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Introduction to Laplace Transforms

XMUT203 Analogue Circuits and Systems

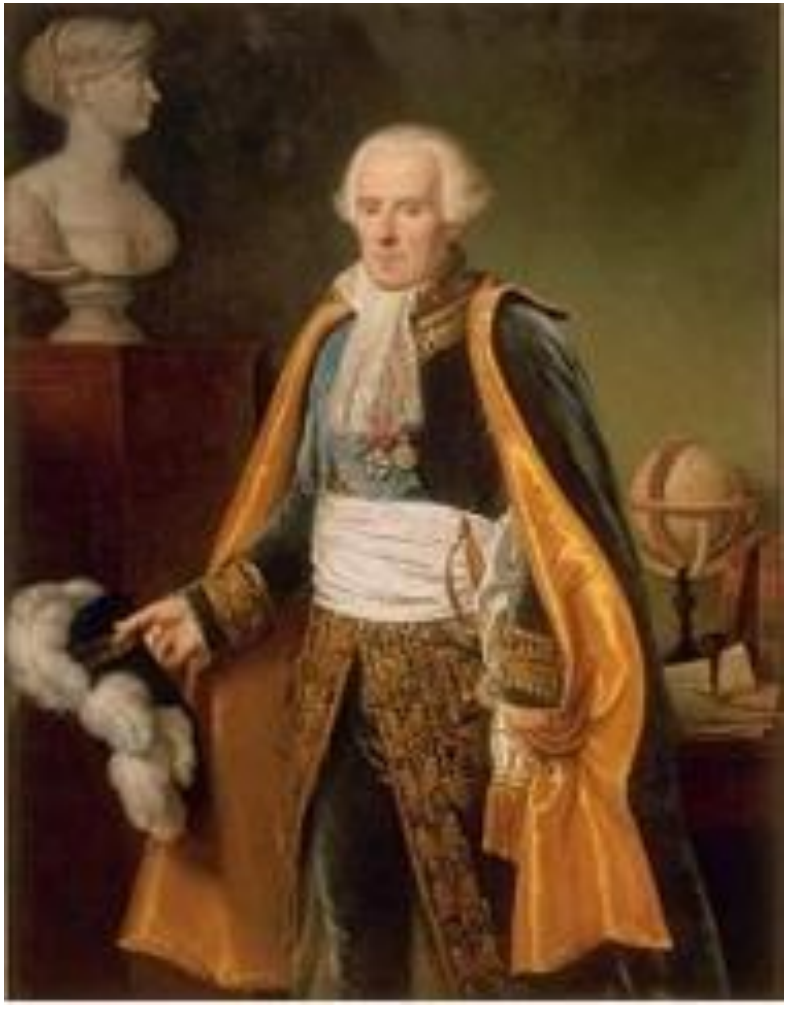
Topics

- Introduction to Laplace transform.
- Forward and reverse (functional) of Laplace transform.
- Properties (operational transforms) of Laplace transform.
- Applying Laplace transform.
- Inverse Laplace transforms.
- Partial fraction expansions.
- Poles and zeros of $F(s)$.
- Initial- and final-value theorems.

Prelude to Laplace Transform

- Laplace transform is a mathematical transformation technique put together by Pierre Simon.
- It is for converting the solving of integro-integral equation from integration and/or differentiation in the time domain to simpler algebraic manipulation.
- To help with conversion from the time domain to the intermediate domain (i.e. s-domain), partial fraction expansion method, formulated by Olivier Heaviside, is used to perform the tasks.

The Hero of Our Story



- Pierre Simon, marquis de Laplace



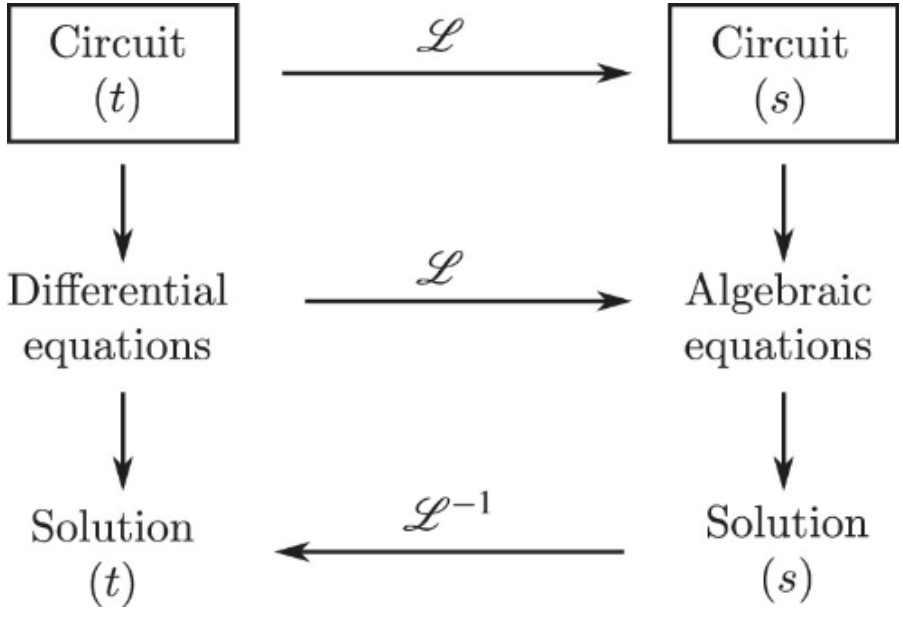
- Oliver Heaviside

Mathematical Transforms

- When analysing circuit, we have learnt to apply a few transforms already:
  - Logarithms are used to change a multiplication or division problem into a simpler addition or subtraction problem.
  - Antilogs are used to carry out the inverse process.
  - The phasor converts a sinusoidal signal into a complex number for easier algebraic computation of circuit values.
  - After determining the phasor value of a signal, we transform it back to its time-domain expression.

Mathematical Transforms

- Both of these examples point out the essential feature of mathematical transforms:
  - They are designed to create a new domain to make the mathematical manipulations easier.
  - After finding the unknown in the new domain, we inverse-transform it back to the original domain.



Circuit Analysis and Laplace Transform

- Consider the transient behavior of circuits whose describing equations consist of more than a single node-voltage or mesh current differential equation e.g. multiple-node and multiple-mesh circuits that are described by sets of linear differential equations.
- Determine the transient response of circuits whose signal sources vary in ways more complicated than the simple DC level jumps.
- Use the Laplace transform to introduce the concept of the transfer function as a tool for analysing the steady-state sinusoidal response of a circuit when the frequency of the sinusoidal source is varied.
- Relate, in a systematic fashion, the time-domain behavior of a circuit to its frequency-domain behavior.

Laplace Transform

- The Laplace transform of a function is given by the expression:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st} dt$$

- The Laplace transform of $f(t)$ is also denoted as:

$$F(s) = \mathcal{L}\{f(t)\}$$

- This notation emphasizes that when the integral in the equation has been evaluated, the resulting expression is a function of s.
- In our applications, t represents the time domain, and, because the exponent of e in the integral of the equation above must be dimensionless, s must have the dimension of reciprocal time, or frequency.

Laplace Transform

- The Laplace transform transforms the problem from the time domain to the frequency domain.
- After obtaining the frequency-domain expression for the unknown, we could inverse-transform it back to the time domain.
- In circuit analysis, we use the Laplace transform to transform a set of integro-differential equations from the time domain to a set of algebraic equations in the frequency domain.
- We therefore simplify the solution for an unknown quantity to the manipulation of a set of algebraic equations.

Laplace Transform Characteristics

- Knowing that a Laplace transform is given as:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st} dt$$

- The integral is improper because the upper limit is infinite.
- Does the integral converge?
- In linear circuit analysis, we only excite circuits with sources that have Laplace transforms.
- The lower limit on the integral is zero, the Laplace transform ignores $f(t)$ for negative values of t. e.g. behaviour of $f(t)$ only for positive values of t (one-sided, or unilateral).

Example of Laplace Transform

With regard to Laplace transform, answer the following questions:

a. Describe the definition of Laplace transform. [2 marks]

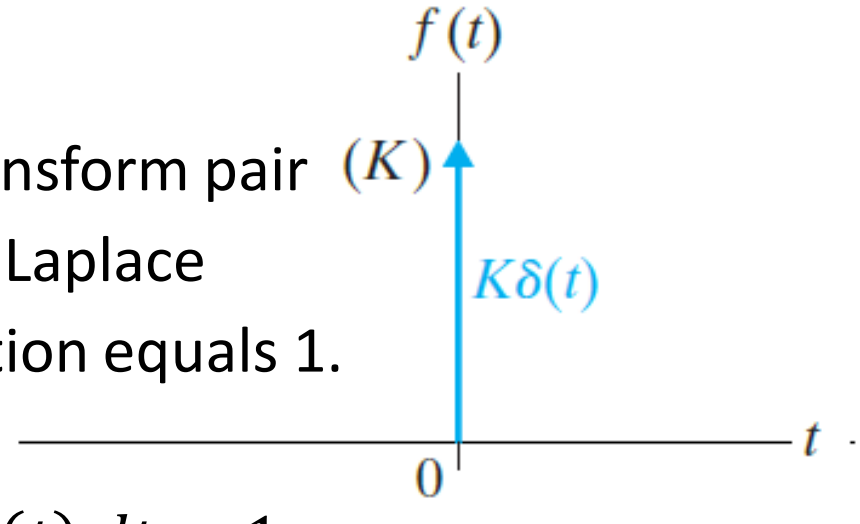
b. Describe three characteristics of the Laplace transform. [3 marks]

Functional Transforms

- A functional transform is simply the Laplace transform of a specified function of t.
- Because we are limiting our introduction to the unilateral, or one-sided, Laplace transform, we define all functions to be zero for $t < 0^-$.

Impulse function:

- First, we derived one functional transform pair before, where we showed that the Laplace transform of the unit impulse function equals 1.



$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)\, e^{-st} dt = \int_{0^-}^{\infty} \delta(t)\, dt = 1$$

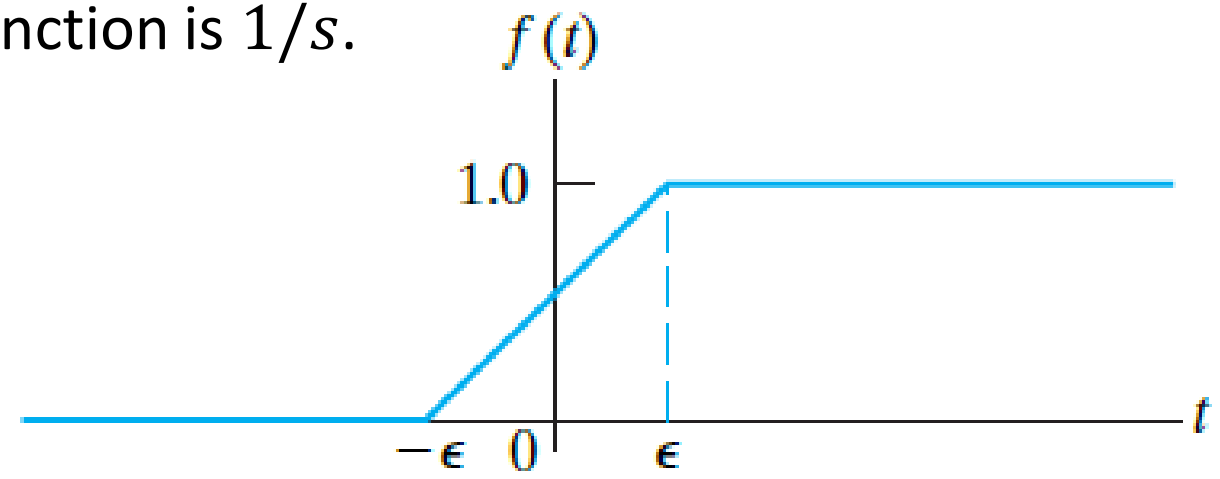
Functional Transforms

Step function:

- From before, a second illustration of the functional transform is the unit step function of the figure below, where:

$$\mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} f(t)\, e^{-st} dt = \int_{0^+}^{\infty} 1\, e^{-st} dt = \frac{e^{-st}}{-s}\Bigg|_{0^+}^{\infty} = \frac{1}{s}$$

- Equation above shows that the Laplace transform of the unit step function is $1/s$.



Functional Transforms

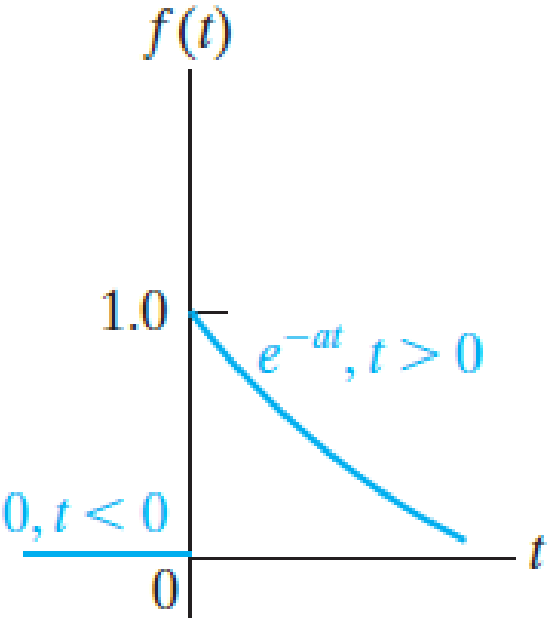
Exponential function:

- The Laplace transform of the decaying exponential function shown in the figure below is:

$$\mathcal{L}\{e^{-at}\} = \int_{0^+}^{\infty} e^{-at}\, e^{-st} dt$$

$$= \int_{0^+}^{\infty} e^{-(a+s)t}$$

$$= \frac{1}{s+a}$$

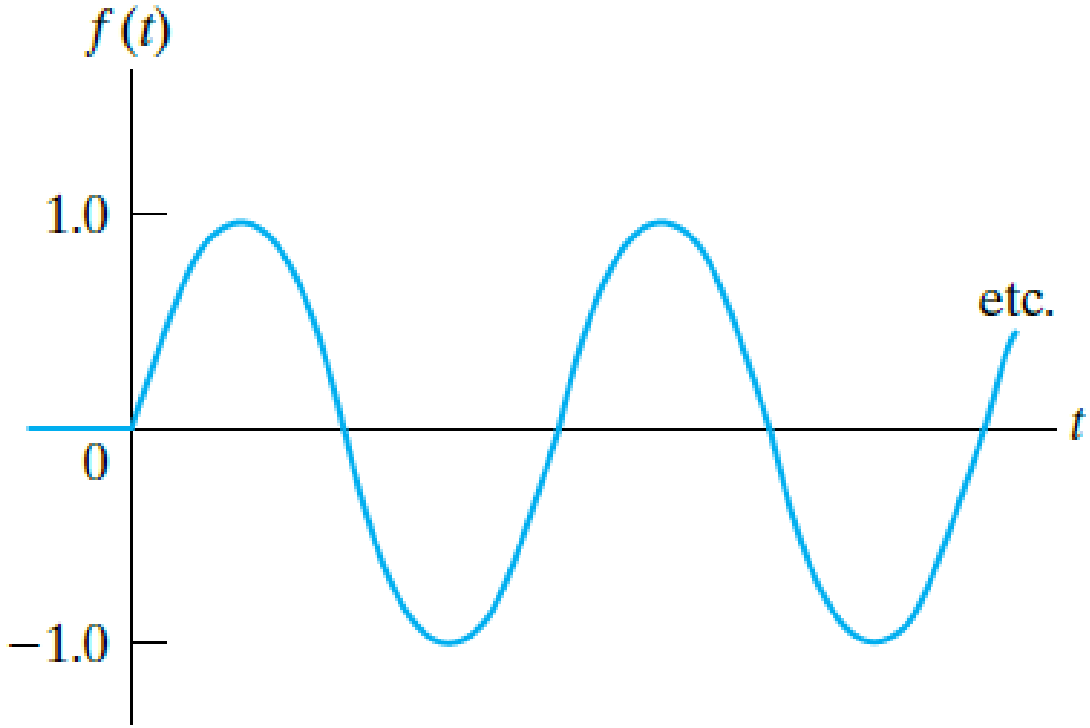


- In deriving the last two equations, we used the fact that integration across the discontinuity at the origin is zero.

Functional Transforms

Sinusoidal function:

- A third illustration of finding a functional transform is a sinusoidal function shown in the figure below.



Functional Transforms

- The expression for $f(t)$ for $t > 0^-$ is $\sin \omega t$, hence its Laplace transform is:

$$\mathcal{L}\{\sin \omega t\} = \int_{0^-}^{\infty} (\sin \omega t)\, e^{-st} dt$$

$$= \int_{0^-}^{\infty} \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right) e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} \left(\frac{e^{-(s-j\omega)t} - e^{-(s+j\omega)t}}{2j} \right) dt$$

$$= \frac{1}{2j} \left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right) = \frac{\omega}{s^2 + \omega^2}$$

Functional Transforms

- Table next gives an abbreviated list of Laplace transform pairs.
- It includes the functions of most interest in an introductory course on circuit applications.

Function	$\boldsymbol{f(t)(t > 0^-)}$	$\boldsymbol{F(s)}$
Impulse	$\delta(t)$	1
Step	$u(t)$	$\frac{1}{s}$
Ramp	t	$\frac{1}{s^2}$
Exponential	e^{-at}	$\frac{1}{s+a}$

Functional Transforms

- Abbreviated list of Laplace transform pairs (continue ….)

Function	$\boldsymbol{f(t)(t > 0^-)}$	$\boldsymbol{F(s)}$
Sine	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
Cosine	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
Damped ramp	te^{-at}	$\frac{1}{(s + a)^2}$
Damped sine	$e^{-at} \sin \omega t$	$\frac{\omega}{(s + a)^2 + \omega^2}$
Damped cosine	$e^{-at} \cos \omega t$	$\frac{s + a}{(s + a)^2 + \omega^2}$

Example of Functional Transforms

There are two types of transformations of the functions in Laplace transform e.g. functional and operational transforms.

a. Describe what is functional Laplace transform. [2 marks]

b. Describe functional Laplace transforms of the following functions:

   i. Impulse function. [2 marks]

   ii. Step function. [2 marks]

   iii. Exponential function. [2 marks]

   iv. Sinusoidal function. [2 marks]

Operational Laplace Transform

- Operational transforms indicate how mathematical operations performed on either or are converted into the opposite domain.
- The operations of primary interest are:
    - Multiplication by a constant.
    - Addition (subtraction).
    - Differentiation.
    - Integration.
    - Translation in the time domain.
    - Translation in the frequency domain.
    - Scale changing.

Operational Laplace Transform

Multiplication by a Constant:

- From the defining integral, if:

$$\mathcal{L}\{f(t)\} = F(s)$$

- If k is a constant, then

$$\mathcal{L}\{kf(t)\} = kF(s)$$

- Thus, multiplication of $f(t)$ by a constant corresponds to multiplying $F(s)$ by the same constant.

Operational Laplace Transform

Addition (Subtraction):

- Addition (subtraction) in the time domain translates into addition (subtraction) in the frequency domain. Thus, if:

$$\mathcal{L}\{f_1(t)\} = F_1(s)$$

$$\mathcal{L}\{f_2(t)\} = F_2(s)$$

$$\mathcal{L}\{f_3(t)\} = F_3(s)$$

- then

$$\mathcal{L}\{f_1(t) + f_2(t) + f_3(t)\} = F_1(s) + F_2(s) + F_3(s)$$

- which is derived by simply substituting the algebraic sum of time-domain functions into the defining integral.

Operational Laplace Transform

Differentiation:

- Differentiation in the time domain corresponds to multiplying $F(s)$ by s and then subtracting the initial value of $f(t)$ - that is, $f(0^-)$ - from this product:

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-) \quad (Eq.1)$$

- which is obtained directly from the definition of the Laplace transform, or:

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty}\left[\frac{df(t)}{dt}\right]e^{-st}dt \qquad (Eq.2)$$

Operational Laplace Transform

- We evaluate the integral given in equation 2 above by integrating by parts.
- Letting $e^{-st}f(t)$ and $dv = [df(t)/dt]dt$ yields:

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = e^{-st}f(t)|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)\,(-se^{-st})dt \qquad (Eq.3)$$

- Because we are assuming that $f(t)$ is Laplace transformable, the evaluation of $e^{-st}f(t)$ at $t = \infty$ is zero.
- Therefore, the right-hand side of equation above reduces to:

$$-f(0^-) + s\int_{0^-}^{\infty} f(t)\,e^{-st}dt = sF(s) - f(0^-)$$

Operational Laplace Transform

- Differentiation property is important because it states that differentiation in the time domain reduces to an algebraic operation in the s domain.
- We determine the Laplace transform of higher-order derivatives by using the differentiation property as the starting point.
- We find the Laplace transform of the n-th derivative by successively applying the preceding process, which leads to the general result:

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0^-) - s^{n-2}\frac{df(0^-)}{dt}$$

$$-s^{n-3}\frac{d^2 f(0^-)}{dt^2} - \cdots - \frac{d^{n-1} f(0^-)}{dt^{n-1}} \qquad (Eq.4)$$

Operational Laplace Transform

Integration:

- Integration in the time domain corresponds to dividing by s in the s domain.
- As before, we establish the relationship by the defining integral:

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)dx\right\} = \int_{0^-}^{\infty}\left[\int_{0^-}^{t} f(x)dx\right]e^{-st}dt \qquad (Eq.\,1)$$

- We evaluate the integral on the right-hand side of Eq. 1 given above by integrating by parts, first letting:

$$u = \int_{0^-}^{t} f(x)dx$$

Operational Laplace Transform

- And

$$dv = e^{-st}dt$$

- Then

$$du = f(t)dt$$

$$v = -\frac{e^{-st}}{s}$$

- The integration-by-parts formula yields:

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)dx\right\} = -\frac{e^{-st}}{s}\int_{0^-}^{t} f(x)dx\Big|_{0^-}^{\infty} + \int_{0^-}^{\infty}\frac{e^{-st}}{s}f(t)dt \quad (Eq.2)$$

Operational Laplace Transform

- The first term on the right-hand side of Eq. 2 is zero at both the upper and lower limits.
- The evaluation at the lower limit obviously is zero, whereas the evaluation at the upper limit is zero because we are assuming that $f(t)$ has a Laplace transform.
- The second term on the right-hand side of Eq. 2 is $F(s)/s$; therefore:

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)dx\right\} = \frac{F(s)}{s} \qquad (Eq.3)$$

- which reveals that the operation of integration in the time domain is transformed to the algebraic operation of multiplying by $1/s$ in the s-domain.

Operational Laplace Transform

- Knowing the given equations

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - s^{n-3}\frac{d^2 f(0^-)}{dt^2} - \cdots - \frac{d^{n-1} f(0^-)}{dt^{n-1}}$$

- And

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)dx\right\} = \frac{F(s)}{s}$$

- Differentiation and integration properties prove that the Laplace transform translates a set of integro-differential equations into a set of algebraic equations.

Operational Laplace Transform

Translation in the Time Domain:

- If we start with any function $f(t)u(t)$, we can represent the same function, translated in time by the constant a, as $f(t-a)u(t-a)$.
- Note that throughout we multiply any arbitrary function by the unit step function to ensure that the resulting function is defined for all positive time.
- Translation in the time domain corresponds to multiplication by an exponential in the frequency domain. Thus:

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s), \qquad a > 0 \qquad (Eq.1)$$

Operational Laplace Transform

- For example, knowing that:

$$\mathcal{L}\{tu(t)\} = \frac{1}{s^2}$$

- Eq. 1 permits writing the Laplace transform of $(t-a)u(t-a)$ directly:

$$\mathcal{L}\{f(t-a)u(t-a)\} = \frac{e^{-as}}{s^2}$$

- The proof of Eq. 1 follows from the defining integral:

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_{0^-}^{\infty} u(t-a)f(t-a)\,e^{-st}dt$$

$$= \int_{a}^{\infty} f(t-a)e^{-st}dt \qquad (Eq.\,2)$$

Operational Laplace Transform

- In writing Eq. 2, we took advantage of $u(t-a)$ for $t > a$. Now, we change the variable of integration.
- Specifically, we let $x = t - a$. Then, $x = 0$ when $t = a$, $x = \infty$ when $t = \infty$ and $dx = dt$.
- Thus, we write the integral in Eq. 2 as:

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_0^\infty f(x)e^{-s(x+a)}dx$$

$$= e^{-sa}\int_0^\infty f(x)e^{-sx}dx$$

$$= e^{-as}F(s)$$

Operational Laplace Transform

Translation in the Frequency Domain:

- Translation in the frequency domain corresponds to multiplication by an exponential in the time domain:

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a) \qquad (Eq.\,1)$$

- which follows from the defining integral. The derivation of Eq. 1 is shown below.

$$\mathcal{L}\{e^{-at}f(t)\} = \int_{0^-}^{\infty} [e^{-at}f(t)]\, e^{-st}dt$$

$$= \int_{0^-}^{\infty} f(t)\, e^{-(s+a)t}dt$$

$$= F(s+a)$$

Operational Laplace Transform

Scale Changing:

- The scale-change property gives the relationship between and when the time variable is multiplied by a positive constant:

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \qquad a > 0 \qquad (Eq.1)$$

- The derivation of Eq. 1 is shown below.

$$\mathcal{L}\{f(at)\} = \int_{0^-}^{\infty} f(at)\, e^{-st}\, dt \qquad (Eq.2)$$

- The form of the equation requires integration by substitution to solve.

Operational Laplace Transform

- Let $u = at$ or $t = u/a$, $du = adt$ or $dt = du/a$, $u = 0^-$ when $t = 0^-$, and $u = \infty$ when $t = \infty$.

$$\int_{0^-}^{\infty} f(at)\, e^{-st} dt = \int_{0^-}^{\infty} f(u)\, e^{-s\left(\frac{u}{a}\right)} \frac{du}{a}$$

- Rearrange the equation

$$\int_{0^-}^{\infty} f(u)\, e^{-s\left(\frac{u}{a}\right)} \frac{du}{a} = \frac{1}{a} \int_{0^-}^{\infty} f(u)\, e^{-\left(\frac{s}{a}\right)u} du$$

- Since

$$\int_{0^-}^{\infty} f(u)\, e^{-\left(\frac{s}{a}\right)u} du = F\left(\frac{s}{a}\right)$$

Operational Laplace Transform

- Therefore, perform substitution of the variables and the above equation becomes:

$$\mathcal{L}\{f(at)\} = \int_{0^-}^{\infty} f(u)\, e^{-s\left(\frac{u}{a}\right)} \frac{du}{a}$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$

- The scale-change property is particularly useful in experimental work, especially where time-scale changes are made to facilitate building a model of a system.

Example of Operational Laplace Transform 1

With regard to operational transforms, attempt the following tasks:

a. Describe operational transforms. [2 marks]

b. Prove the following operational transforms:

i. Multiplication by a constant. [4 marks]

$$\mathcal{L}\{Kf(t)\} = KF(s)$$

ii. Addition (subtraction). [4 marks]

$$\mathcal{L}\{f_1(t) + f_2(t) + f_3(t)\} = F_1(s) + F_2(s) + F_3(s)$$

iii. Differentiation. [4 marks]

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$$

Example of Operational Laplace Transform 1

iv. Integration. [4 marks]

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)dx\right\} = \frac{F(s)}{s}$$

v. Translation in the time domain. [4 marks]

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-sa}F(s), \qquad a > 0$$

vi. Translation in the frequency domain. [4 marks]

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

vii. Scale changing. [4 marks]

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F(s/a), \qquad a > 0$$

Example of Operational Laplace Transform 2

Some of the properties (operational transforms) of Laplace transform could be used to solve problems in the Laplace transform of function.

a. Use the differentiation property of Laplace transform to prove that: [12 marks]

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2 F(s) - sf(0^-) - \frac{df(0^-)}{dt}$$

b. Use to translation in the frequency domain property of Laplace transform to prove that: [8 marks]

$$\mathcal{L}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2+\omega^2}$$

Example of Operational Laplace Transform 2

c. Use scale changing property of Laplace transform to prove that: [8 marks]

$$\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}$$

Inverse Laplace Transform

- In general, we need to find the inverse transform of a function that has the form

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}$$

- The coefficients a and b are real constants, and the exponents m and n are positive integers.
- The ratio $N(s)/D(s)$ is called a proper rational function if $m > n$ and an improper rational function if $m \leq n$.
- Only a proper rational function can be expanded as a sum of partial fractions.

Inverse Laplace Transform

- Converting the equation into its equation with the roots.

$$F(s) = \frac{(s^n + z_n)(s^n + z_{n-1}) \dots (s + \mathrm{z})}{(s^n + p_n)(s^n + p_{n-1}) \dots (s + p)}$$

- Applying partial fraction, we could expand the equation given above.

$$F(s) = \frac{K_n}{(s^n + p_n)} + \frac{K_{n-1}}{(s^n + p_{n-1})} + \cdots \frac{K_{nth}}{(s + p)}$$

- By referring to the table of Laplace transform pairs, the inverse Laplace transform could be found.
- In the end, we can determine the time-domain equation of the circuit ($f(t)$).

Partial Fraction Expansions

Partial fraction expansions:

- Partial fraction expansion helps to simplify complex forms of Laplace transform equation into several standard functions that we can find its inverse from the Laplace transform table pairs.
- Proper Rational Functions.
    - Distinct Real Roots of Denominator.
    - Distinct Complex Roots of Denominator.
    - Repeated Real Roots of Denominator.
    - Repeated Complex Roots of Denominator.
- Improper Rational Functions.

Partial Fraction Expansions

Proper Rational Functions:

- A proper rational function is expanded into a sum of partial fractions by writing a term or a series of terms for each root of $D(s)$.
- Thus, $D(s)$ must be in factored form before we can make a partial fraction expansion.
- For each distinct root of $D(s)$, a single term appears in the sum of partial fractions.
- For each multiple root of $D(s)$ of multiplicity r, the expansion contains r terms.

Partial Fraction Expansions

- For example, in the rational function:

$$\frac{s+z}{s(s+p_1)(s+p_2)^2}$$

- the denominator has four roots.
- Two of these roots are distinct—namely, at $s = 0$ and $s = -p_1$.
- A multiple root of multiplicity 2 occurs at $s = -p_2$.
- Thus, the partial fraction expansion of this function takes the form:

$$\frac{s+z}{s(s+p_1)(s+p_2)^2} \equiv \frac{K_1}{s} + \frac{K_2}{s+p_1} + \frac{K_3}{(s+p_2)^2} + \frac{K_4}{s+p_2} \qquad (Eq.\,1)$$

Partial Fraction Expansions

- The key to the partial fraction technique for finding inverse transforms lies in recognizing the corresponding to each term in the sum of partial fractions.
- From Laplace transform pair table, you should be able to verify that:

$$\mathcal{L}^{-1}\left\{\frac{s+z}{s(s+p_1)(s+p_2)^2}\right\}$$

$$= (K_1 + K_2 e^{-p_1 t} + K_3 t e^{-p_2 t} + K_4 e^{-p_2 t})u(t) \qquad (Eq.2)$$

- All that remains is to establish a technique for determining the coefficients (K_1, K_2, K_3,…) generated by making a partial fraction expansion.

Partial Fraction Expansions

- There are four general forms this problem can take.
- Specifically, the roots of $D(s)$ are either (1) real and distinct; (2) complex and distinct; (3) real and repeated; or (4) complex and repeated.
- Before we consider each situation in turn, a few general comments are in order.
- We used the identity sign ($\equiv$) in Eq. 1 to emphasize that expanding a rational function into a sum of partial fractions establishes an identical equation.
- Thus, both sides of the equation must be the same for all values of the variable s.

Partial Fraction Expansions

- Also, the identity relationship must hold when both sides are subjected to the same mathematical operation.
- These characteristics are pertinent to determining the coefficients, as we will see.
- Be sure to verify that the rational function is proper. This check is important because nothing in the procedure for finding the various K's will alert you to nonsense results if the rational function is improper.
- We present a procedure for checking the Ks, but you can avoid wasted effort by forming the habit of asking yourself, "Is a proper rational function?"

Partial Fraction Expansions

Distinct Real Roots of $D(s)$:

- We first consider determining the coefficients in a partial fraction expansion when all the roots of $D(s)$ are real and distinct.
- To find a K associated with a term that arises because of a distinct root of $D(s)$, we multiply both sides of the identity by a factor equal to the denominator beneath the desired K.
- Then when we evaluate both sides of the identity at the root corresponding to the multiplying factor, the right-hand side is always the desired K, and the left-hand side is always its numerical value.

Partial Fraction Expansions

For example,

$$F(s) = \frac{K(s+z_1)(s+z_2)}{s(s+p_1)(s+p_2)} = \frac{K_1}{s} + \frac{K_2}{s+p_1} + \frac{K_3}{s+p_2}$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{K(s+z_1)(s+z_2)}{s(s+p_1)(s+p_2)}\right\}$$
$$= (K_1 + K_2 e^{-p_1 t} - K_3 e^{-p_2 t})u(t)$$

Partial Fraction Expansions

Distinct Complex Roots of $\boldsymbol{D(s)}$:

- The only difference between finding the coefficients associated with distinct complex roots and finding those associated with distinct real roots is that the algebra in the former involves complex numbers.

Partial Fraction Expansions

For example,

$$F(s) = \frac{K(s+z)}{(s+p)(s^2+xs+y)}$$

$$= \frac{K_1}{s+p} + \frac{K_2\angle\theta}{(s+a-jb)} + \frac{K_3\angle\theta}{(s+a+jb)}$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{K(s+z)}{(s+p)(s^2+xs+y)}\right\}$$

$$= \left(K_1 e^{-pt} + K_2 e^{-j\theta} e^{-(a-jb)t} + K_3 e^{j\theta} e^{-(a+jb)t}\right)u(t)$$

Where:

$$x = 2a \qquad \text{and} \qquad y = \sqrt{a^2+b^2}$$

Partial Fraction Expansions

- Because distinct complex roots appear frequently in lumped-parameter linear circuit analysis, we need to summarize these results with a new transform pair.
- Whenever $D(s)$ contains distinct complex roots—that is, factors of the form $(s+\alpha-j\beta)(s+\alpha+j\beta)$ - a pair of terms of the form:

$$\frac{K}{(s+\alpha-j\beta)}+\frac{K^*}{(s+\alpha+j\beta)} \qquad (Eq.1)$$

- appears in the partial fraction expansion, where the partial fraction coefficient is, in general, a complex number.

Partial Fraction Expansions

- In polar form,

$$K = |K|e^{j0} = |K|\angle\theta^\circ \qquad (Eq.2)$$

- where $|K|$ denotes the magnitude of the complex coefficient. Then:

$$K^* = |K|e^{-j\theta} = |K|\angle -\theta^\circ \qquad (Eq.3)$$

- The complex conjugate pair in Eq. 1 always inverse-transforms as:

$$\mathcal{L}^{-1}\left\{\frac{K}{(s+\alpha-j\beta)} + \frac{K^*}{(s+\alpha+j\beta)}\right\}$$
$$= [2|K|e^{-\alpha t}\cos(\beta t+\theta)]u(t) \qquad (Eq.4)$$

Partial Fraction Expansions

- In applying Eq. 4 it is important to note that K is defined as the coefficient associated with the denominator term $s + \alpha - j\beta$ and K^* is defined as the coefficient associated with the denominator $s + \alpha + j\beta$.

Partial Fraction Expansions

Repeated Real Roots of $\boldsymbol{D(s)}$:

- To find the coefficients associated with the terms generated by a multiple root of multiplicity r, we multiply both sides of the identity by the multiple root raised to its rth power.
- We find the K appearing over the factor raised to the rth power by evaluating both sides of the identity at the multiple root.
- To find the remaining $(r-1)$ coefficients, we differentiate both sides of the identity $(r-1)$ times.
- At the end of each differentiation, we evaluate both sides of the identity at the multiple root.
- The right-hand side is always the desired K, and the left-hand side is always its numerical value.

Partial Fraction Expansions

For example,

$$F(s) = \frac{K(s+z)}{s(s+p)^3} = \frac{K_1}{s} + \frac{K_2}{(s+p)^3} + \frac{K_3}{(s+p)^2} + \frac{K_4}{s+p}$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{K(s+z)}{s(s+p)^3}\right\}$$

$$= (K_1 + K_2 t^2 e^{-pt} + K_3 t e^{-pt} + K_4 e^{-pt})u(t)$$

Partial Fraction Expansions

Repeated Complex Roots of $D(s)$:

- We handle repeated complex roots in the same way that we did repeated real roots.
- The only difference is that the algebra involves complex numbers.
- Recall that complex roots always appear in conjugate pairs and that the coefficients associated with a conjugate pair are also conjugates, so that only half the K's need to be evaluated.

Partial Fraction Expansions

For example:

$$F(s) = \frac{K}{(s+a-jb)^2(s+a+jb)^2}$$

$$= \frac{K_1}{(s+a-jb)^2} + \frac{K_2}{s+a-jb} + \frac{K_1^*}{(s+a+jb)^2} + \frac{K_2^*}{s+a+jb}$$

Thus, we now group the partial fraction expansion by conjugate terms to obtain:

$$F(s) = \frac{K_1}{(s+a-jb)^2} + \frac{K_2\angle\theta}{s+a-jb} + \frac{K_1^*}{(s+a+jb)^2} + \frac{K_2^*\angle\theta}{s+a+jb}$$

We now write the inverse transform of $F(s)$:

$$f(t) = [2K_1 te^{-at}\cos(bt) + 2K_2 e^{-at}\cos(bt-\theta)]u(t)$$

Partial Fraction Expansions

- Note that if $F(s)$ has a real root a of multiplicity r in its denominator, the term in a partial fraction expansion is of the form:

$$\frac{K}{(s+a)^r}$$

- The inverse transform of this term is:

$$\mathcal{L}^{-1}\left\{\frac{K}{(s+a)^r}\right\} = \left[\frac{Kt^{r-1}e^{-at}}{(r-1)!}\right]u(t) \qquad (Eq.\,1)$$

Partial Fraction Expansions

- If $F(s)$ has a complex root of $\alpha + j\beta$ of multiplicity r in its denominator, the term in partial fraction expansion is the conjugate pair:

$$\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}$$

- The inverse transform of this pair is:

$$\mathcal{L}^{-1}\left\{\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}\right\}$$

$$= \left[\frac{2|K|t^{r-1}}{(r-1)!}e^{-\alpha t}\cos(\beta t+\theta)\right]u(t) \qquad (Eq.2)$$

Partial Fraction Expansions

- Equations 1 and 2 are the key to being able to inverse-transform any partial fraction expansion by inspection.
- One further note regarding these two equations: In most circuit analysis problems, r is seldom greater than 2.
- Therefore, the inverse transform of a rational function can be handled with four transform pairs.

Partial Fraction Expansions

Improper Rational Functions:

- We conclude the discussion of partial fraction expansions by returning to an observation made at the beginning of this section, namely, that improper rational functions pose no serious problem in finding inverse transforms.
- An improper rational function can always be expanded into a polynomial plus a proper rational function.
- The polynomial is then inverse-transformed into impulse functions and derivatives of impulse functions.
- The proper rational function is inverse-transformed by the techniques outlined in this section.

Example of Partial Fraction Expansions

For a given inverse Laplace transform, attempt the following tasks:

a. Describe the role of partial fraction expansion in the inverse Laplace transform. [2 marks]

b. Describe and solve the following partial fraction expansions:

i. Proper Rational Functions. [20 marks]

$$D(s) = \frac{s+6}{s(s+3)(s+1)^2}$$

ii. Distinct Real Roots of $D(s)$. [30 marks]

$$D(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)}$$

Example of Partial Fraction Expansions

iii. Distinct Complex Roots of $D(s)$. [30 marks]

$$D(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$

iv. Repeated Real Roots of $D(s)$. [24 marks]

$$D(s) = \frac{100(s+25)}{s(s+5)^3}$$

v. Repeated Complex Roots of $D(s)$. [24 marks]

$$D(s) = \frac{768}{(s^2+6s+25)^2}$$

vi. Improper Rational Functions of $D(s)$. [10 marks]

$$D(s) = \frac{s^4+13s^3+66s^2+200s+300}{s^2+9s+20}$$

Poles and Zeros

- Given an inverse Laplace transform function that has the form:

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0} \qquad (Eq. 1)$$

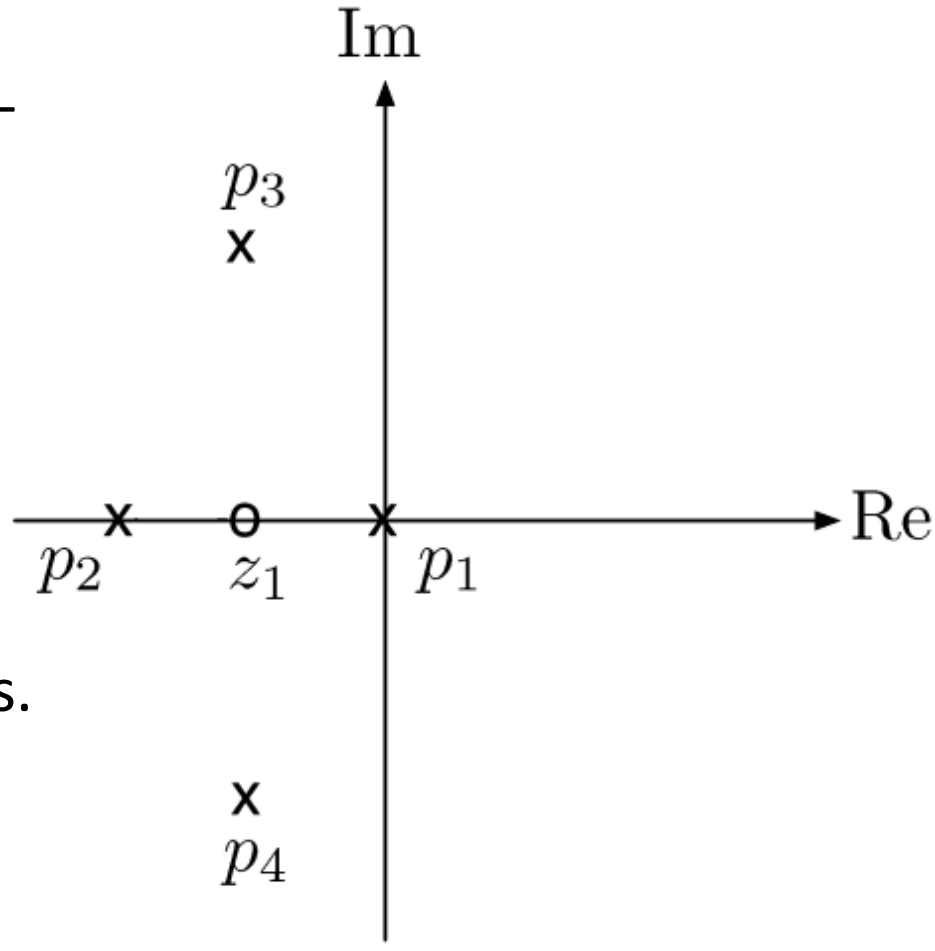
- The rational function of the equation above also may be expressed as the ratio of two factored polynomials.
- In other words, we may write $F(s)$as:

$$F(s) = \frac{K(s + z_1)(s + z_2) \ldots (s + z_n)}{(s + p_1)(s + p_2) \ldots (s + p_m)} \qquad (Eq. 2)$$

- Where K is the constant a_n / b_m.

Poles and Zeros

Poles and zeros locations in the s-plane diagram:



Poles:

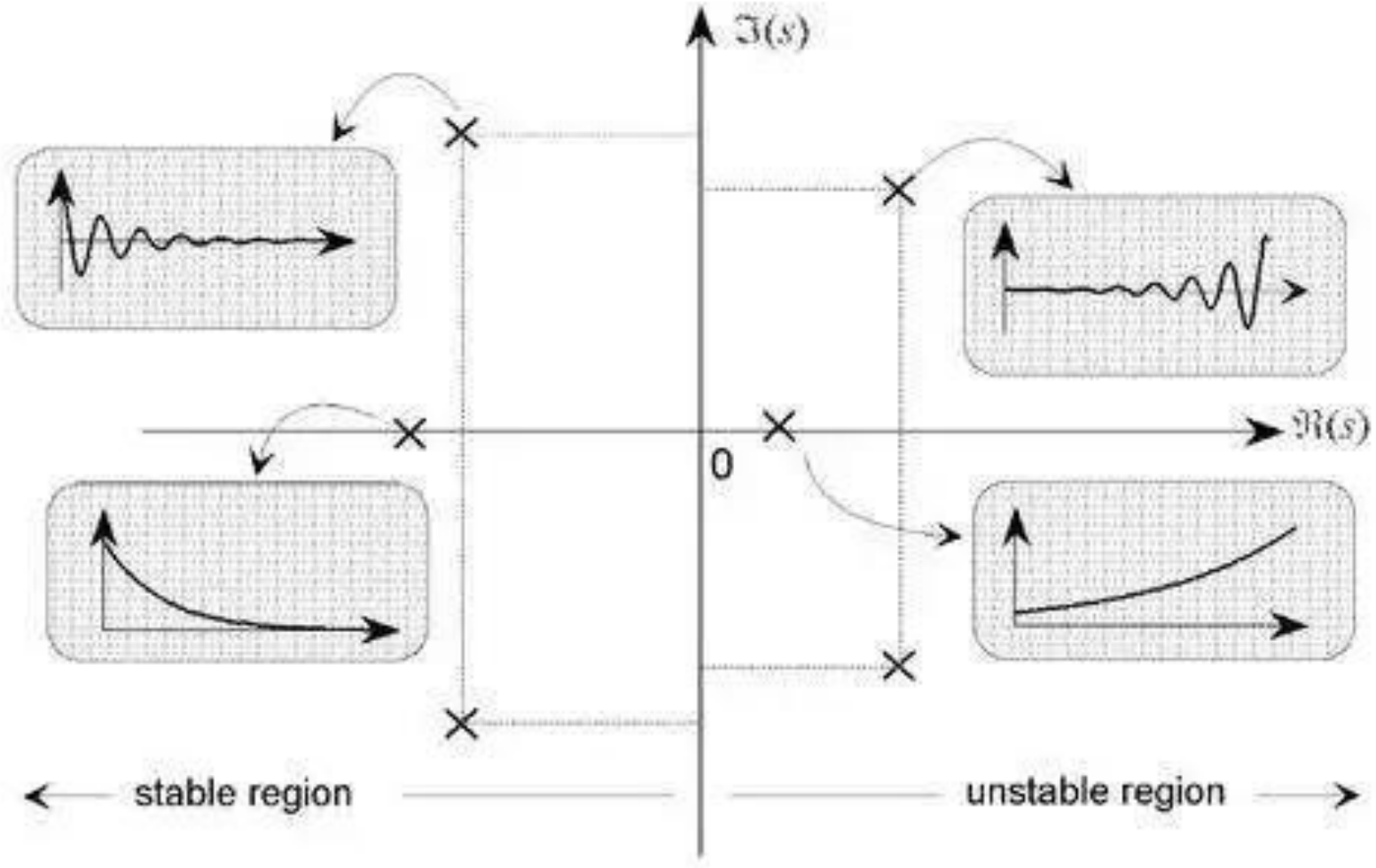
- P_1 – pole at origin.
- P_2 – simple (real) pole.
- P_3 and P_4 – complex pair poles.

Zeros:

- Z_1 – simple (real) zero.

Poles and Zeros

- Location of poles and zeros in the s-diagram and their relevant transient responses.



Poles and Zeros

- For example, we may also write the function:

$$F(s) = \frac{8s^2 + 120s + 400}{2s^4 + 20s^3 + 70s^2 + 100s + 48} \qquad (Eq.1)$$

- as

$$F(s) = \frac{8(s^2 + 15s + 50)}{2(s^4 + 10s^3 + 35s^2 + 50s + 24)}$$

$$= \frac{4(s + 5)(s + 10)}{(s + 1)(s + 2)(s + 3)(s + 4)} \qquad (Eq.2)$$

Poles and Zeros

- The roots of the denominator polynomial, that is, $-p_1$, $-p_2$, $-p_3$, …, $-p_m$ are called the poles of $F(s)$.

  They are the values of s at which $F(s)$ becomes infinitely large.

  In the function described by equation 2, the poles of $F(s)$ are -1, -2, -3, and -4.

- The roots of the numerator polynomial, that is, $-z_1$, $-z_2$, $-z_3$, …, $-z_n$ are called the zeros of $F(s)$.

  They are the values of s at which $F(s)$ becomes zero.

  In the function described by the equation 2, the zeros of are -5 and -10.

Poles and Zeros

- In what follows, you may find that being able to visualize the poles and zeros of $F(s)$ as points on a complex s-plane is helpful.
- A complex plane is needed because the roots of the polynomials may be complex.
- In the complex s plane, we use the horizontal axis to plot the real values of s and the vertical axis to plot the imaginary values of s.

Poles and Zeros

- Note that the poles and zeros for equation given below are located in the finite s-plane.
- $F(s)$ can also have either an rth-order pole or an rth-order zero at infinity.

$$F(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_n)}{(s+p_1)(s+p_2)\ldots(s+p_m)}$$

- For example, the function described by equation below has a second-order zero at infinity, because for large values of s the function reduces to $4/s^2$ and $F(s) = 0$ when $s = \infty$.

$$F(s) = \left.\frac{4(s+5)(s+10)}{(s+1)(s+2)(s+3)(s+4)}\right| \; s \to \infty \cong \frac{4}{s^2}$$

Poles and Zeros

- In this case, we are interested in the poles and zeros located in the finite s-plane.
- Therefore, when we refer to the poles and zeros of a rational function of s, we are referring to the finite poles and zeros.

Example of Poles and Zeros

For given inverse Laplace transform, attempt the following tasks:

a. Describe the poles and zeros. [2 marks]

b. Determine the poles and zeros of the following system. [6 marks]

$$F(s) = \frac{8s^2 + 120s + 400}{2s^4 + 20s^3 + 70s^2 + 100s + 48}$$

c. For plotting the poles and zeros, consider the function: [4 marks]

$$F(s) = \frac{10(s+5)(s+3-j4)(s+3+j4)}{s(s+10)(s+6-j8)(s+6+j8)}$$

Initial-Value and Final-Value Theorems

- The initial- and final-value theorems are useful because they enable us to determine from $F(s)$ the behaviour of $f(t)$ at 0 and ∞.
- Hence, we can check the initial and final values of $f(t)$ to see if they conform with known circuit behaviour, before actually finding the inverse transform of $F(s)$.

Initial-Value and Final-Value Theorems

Initial-Value Theorem:

- The initial-value theorem states that:

$$\lim_{t\to 0} f(t) = \lim_{s\to \infty} sF(s) \qquad (Eq.A)$$

- The initial-value theorem is based on the assumption that $f(t)$ contains no impulse functions.
- In the equation for the final value theorem, we must add the restriction that the theorem is valid only if the poles of $F(s)$, except for a first-order pole at the origin, lie in the left half of the s-plane.

Initial-Value and Final-Value Theorems

- To prove the equation for the initial value theorem, we start with the operational transform of the first derivative:

$$\mathcal{L}^{-1}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-) = \int_0^\infty \left(\frac{df}{dt}\right) e^{-st} dt \qquad (Eq.1)$$

- Now we take the limit as $s \to \infty$:

$$\lim_{s\to\infty}[sF(s) - f(0^-)] = \lim_{s\to\infty} \int_0^\infty \left(\frac{df}{dt}\right) e^{-st} dt \qquad (Eq.2)$$

- Observe that the right-hand side of the Eq. 2 given above may be written as:

$$\lim_{s\to\infty} \left(\int_{0^-}^{0^+} \left(\frac{df}{dt}\right) e^{0} dt + \int_{0^+}^{\infty} \left(\frac{df}{dt}\right) e^{-st} dt \right)$$

Initial-Value and Final-Value Theorems

- As $s \to \infty$, $(df/dt)e^{-st} \to 0$; hence the second integral vanishes in the limit.
- The first integral reduces to $f(0^+) - f(0^-)$ which is independent of s.
- Thus, the right-hand side of Eq. 2 becomes:

$$\lim_{s\to\infty} \int_{0^-}^{0^+} \left(\frac{df}{dt}\right) e^{-st} dt = f(0^+) - f(0^-) \qquad (Eq. 3)$$

- Because $f(0^-)$ is independent of s, the left-hand side of Eq. 2 may be written:

$$\lim_{s\to\infty} [sF(s) - f(0^-)] = \lim_{s\to\infty} [sF(s)] - f(0^-) \qquad (Eq. 4)$$

Initial-Value and Final-Value Theorems

- From Eqs. 3 and 4,

$$\lim_{s\to\infty}[sF(s)] = f(0^+) = \lim_{t\to 0^-} f(t) \qquad (Eq.5)$$

- which completes the proof of the initial-value theorem.

Initial-Value and Final-Value Theorems

Final-Value Theorem:

- The final-value theorem states that:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \qquad (Eq.B)$$

- The proof of the final-value theorem also starts with the operational transform of the first derivative.

$$\mathcal{L}^{-1}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-) = \int_0^\infty \left(\frac{df}{dt}\right) e^{-st} dt \qquad (Eq.1)$$

- Here, we take the limit as $s \to 0$:

$$\lim_{s\to 0}[sF(s) - f(0^-)] = \lim_{s\to 0}\left(\int_{0^-}^\infty \left(\frac{df}{dt}\right) e^{-st}\, dt\right) \qquad (Eq.2)$$

Initial-Value and Final-Value Theorems

- The integration is with respect to t and the limit operation is with respect to s, so the right-hand side of Eq. 2 reduces to:

$$\lim_{s\to 0}\left(\int_{0-}^{\infty}\left(\frac{df}{dt}\right)e^{-st}\,dt\right)=\int_{0-}^{\infty}\left(\frac{df}{dt}\right)dt \qquad (Eq.3)$$

- Because the upper limit on the integral is infinite, this integral may also be written as a limit process:

$$\int_{0-}^{\infty}\left(\frac{df}{dt}\right)dt=\lim_{t\to\infty}\int_{0-}^{t}\left(\frac{df}{dy}\right)dy \qquad (Eq.4)$$

- where we use y as the symbol of integration to avoid confusion with the upper limit on the integral.

Initial-Value and Final-Value Theorems

- Carrying out the integration process yields proof of derivation:

$$\lim_{t\to\infty}[f(t)-f(0^-)]=\lim_{t\to\infty}|f(t)|-f\,(0^-) \qquad (Eq.5)$$

- Substituting Eq. 5 into Eq. 2 gives:

$$\lim_{s\to 0}[sF(s)]-f(0^-)=\lim_{t\to\infty}|f(t)|-f\,(0^-) \qquad (Eq.6)$$

- Because $f(0^-)$ cancels, Eq. 6 reduces to the final-value theorem, namely,

$$\lim_{s\to 0}sF(s)=\lim_{t\to\infty}f(t)$$

- The final-value theorem is useful only if $f(\infty)$ exists. This condition is true only if all the poles of $F(s)$, except for a simple pole at the origin, lie in the left half of the s-plane.

Example of Initial-Value and Final-Value Theorems

For a given Laplace transform, attempt the following tasks.

a. Why are initial and final values theorems useful? [2 marks]

b. Describe and prove initial-value theorem. [6 marks]

c. Describe and prove final-value theorem. [6 marks]

Example of Initial-Value and Final-Value Theorems

Apply the initial and final value theorems to the following transform pair.

a. Transform pair 1. [6 marks]

$$\mathcal{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\}$$

$$= (-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13°))u(t)$$

b. Transform pair 2. [6 marks]

$$V(s) = \frac{\left(\frac{I_{dc}}{C}\right)}{s^2 + \left(\frac{1}{RC}\right)s + \left(\frac{1}{LC}\right)}$$

Appendix

$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$	**Operation**
$Kf(t)$	$KF(s)$	Multiplication by a constant
$f_1(t) + f_2(t) + f_3(t)$	$F_1(s) + F_2(s) + F_3(s)$	Addition/subtraction
$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$	First derivative (time)
$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$	Second derivative (time)
$\frac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - s^{n-3}\frac{d^2f(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$	nth derivative (time)
$\int_0^t f(x)\,dx$	$\frac{F(s)}{s}$	Time integral

Appendix

$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$	**Operation**
$f(t-a)u(t-a), \qquad a>0$	$e^{-as}F(s)$	Translation in time
$f(at), \qquad a>0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	Scale changing
$tf(t)$	$-\frac{dF(s)}{ds}$	First derivative (s)
$t^n f(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$	nth derivative (s)
$\frac{f(t)}{t}$	$\int_s^\infty F(u)\,du$	s integral

Appendix

#Pair	$F(s)$	$f(t)$	Description
1	$\frac{K}{s+a}$	$Ke^{-at}u(t)$	Distinct real
2	$\frac{K}{(s+a)^2}$	$Kte^{-at}u(t)$	Repeated real
3	$\frac{K}{s+\alpha-j\beta}+\frac{K^*}{s+\alpha+j\beta}$	$2\lvert K\rvert e^{-at}\cos(\beta t+\theta)\,u(t)$	Distinct complex
4	$\frac{K}{(s+\alpha-j\beta)^2}+\frac{K^*}{(s+\alpha+j\beta)^2}$	$2t\lvert K\rvert e^{-at}\cos(\beta t+\theta)\,u(t)$	Repeated complex