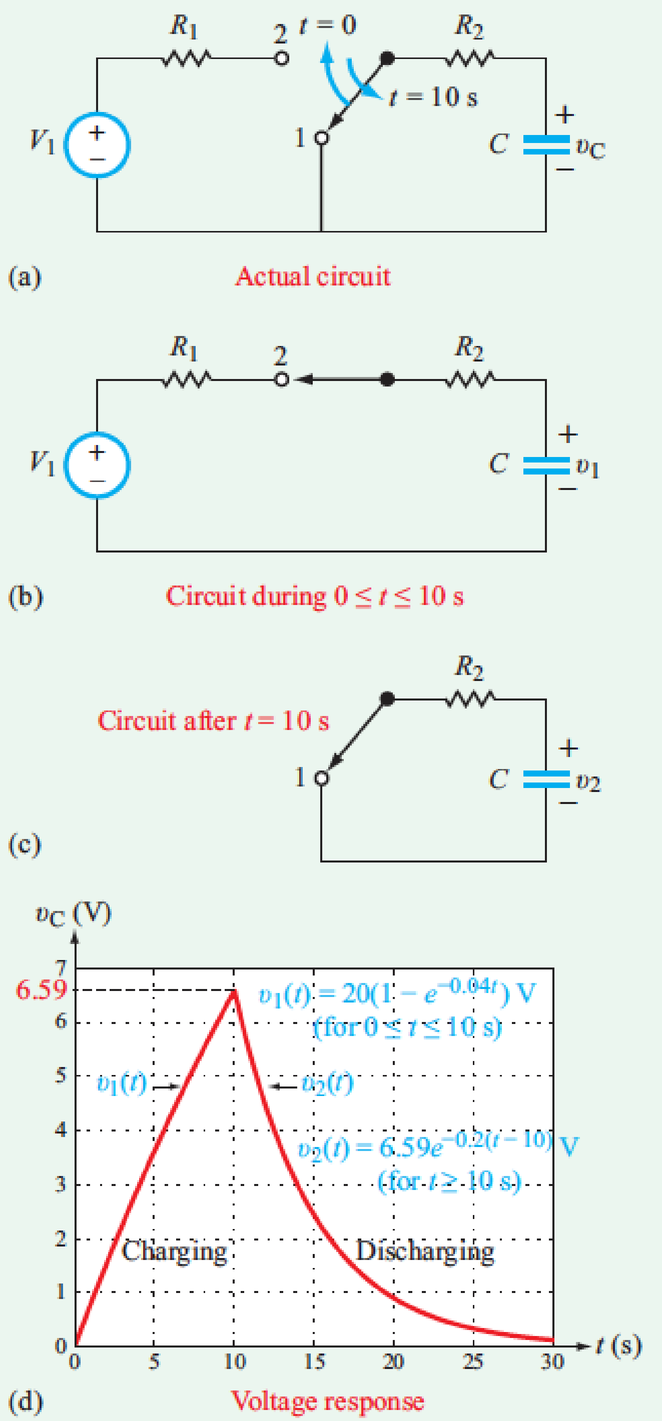
Time Segment 1: $0 \le t \le 10$ s

When the switch is in position 2 (Fig. 5-32(b)), the resistance of the circuit is $R = R_1 + R_2$. Hence, the time constant during this first time segment is

$$\begin{aligned}\tau_1 &= (R_1 + R_2)C \\ &= (80 + 20) \times 10^3 \times 0.25 \times 10^{-3} = 25 \text{ s}.\end{aligned}$$

$$\begin{aligned}v_1(t) &= v_1(\infty) + [v_1(0) - v_1(\infty)]e^{-t/\tau_1} \\ &= 20(1 - e^{-0.04t}) \quad \text{V} \quad (\text{for } 0 \le t \le 10 \text{ s}).\end{aligned}$$

Example 5-11 (cont.)



Time Segment 2: $t \geq 10$ s

Voltage $v_2(t)$, corresponding to the second time segment (Fig. 5-32(c)), is given by Eq. (5.98) with a new time constant τ_2 as

$$v_2(t) = v_2(\infty) + [v_2(10) - v_2(\infty)]e^{-(t-10)/\tau_2}.$$

The new time constant is associated with the capacitor circuit remaining after returning the switch to position 1,

$$\tau_2 = R_2 C$$
$$= 20 \times 10^3 \times 0.25 \times 10^{-3} = 5 \text{ s}.$$

The initial voltage $v_2(10)$ is equal to the capacitor voltage v_1 at the end of time segment 1, namely

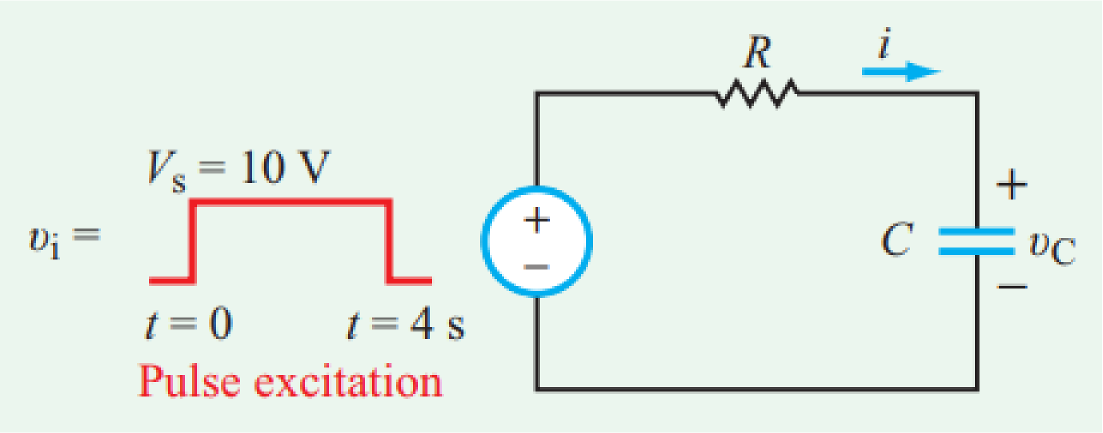
$$v_2(10) = v_1(10) = 20(1 - e^{-0.04 \times 10})$$
$$= 6.59 \text{ V}.$$

With no voltage source present in the R_2C circuit, the charged capacitor will dissipate its energy into R_2, exhibiting a ***natural response*** with a final voltage of $v_2(\infty) = 0$. Consequently,

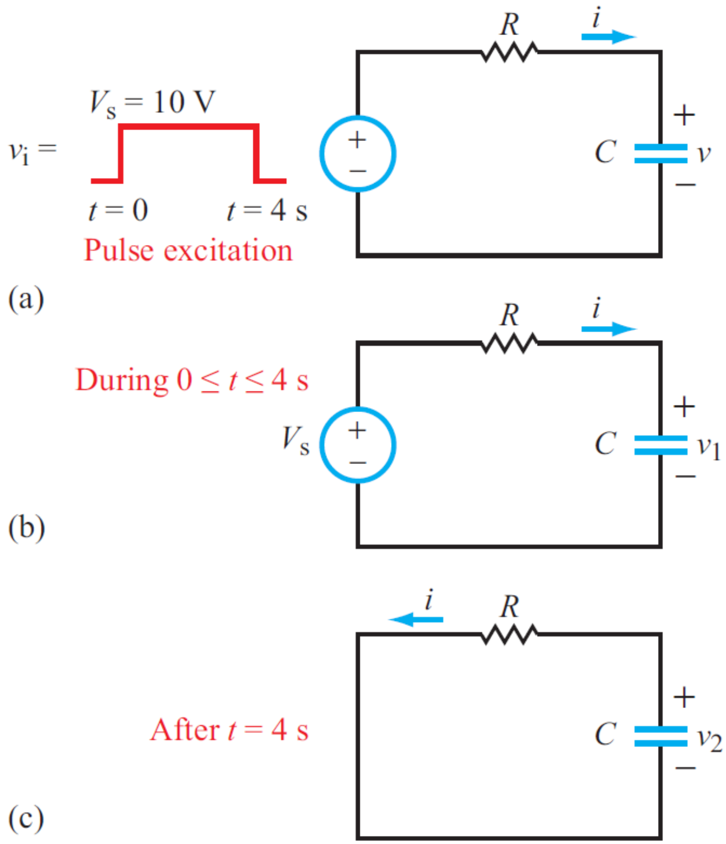
$$v_2(t) = v_2(10)\, e^{-(t-10)/\tau_2}$$
$$= 6.59e^{-0.2(t-10)} \text{ V} \quad (\text{for } t \geq 10 \text{ s}).$$

Example 5-12: Response to Rectangular Pulse

- Determine the voltage response of a previously uncharged RC circuit to a rectangular pulsi $v_i(t)$ of amplitude V_s and duration T_0, as depicted here. Evaluate and plot the response for R = 25 kΩ, C = 0.2 mF, V_s = 10 V, and T_0 = 4 s.



Example 5-12: Rectangular Pulse



$$v_i(t) = V_s\, u(t) - V_s\, u(t-4).$$

Since the circuit is linear, we can apply the superposition theorem to determine the capacitor response $v(t)$. Thus,

$$v(t) = v_1(t) + v_2(t),$$

$$v_1(t) = v_1(\infty) + [v_1(0) - v_1(\infty)]e^{-t/\tau}$$
$$= V_s(1 - e^{-t/\tau}) \qquad (\text{for } t \ge 0).$$

For $V_s = 10$ V and $\tau = RC = 25 \times 10^3 \times 0.2 \times 10^{-3} = 5$ s,

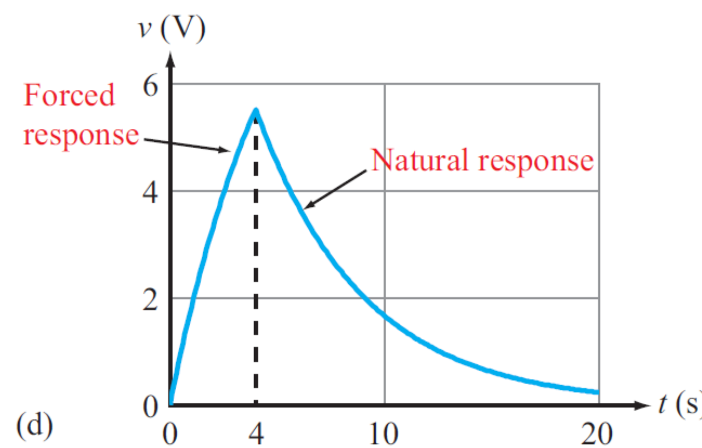
$$v_1(t) = 10(1 - e^{-0.2t}) \text{ V} \qquad (\text{for } t \ge 0).$$

The second step function has an amplitude of $-V_s$ and is delayed in time by 4 s. Upon reversing the polarity of V_s and replacing t with $(t-4)$, we have

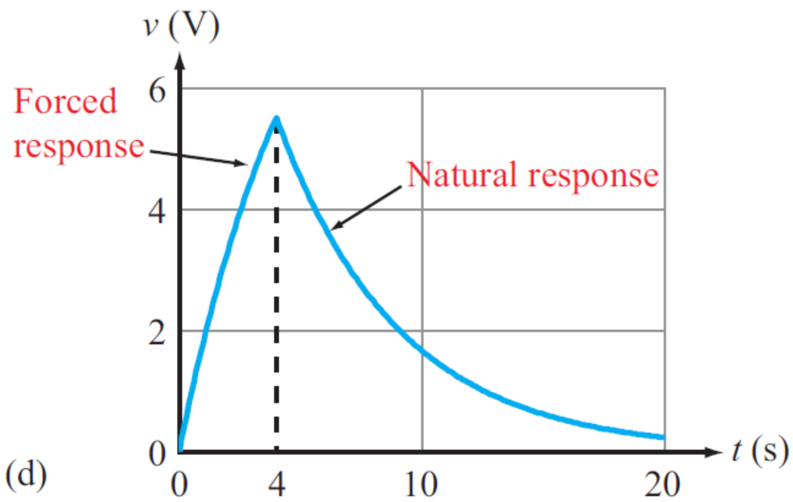
$$v_2(t) = -10[1 - e^{-0.2(t-4)}] \text{ V} \qquad (\text{for } t \ge 4 \text{ s}).$$

The total response for $t \ge 0$ therefore is given by

$$v(t) = v_1(t) + v_2(t)$$
$$= 10[1 - e^{-0.2t}] - 10[1 - e^{-0.2(t-4)}]\, u(t-4) \text{ V}$$



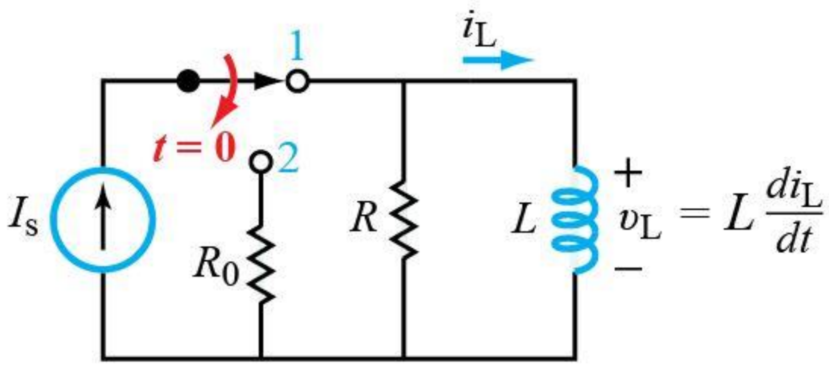
Example 5-12: Rectangular Pulse



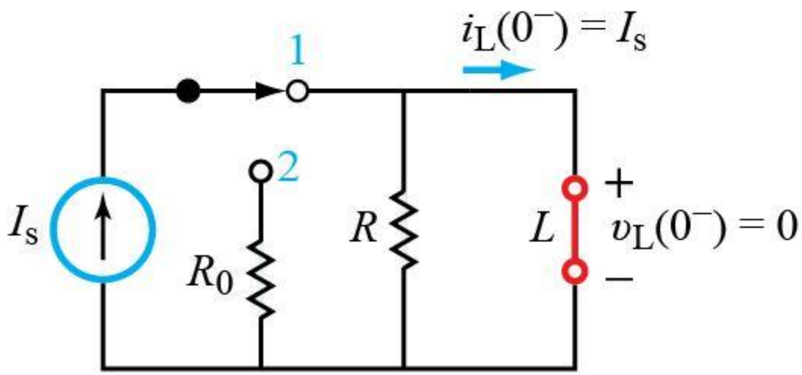
Explanation of the plot:

- $v_c(t)$ builds up to a maximum of 5.5 V by the end of the pulse (t = 4) and then decays exponentially back to zero thereafter.
- The build-up part is due to external excitation and is called *forced response*.
- For t≥4, $v_c(t)$ exhibits a *natural decay response* as the capacitor discharges its energy into the resistor, and i(t) flows in a counterclockwise direction.

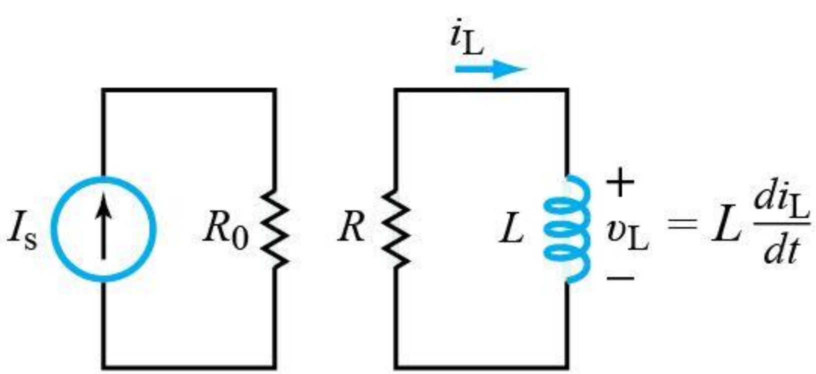
Natural Response of the RL Circuit



(a) Switch is moved at $t = 0$



(b) Initial condition at $t = 0^-$



(c) Circuit at $t \geq 0$ (natural response)

$$i_L(0) = i_L(0^-) = I_s.$$

For the time period $t \geq 0$, the loop equation for the RL circuit in Fig. 5-36(c) is given by

$$Ri_L + L\frac{di_L}{dt} = 0,$$

which can be cast in the form

$$\frac{di_L}{dt} + ai_L = 0, \qquad (5.101)$$

where a is a temporary constant given by

$$a = \frac{R}{L}.$$

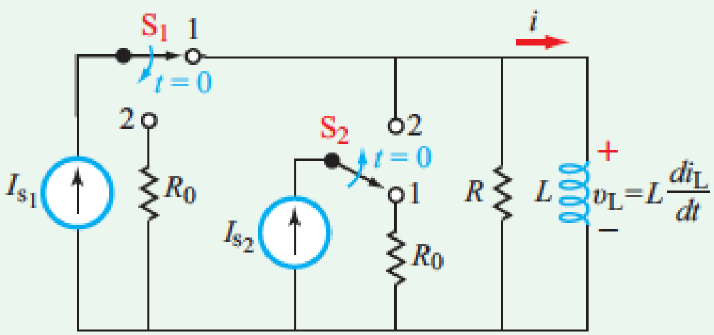
$$i_L(t) = i_L(0)\, e^{-t/\tau}\, u(t),$$

(natural response discharging)

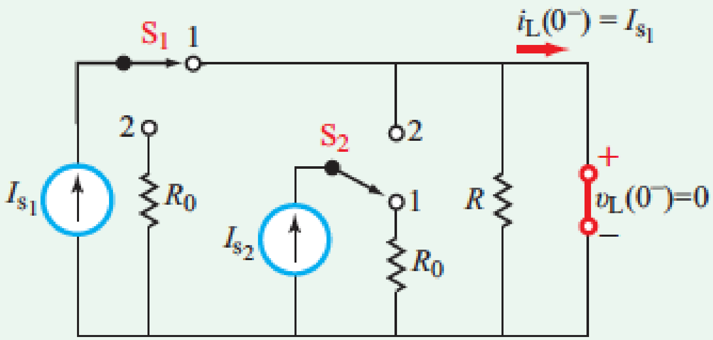
where for the RL circuit, the *time constant* is given by

$$\tau = \frac{1}{a} = \frac{L}{R}.$$

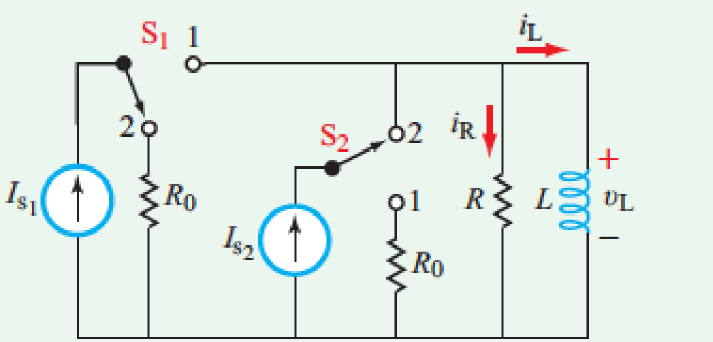
General Response of the RL Circuit



(a)



(b) Initial condition at $t = 0^-$



(c) At $t \geq 0$ (natural response)

$$i_L(0) = i_L(0^-) = I_{s_1}.$$

The circuit in Fig. 5-37(c) represents the arrangement at $t \geq 0$. Application of KCL at the common node gives

$$-I_{s_2} + i_R + i_L = 0.$$

Since υ is common to R and L, $i_R = \upsilon/R$, and by applying $\upsilon_L = L\, di_L/dt$, the KCL equation becomes

$$\frac{di_L}{dt} + a i_L = b, \qquad (5.105)$$

where a is as given previously by Eq. (5.102) and

$$b = a I_{s_2} = \frac{R}{L} I_{s_2}. \qquad (5.106)$$

$$i_L(t) = \left[i_L(\infty) + [i_L(0) - i_L(\infty)]e^{-t/\tau}\right] u(t), \qquad (5.107)$$

(switch action at $t = 0$)

with *time constant* $\tau = L/R$. For the specific circuit in Fig. 5-37(a), $i_L(0) = I_{s_1}$ and $i_L(\infty) = I_{s_2}$.

Parallel RL circuit solution

Parallel RL Circuit Solution

1: If switch action is at $t = 0$, analyze circuit at $t = 0^-$ (by replacing L with a short circuit) to determine initial conditions $i_L(0^-)$ and $\upsilon_L(0^-)$. Use this information to determine $i_L(0)$ and $i_L(0)$, at t immediately after the switch action. Remember that the current through an inductor cannot change instantaneously (between $t = 0^-$ and $t = 0$), but the voltage can.

2: Analyze the circuit to determine $i_L(\infty)$, the current through the inductor long after the switch action.

3: Determine the time constant $\tau = L/R$.

4: Incorporate the information obtained in the previous three steps in Eq. (5.107):

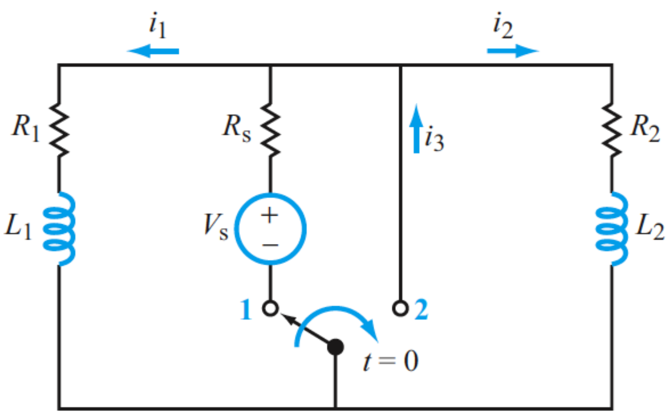
$$i_L(t) = \left[i_L(\infty) + [i_L(0) - i_L(\infty)]e^{-t/\tau}\right] u(t).$$

5: If the switch action is at $t = T_0$ instead of $t = 0$, replace 0 with T_0 everywhere and use Eq. (5.108):

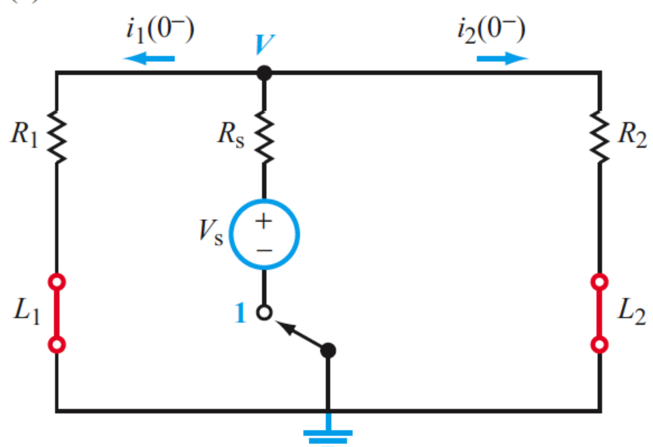
$$i_L(t) = \left\{i_L(\infty) + [i_L(T_0) - i_L(\infty)]e^{-(t-T_0)/\tau}\right\} u(t-T_0).$$

Example 5-13: Two RL Branches

After having been in position 1 for a long time, the SPDT switch in Fig. 5-36(a) was moved to position 2 at $t = 0$. Determine i_1, i_2, and i_3 for $t \geq 0$ given that $V_s = 9.6$ V, $R_s = 4$ kΩ, $R_1 = 6$ kΩ, $R_2 = 12$ kΩ, $L_1 = 1.2$ H, and $L_2 = 0.36$ H.



(a) Circuit with two inductors



(b) At $t = 0^-$

At $t=0^-$

Application of KCL to node V gives

$$\frac{V}{R_1} + \frac{V - V_s}{R_s} + \frac{V}{R_2} = 0,$$

whose solution is

$$V = \frac{R_1 R_2 V_s}{R_1 R_2 + R_1 R_s + R_2 R_s}$$
$$= \frac{6 \times 12 \times 9.6}{6 \times 12 + 6 \times 4 + 12 \times 4} = 4.8 \text{ V}.$$

Hence, the initial currents $i_1(0)$ and $i_2(0)$ are given by

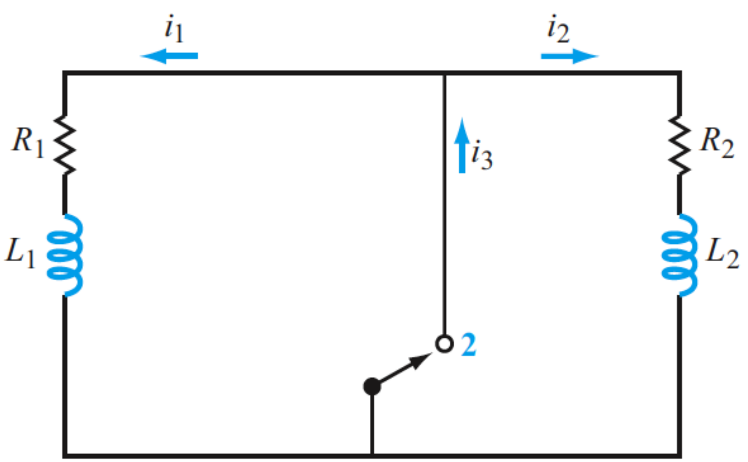
$$i_1(0) = i_1(0^-) = \frac{V}{R_1} = \frac{4.8}{6 \times 10^3} = 0.8 \text{ mA}$$

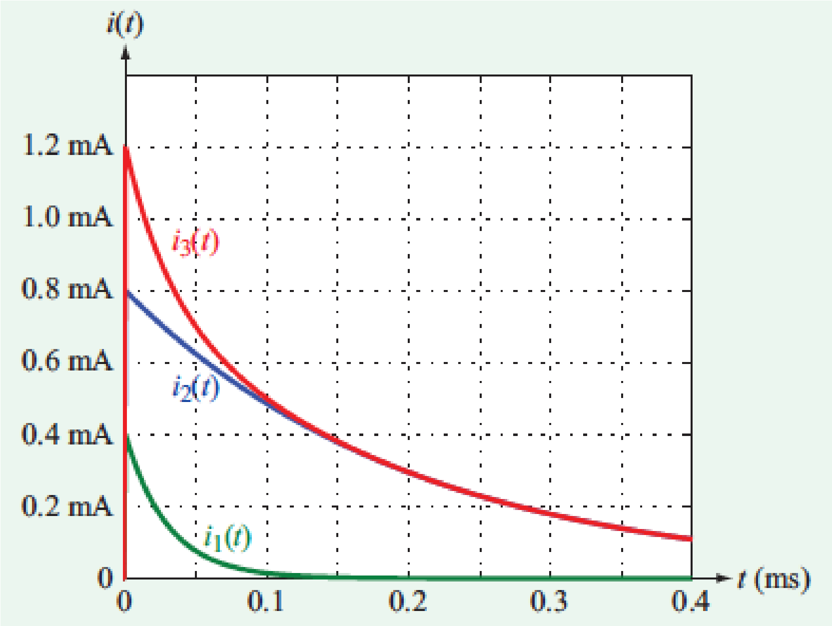
and

$$i_2(0) = i_2(0^-) = \frac{V}{R_2} = \frac{4.8}{12 \times 10^3} = 0.4 \text{ mA}.$$

Cont.

Example 5-13: Two RL Branches (cont.)





After t=0:

$$i_1(t) = [i_1(\infty) + [i_1(0) - i_1(\infty)]e^{-t/\tau_1}]$$
$$= 0.8e^{-t/\tau_1} \text{ mA}$$

and

$$i_2(t) = [i_2(\infty) + [i_2(0) - i_2(\infty)]e^{-t/\tau_2}]$$
$$= 0.4e^{-t/\tau_2} \text{ mA},$$

where τ_1 and τ_2 are the time constants of the two RL circuits, namely

$$\tau_1 = \frac{L_1}{R_1} = \frac{1.2}{6 \times 10^3} = 2 \times 10^{-4} \text{ s}$$

and

$$\tau_2 = \frac{L_2}{R_2} = \frac{0.36}{12 \times 10^3} = 3 \times 10^{-5} \text{ s}.$$

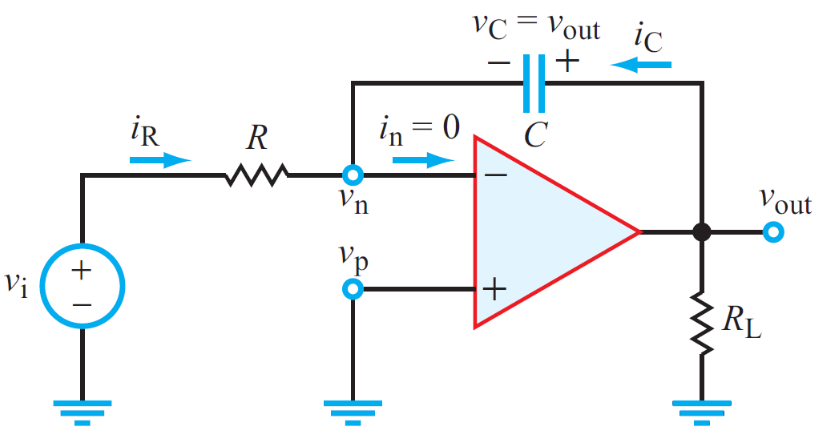
The current flowing through the short circuit is simply

$$i_3 = i_1 + i_2$$
$$= (0.8e^{-t/\tau_1} + 0.4e^{-t/\tau_2}) \text{ mA} \quad (\text{for } t \geq 0).$$

RC Op-Amp Circuits:

Ideal Integrator

RC Integrator



The output voltage v_{out} of such an integrator circuit is directly proportional to the time integral of the input signal v_i.

$$i_R = \frac{v_i}{R}. \quad (5.123)$$

Given that $v_n = 0$, the voltage v_C across C is simply v_{out}, and the current flowing through it is

$$i_C = C\,\frac{dv_{out}}{dt}. \quad (5.124)$$

At node v_n,

$$i_R + i_C - i_n = 0. \quad (5.125)$$

In view of the second op-amp constraint, namely $i_n = i_p = 0$, it follows that

$$i_C = -i_R \quad (5.126)$$

or

$$\frac{dv_{out}}{dt} = -\frac{1}{RC}\,v_i. \quad (5.127)$$

Upon integrating both sides of Eq. (5.127) from a reference time t_0 to time t, we have

$$\int_{t_0}^{t} \left(\frac{dv_{out}}{dt}\right) dt = -\frac{1}{RC}\int_{t_0}^{t} v_i\,dt, \quad (5.128)$$

which leads to

$$v_{out}(t) = -\frac{1}{RC}\int_{t_0}^{t} v_i\,dt + v_{out}(t_0). \quad (5.129)$$

Example 5-15: Square-Wave Signal

The square-wave signal shown in Fig. 5-39(a) is applied at the input of an ideal integrator circuit with an initial capacitor voltage of zero at $t = 0$. If $R = 200$ kΩ and $C = 2.5$ μF, determine the waveform of the corresponding output voltage for an amp with (a) $V_{cc} = 14$ V and (b) $V_{cc} = 9$ V.

Solution:

(a) The scaling factor is given by

$$-\frac{1}{RC} = -\frac{1}{2 \times 10^5 \times 2.5 \times 10^{-6}} = -2 \quad \text{s}^{-1}.$$

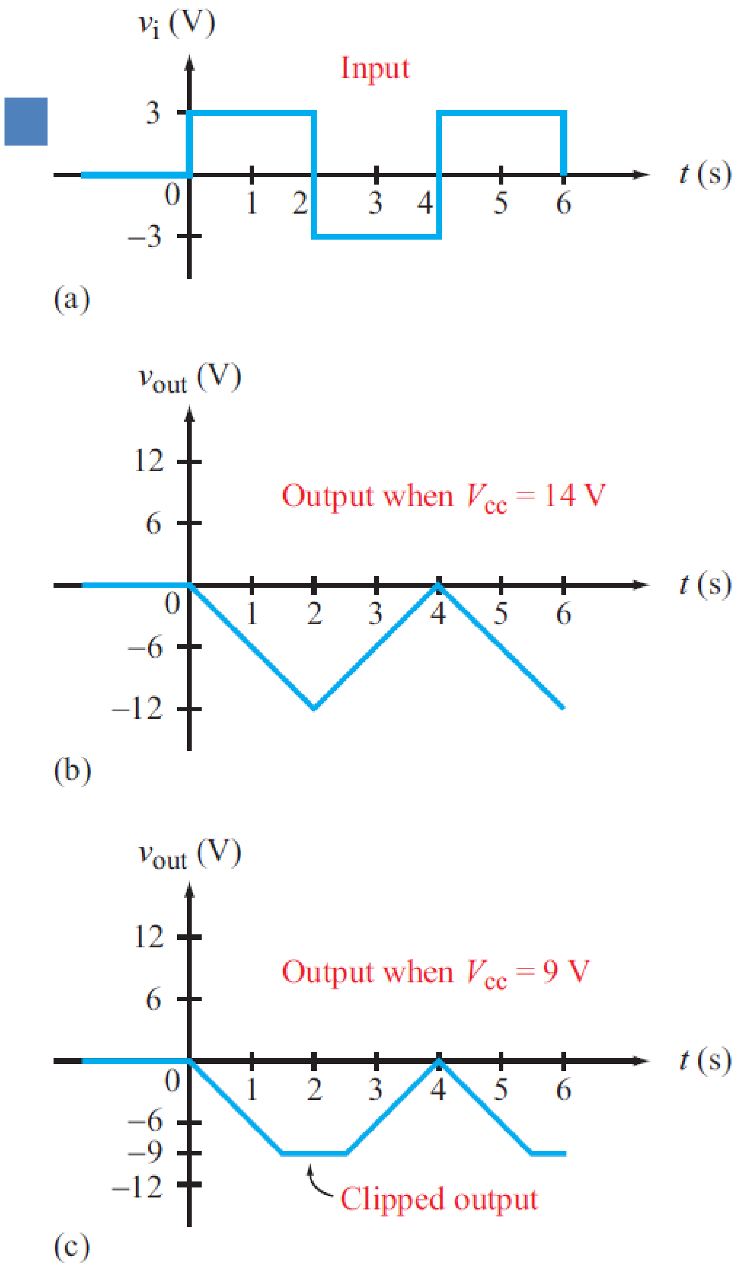
For the time period $0 \leq t \leq 2$ s (first half of the first cycle),

$$v_{out}(t) = -2 \int_0^t v_i \, dt = -2 \int_0^t 3 \, dt = -6t \text{ V}$$

$$(0 \leq t \leq 2 \text{ s}),$$

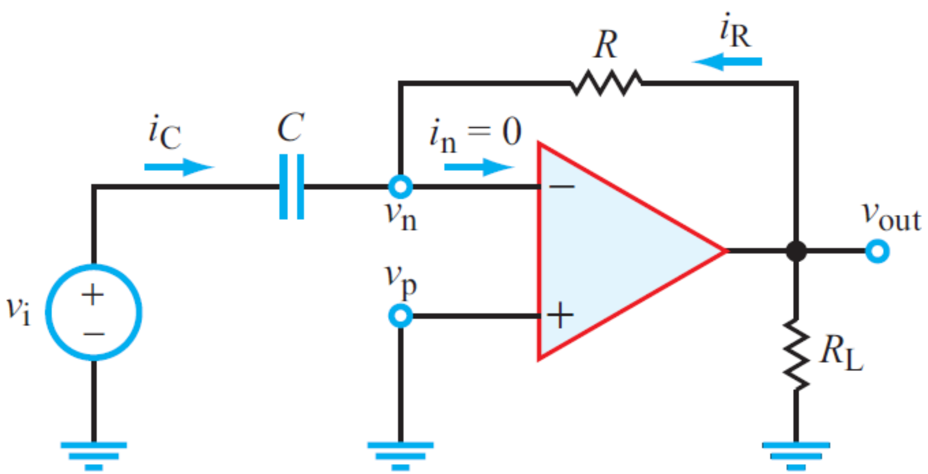
We note that because $|v_{out}|$ never exceeds $|V_{cc}| = 14$ V, no saturation occurs in the op amp.

(b) For the op amp with $V_{cc} = 9$ V, the waveform shown in Fig. 5-39(c) is the same as that in Fig. 5-39(b), except that it is ***clipped*** at −9 V.



RC Op-Amp Circuits: Ideal Differentiator

RC Differentiator



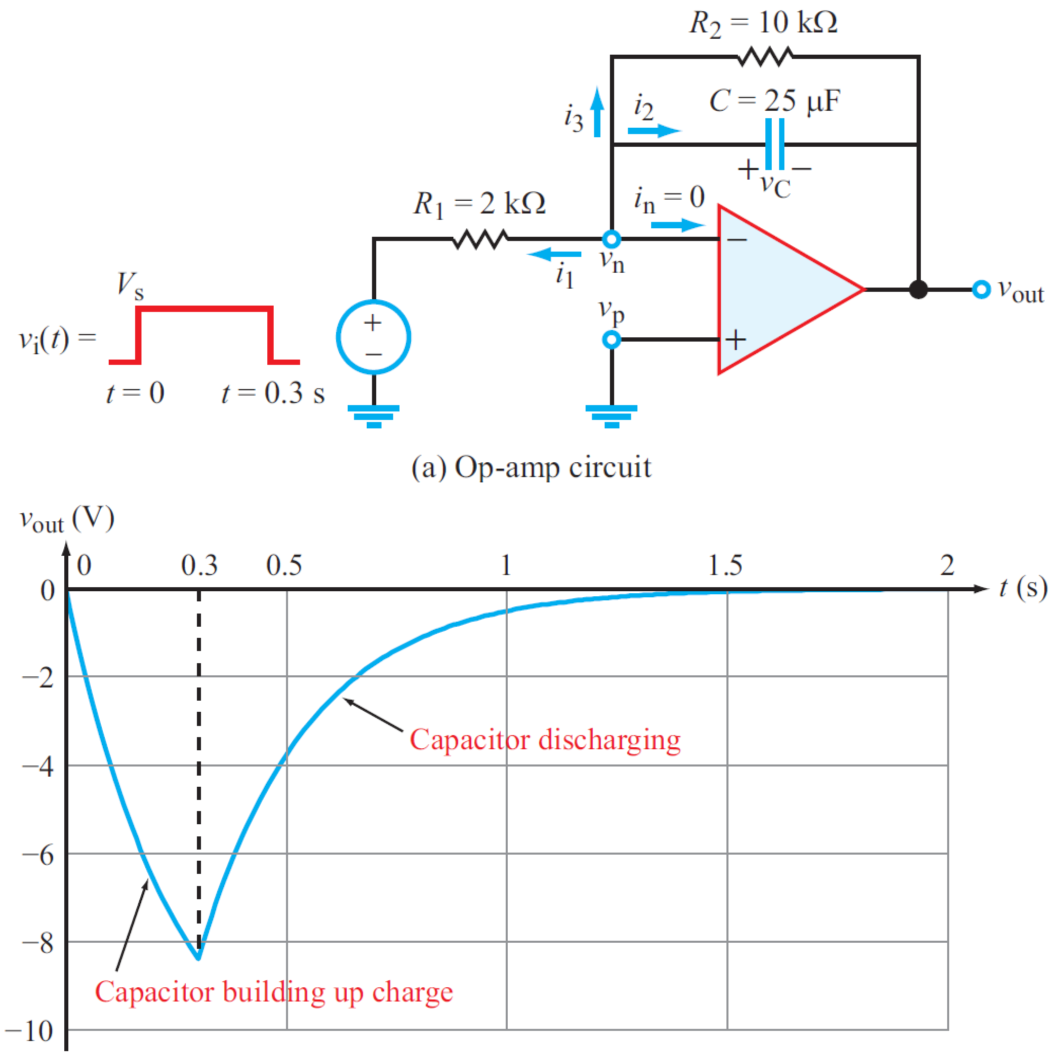
$$i_C = C\,\frac{dv_i}{dt},$$

$$i_R = \frac{v_{out}}{R},$$

$$i_C = -i_R.$$

$$\boxed{v_{out} = -RC\,\frac{dv_i}{dt},}$$

Example 5-16: Pulse Response



(b) $v_{out}(t)$

Example 5-18: Differential Equation Solver

Design an op-amp circuit whose output is the solution of the differential equation

$$\frac{d^2\upsilon}{dt^2} + 8\,\frac{d\upsilon}{dt} + 2\upsilon = 4\upsilon_s(t), \tag{5.140}$$

where $\upsilon_s(t)$ is a sinusoidal source given by

$$\upsilon_s(t) = 3\sin(200t)\,u(t).$$

Solution:

$$\begin{aligned}\upsilon &= -\frac{1}{2}\,\upsilon'' - 4\upsilon' + 2\upsilon_s(t)\\ &= -\frac{1}{2}\,\upsilon'' - 4\upsilon' + 6\sin(200t)\,u(t).\end{aligned}$$

$$v = -\frac{1}{2} v'' - 4v' + 2v_s(t)$$
$$= -\frac{1}{2} v'' - 4v' + 6\sin(200t)\, u(t).$$

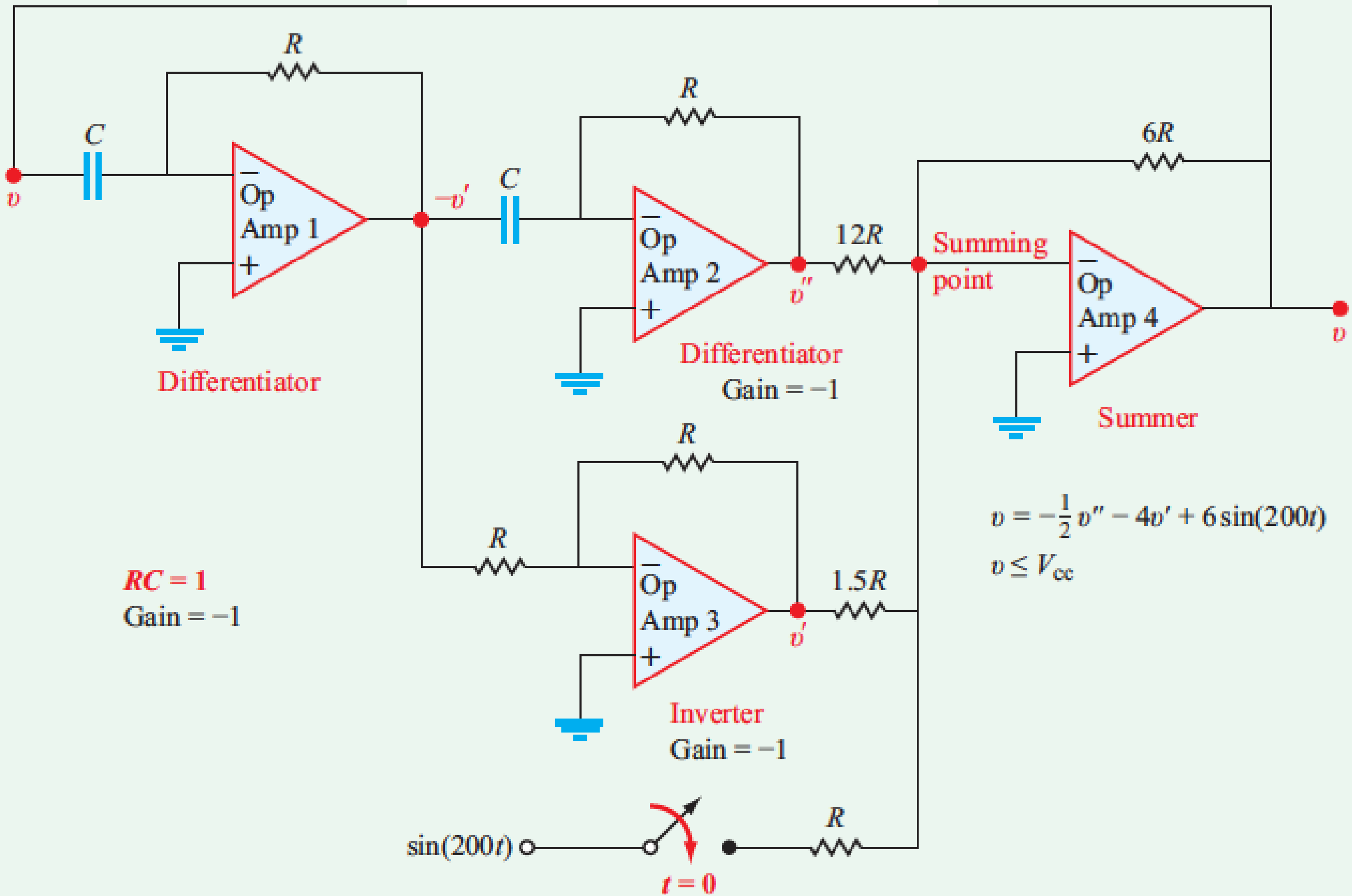


Figure 5-45: Op-amp circuit whose output $v(t)$ is a solution to $v'' + 8v' + 2v = 12\sin(200t)\, u(t)$.

Summary

Mathematical and Physical Models

Unit step function

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

Time-shifted step function

$$u(t-T) = \begin{cases} 0 & \text{for } t < T \\ 1 & \text{for } t > T \end{cases}$$

Unit ramp function

$$r(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ t & \text{for } t \geq 0 \end{cases}$$

Time-shifted ramp function

$$r(t-T) = \begin{cases} 0 & \text{for } t \leq T \\ (t-T) & \text{for } t \geq T \end{cases}$$

Unit rectangular function
(pulse center at $t = T$; pulse length $= \tau$)

$$\text{rect}\left[\frac{(t-T)}{\tau}\right] = \begin{cases} 0 & \text{for } t < (T-\tau/2), \\ 1 & \text{for } (T-\tau/2) \leq t \leq (T+\tau/2), \\ 0 & \text{for } t > (T+\tau/2). \end{cases}$$

Capacitor

$$i = C\,\frac{d\upsilon}{dt}$$

$$\upsilon(t) = \upsilon(t_0) + \frac{1}{C}\int_{t_0}^{t} i\,dt'$$

$$w = \tfrac{1}{2}\,C\upsilon^2 \qquad \text{(stored electrical energy)}$$

Parallel plate $\quad C = \dfrac{\varepsilon A}{d}$

Inductor

$$\upsilon = L\,\frac{di}{dt}$$

$$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} \upsilon\,dt'$$

$$w = \tfrac{1}{2}\,Li^2 \qquad \text{(stored magnetic energy)}$$

Solenoid $\quad L = \dfrac{\mu N^2 S}{\ell}$

Series RC circuit response (sudden change at $t = 0$)

$$\upsilon_{\text{C}}(t) = \upsilon_{\text{C}}(\infty) + [\upsilon(0) - \upsilon(\infty)]e^{-t/\tau}$$
$$\tau = RC$$

Parallel RL circuit response (sudden change at $t = 0$)

$$i_{\text{L}}(t) = i_{\text{L}}(\infty) + [i_{\text{L}}(0) - i_{\text{L}}(\infty)]e^{-t/\tau}$$
$$\tau = L/R$$

Op-amp integrator

$$\upsilon_{\text{out}}(t) = -\frac{1}{RC}\int_{t_0}^{t} \upsilon_{\text{i}}\,dt' + \upsilon_{\text{out}}(t_0)$$

Op-amp differentiator

$$\upsilon_{\text{out}}(t) = -RC\,\frac{d\upsilon_{\text{i}}}{dt}$$