XMUT203 Analogue Circuits and Systems

Note 6b: Further Second-Order Circuits

Topics:

- Forms of response of RLC circuit.
- Overdamped response of RLC circuit.
- Underdamped response of RLC circuit.
- Critically damped response of RLC circuit.
- Resonance characteristics in RLC circuit.
- Transfer function, cut-off frequency, bandwidth, and Q-factor of series RLC circuit.
- Transfer function, cut-off frequency, bandwidth, and Q-factor of parallel RLC circuit.

1. Forms of Natural Response of RLC Circuit

Behavior of a second-order RLC circuit depends on the values of $j\omega_1$ and $j\omega_2$ and hence the circuit parameters R, L, and C. Therefore, the first step is to calculate these values and, relatedly, determine whether the response is over-, under-, or critically damped.

We match the solution for the response to the initial conditions imposed by the circuit, which are the initial value of the current (or voltage) and the initial value of the first derivative of the current (or voltage). Note that these same initial conditions, plus the final value of the variable, will also be needed when finding the step response of a second-order circuit.

We would investigate overdamped, underdamped and critical damped response of the (parallel) RLC circuit in more detail.

1.2. Overdamped Response of Parallel RLC Circuit

When the roots of the characteristic equation are real and distinct (also $\omega_0^2 < \alpha^2$), the voltage response of a parallel RLC circuit is said to be overdamped.

$$v = A_1 e^{j\omega_1 t} + A_2 e^{j\omega_2 t} \qquad (Eq. 1)$$

Where: $j\omega_1$ and $j\omega_2$ are the roots of the characteristic equation.

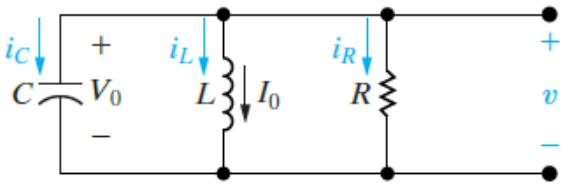


Figure 1: Natural response of parallel RLC circuit

The constants A_1 and A_2 are determined by the initial conditions, specifically from the values of $v(0^+)$ and $dv(0^+)/dt$ which in turn are determined from the initial voltage on the capacitor, V_0 and the initial current in the inductor, I_0.

Knowing the first and second derivatives of the voltage are:

$$v(0^+) = A_1 + A_2 \qquad (Eq.2)$$

Thus

$$\frac{dv(0^+)}{dt} = j\omega_1 A_1 + j\omega_2 A_2 \qquad (Eq.3)$$

With $j\omega_1$ and $j\omega_2$ known, the task of finding A_1 and A_2 reduces to finding $v(0^+)$ and $dv(0^+)/dt$. The value of $v(0^+)$ is the initial voltage on the capacitor V_0. We get the initial value of dv/dt by first finding the current in the capacitor branch at $t = 0^+$. Then,

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} \qquad (Eq.4)$$

For the given parallel RLC circuit, we use Kirchhoff's current law to find the initial current in the capacitor branch. The sum of the three branch currents at $t = 0^+$ must be zero.

$$i_C(0^-) + i_L(0^+) + i_R(0^+) = 0 \qquad (Eq.5)$$

The current in the resistive branch at $t = 0^+$ is the initial voltage V_0 divided by the resistance, and the current in the inductive branch is I_0.

$$i_C(0^+) = \frac{-V_0}{R} - I_0 \qquad (Eq.6)$$

After finding the numerical value of $i_C(0^+)$, we use equation above to find the initial value of dv/dt:

We can summarize the process for finding the overdamped response $v(t)$, as follows:

1. Find the roots of the characteristic equation, and using the values of R, L, and C.
2. Find $v(0^+)$ and $dv(0^+)/dt$ using circuit analysis.
3. Find the values of A_1 and A_2 by solving the following equations simultaneously:

$$v(0^+) = A_1 + A_2 \qquad (Eq.7)$$

Thus

$$\frac{dv(0^+)}{dt} = j\omega_1 A_1 + j\omega_2 A_2 \qquad (Eq.8)$$

4. Substitute the values for $j\omega_1$, $j\omega_2$, A_1 and A_2 into the equation below to determine the expression for $v(t)$ for $t \geq 0$.

$$v = A_1 e^{j\omega_1 t} + A_2 e^{j\omega_2 t} \qquad (Eq. 9)$$

Example 1: Overdamped Parallel RLC

For circuit in the figure below, $v(0^+)$ = 12 V, and $i_L(0^+)$ = 30 mA.

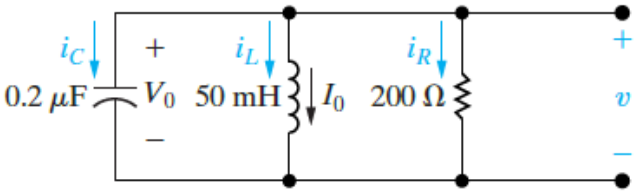
a. Find the initial current in each branch of the circuit. [6 marks]

b. Find the initial value of $dv(t)/dt$. [2 marks]

c. Find the expression for $v(t)$. [12 marks]

d. Sketch $v(t)$ in the interval $0 \leq t \leq 250$ ms. [4 marks]

e. Derive the expressions that describe the three branch currents i_R, i_L, and i_C during the time the stored energy is being released. [8 marks]



Solution

a. The inductor prevents an instantaneous change in its current, so the initial value of the inductor current is 30 mA:

$$i_L(0^-) = i_L(0) = i_L(0^+) = 30 \text{ mA}$$

The capacitor holds the initial voltage across the parallel elements to 12 V. Thus, the initial current in the resistive branch, $i_R(0^+)$ is:

$$i_R(0^+) = \frac{v}{R} = \frac{12}{200} = 60 \text{ mA}$$

Kirchhoff's current law requires the sum of the currents leaving the top node to equal zero at every instant. Hence

$$i_C(0^-) = -i_L(0^+) - i_R(0^+) = -30 \text{ mA} - 60 \text{ mA} = -90 \text{ mA}$$

Note that if we assumed the inductor current and capacitor voltage had reached their dc values at the instant that energy begins to be released, $i_C(0^-)$.

In other words, there is an instantaneous change in the capacitor current at t = 0.

b. Because $i_C = C(dv(t)/dt)$, thus:

$$\frac{dv(0^+)}{dt} = \frac{i_C}{C} = -\frac{90 \times 10^{-3}}{0.2 \times 10^{-6}} = -450 \text{ kV/s}$$

c. The roots of the characteristic equation come from the values of R, L, and C.

$$j\omega_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

And

$$j\omega_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Where:

$$\alpha = \frac{1}{2RC} \qquad \text{and} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

For the values specified and from the equations above, the roots are:

$$j\omega_1 = -1.25 \times 10^4 + \sqrt{1.5625 \times 10^8 - 10^8}$$
$$= -12{,}500 + 7500 = -5000 \text{ rad/s}$$

And

$$j\omega_2 = -1.25 \times 10^4 - \sqrt{1.5625 \times 10^8 - 10^8}$$
$$= -12{,}500 - 7500 = -20{,}000 \text{ rad/s}$$

Because the roots are real and distinct, we know that the response is overdamped and hence has the form of the equation below.

$$v = A_1 e^{j\omega_1 t} + A_2 e^{j\omega_2 t}$$

We find the coefficients and from the following equations.

$$v(0^+) = A_1 + A_2$$

We have already determined and so:

$$12 = A_1 + A_2 \qquad (Eq.1)$$

And

$$\frac{dv(0^+)}{dt} = j\omega_1 A_1 + j\omega_2 A_2$$

Thus

$$-450 \times 10^3 = -5000A_1 - 20{,}000A_2 \qquad (Eq.2)$$

We solve two equations (1) and (2) for A_1 and A_2 to obtain A_1 = -14 V and A_2 = 26 V.

$$v(t) = A_1 e^{j\omega_1 t} + A_2 e^{j\omega_2 t}$$

Substituting these values into the equation above yields the overdamped voltage response:

$$v(t) = (-14e^{-5000t} + 26e^{-20{,}000t}) \text{ V}, \quad t \geq 0$$

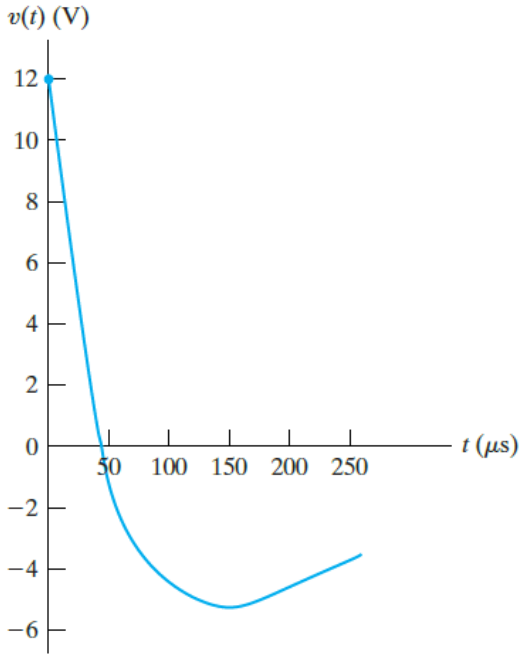
As a check on these calculations, we note that the solution yields:

$$v(0) = -14e^0 + 26e^0 = 12 \text{ V}$$

And

$$\frac{dv(0^+)}{dt} = (-14)(5000)e^0 + (26)(-20{,}000)e^0 = -450{,}000 \text{ V/s}$$

d. Figure below shows a plot of versus t over the interval $0 \leq t \leq 250$ ms.



e. We know the voltage across the three branches from the solution in parts (a)-(d), namely,

$$v(t) = (-14e^{-5000t} + 26e^{-20{,}000t}) \text{ V}, \qquad t \geq 0$$

The current in the resistive branch is then

$$i_R(t) = \frac{v(t)}{200} = (-70e^{-5000t} + 130e^{-20{,}000t}) \text{ mA}, \qquad t \geq 0$$

There are two ways to find the current in the inductive branch.

First is to use the integral relationship that exists between the current and the voltage at the terminals of an inductor:

$$i_L(t) = \frac{1}{L}\int_0^t v_L(x)\,dx + I_0$$

A second approach is to find the current in the capacitive branch first and then use the fact that $i_R + i_L + i_C = 0$. Let's use this approach. The current in the capacitive branch is:

$$\begin{aligned} i_C(t) = C\frac{dv(t)}{dt} &= 0.2 \times 10^{-6}(70{,}000e^{-5000t} - 520{,}000e^{-20{,}000t}) \\ &= (14e^{-5000t} - 104e^{-20{,}000t}) \text{ mA}, \qquad t \geq 0^+ \end{aligned}$$

Note that $i_C(t)$ = -90 mA which agrees with the result in previous question. Now, we obtain the inductive branch current from the relationship:

$$i_L(t) = -i_R(t) - i_C(t)$$

$$= (-70e^{-5000t} + 130e^{-20{,}000t})\ \text{mA} - (14e^{-5000t} - 104e^{-20{,}000t})\ \text{mA}$$

$$= (56e^{-5000t} - 26e^{-20{,}000t})\ \text{mA}, \qquad t \geq 0$$

Note that the expression for i_L agrees with the initial inductor current, as it must.

2.1.3. Underdamped Response of Parallel RLC

When $\omega_0^2 > \alpha^2$, the roots of the characteristic equation are complex, and the response is underdamped. For convenience, we express the roots $j\omega_1$ and $j\omega_2$ as:

$$j\omega_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)}$$

$$= -\alpha + j\sqrt{\omega_0^2 - \alpha^2}$$

$$= -\alpha + j\omega_d \qquad (Eq.10a)$$

And

$$j\omega_2 = -\alpha - j\omega_d \qquad (Eq.10b)$$

Where:

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \qquad (Eq.11)$$

The ω_d term is called the damped radian frequency. The underdamped voltage response of a parallel RLC circuit is:

$$v(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t \qquad (Eq.12)$$

Which follows from:

$$v = A_1 e^{j\omega_1 t} + A_2 e^{j\omega_2 t} \qquad (Eq.13)$$

Use the Euler identity:

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta \qquad (Eq.14)$$

For the given underdamped response equation of the parallel RLC circuit:

$$v(t) = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t} \qquad (Eq.15)$$

Thus

$$v(t) = A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha} e^{-j\omega_d t}$$

$$= e^{-\alpha t}(A_1\cos\omega_d t + jA_1\sin\omega_d t + A_2\cos\omega_d t - jA_2\sin\omega_d t)$$

$$= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t] \qquad (Eq.16)$$

Or

$$v(t) = e^{-\alpha t}(B_1\cos\omega_d t + B_2\sin\omega_d t)$$

$$= B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t \qquad (Eq.17)$$

With:

$$B_1 = A_1 + A_2 \qquad \text{and} \qquad B_2 = j(A_1 - A_2) \qquad (Eq.18)$$

As with $j\omega_1$ and $j\omega_2$, α and ω_d are fixed by the circuit parameters R, L, and C, for the underdamped response, the two simultaneous equations that determine B_1 and B_1 are:

$$v(0^+) = V_o = B_1 \qquad (Eq.19)$$

And

$$\frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = -\alpha B_1 + \omega_d B_2 \qquad (Eq.20)$$

Example 2: Underdamped Parallel RLC

In circuit shown in the figure below, $V_0 = 0$ and I_0 = -12.25 mA.

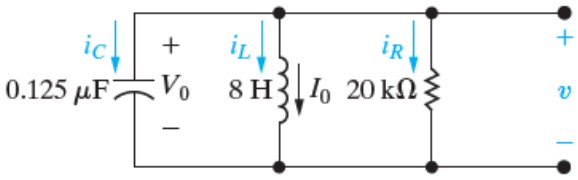
a. Calculate the roots of the characteristic equation. [8 marks]

b. Calculate v and $dv(t)/dt$ at t = 0^+. [6 marks]

c. Calculate the voltage response for $t \geq 0$. [10 marks]

d. If R were increased to infinity, find the solution for $v(t)$. [6 marks]

e. Plot $v(t)$ versus t for the time interval $0 \leq t \leq 11$ ms. [4 marks]



Solution

a. Because the values of all components are provided, the neper and resonant frequencies are:

$$\alpha = \frac{1}{2RC} = \frac{10^6}{2(20)10^3(0.125)} = 200 \text{ rad/s}$$

And

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{\frac{10^6}{(8)(0.125)}} = 10^3 \text{ rad/s}$$

we have $\omega_0^2 > \alpha^2$. Therefore, the response is underdamped. Now, the damped frequency is:

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{10^6 - 4 \times 10^4} = 100\sqrt{96} = 979.80 \text{ rad/s}$$

Thus, the roots of the equation are:

$$j\omega_1 = -\alpha + j\omega_d = -200 + j979.80 \text{ rad/s}$$

And

$$j\omega_2 = -\alpha - j\omega_d = -200 - j979.80 \text{ rad/s}$$

For the underdamped case, we do not ordinarily solve for s_1 and s_2 because we do not use them explicitly. However, this example emphasizes why $j\omega_1$ and $j\omega_2$ are known as complex frequencies.

b. Because v is the voltage across the terminals of a capacitor, we have:

$$v(0) = v(0^+) = V_0 = 0$$

Because $v(0^+) = 0$, the current in the resistive branch is zero at $t = 0^+$. Hence, the current in the capacitor at $t = 0^+$ is the negative of the inductor current:

$$i_C(0^+) = -(-12.25) = 12.25 \text{ mA}$$

Therefore, the initial value of the derivative is:

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{(12.25)(10^{-3})}{(0.125)(10^{-6})} = 98{,}000 \text{ V/s}$$

c. From the equations given below, the initial voltage of the circuit is:

$$v(0^+) = V_o = B_1$$

And

$$\frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = -\alpha B_1 + \omega_d B_2$$

Values of B_1 and B_2 are determined as follows:

$$B_1 = 0$$

And

$$B_2 = \frac{98{,}000}{\omega_d} \approx 100 \text{ V}$$

Substituting the numerical values of α, ω_d, B_1 and B_2 into the expression for $v(t)$ gives:

$$v(t) = 100e^{-200t} \sin 979.80t \text{ V}, \qquad t \geq 0$$

d. If R were increased to infinity, since $\alpha = 1/2RC$, thus the neper frequency of the circuit, α = 0.

Thus, the damped frequency becomes:

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{10^6 - 0} = 1000 \text{ rad/s}$$

The value of B_1 is still 0 and the value of B_2 is now:

$$B_2 = \frac{\left[\frac{dv(0^+)}{dt}\right]}{\omega_d} = \frac{98{,}000}{1000} \approx 98 \text{ V}$$

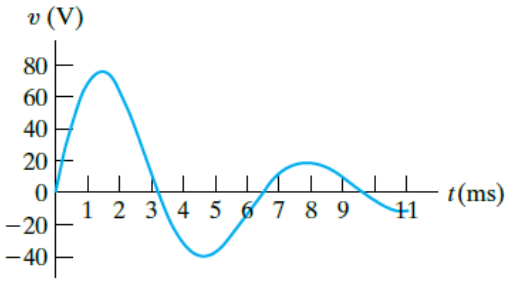
The solution for $v(t)$ would become:

$$v(t) = 98 \sin 1000t \ V, \quad t \geq 0$$

As a result, in this case, the oscillation is sustained, the maximum amplitude of the voltage is 98 V, and the frequency of oscillation is 1000 rad/s.

e. Figure below shows the plot of $v(t)$ versus t for the first 11 ms after the stored energy is released. It clearly indicates the damped-oscillatory nature of the underdamped response.

The voltage $v(t)$ approaches its final value, alternating between values that are greater than and less than the final value. Furthermore, these swings about the final value decrease exponentially with time.



2.1.5. Critically Damped Response of Parallel RLC

The second-order parallel RLC circuit is critically damped when $\omega_0^2 = \alpha^2$, or $\omega_0 = \alpha$. When a circuit is critically damped, the response is on the verge of oscillating. In addition, the two roots of the characteristic equation are real and equal; that is,

$$j\omega_1 = j\omega_2 = -\alpha = -\frac{1}{2RC} \qquad (Eq. 21)$$

When this occurs, the solution for the voltage no longer takes the form of equation:

$$v = A_1 e^{j\omega_1 t} + A_2 e^{j\omega_2 t} \qquad (Eq. 22)$$

This equation breaks down because if $j\omega_1 = j\omega_2 = -\alpha$, it predicts that:

$$v = (A_1 + A_2)e^{-\alpha t} = A_0 e^{-\alpha t} \qquad (Eq. 23)$$

Where: A_0 is an arbitrary constant.

When the roots of the characteristic equation are equal, the solution for the differential equation takes a different form, namely:

$$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t} \qquad (Eq. 24)$$

As a result, the two simultaneous equations needed to determine D_1 and D_2 are:

$$v(0^+) = V_0 = D_2 \qquad (Eq. 25)$$

And

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = D_1 - \alpha D_2 \qquad (Eq.26)$$

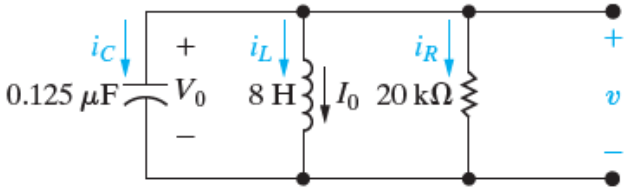
Example 3: Critically Damped Parallel RLC

For the circuit in previous question, perform the following tasks.

a. Find the value of R that results in a critically damped voltage response. [2 marks]

b. Calculate $v(t)$ for $t \geq 0$. [8 marks]

c. Plot $v(t)$ versus t for $0 \leq t \leq 7$ ms. [4 marks]



Solution

a. From previous question, we know that $\omega_0^2 = 10^6$. Therefore, for critical damping, the neper frequency is:

$$\alpha = 10^3 = \frac{1}{2RC}$$

Rearrange the equation above:

$$R = \frac{1}{2(10^3)(0.125 \times 10^{-6})} = 4000\ \Omega$$

b. From the solution of previous question, we know that $v(0^+) = 0$ and $dv(0^+)/dt = 98{,}000$ V/s.

$$v(0^+) = V_0 = D_2$$

And

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = D_1 - \alpha D_2$$

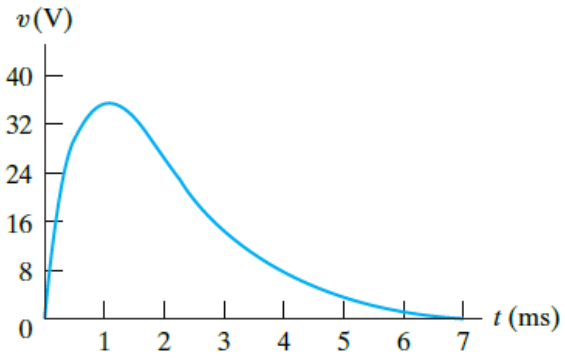
From the equations given above, $D_2 = 0$ and $D_1 = 98{,}000$ V/s.

$$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

Substituting these values for a, D_1, and D_2 into the equation above gives:

$$v(t) = 98{,}000 t e^{-1000t}\ V, \quad t \geq 0$$

c. Figure below shows a plot of $v(t)$ versus t in the interval $0 \leq t \leq 7$ ms.



2. Resonance Characteristics of Second-Order Circuits

For the characteristics of the second-order circuits, we will cover about its resonant properties, circuit characteristics and parameters, and steady-state analysis in this section.

2.1. Resonance in Second-Order Circuit

Resonance in circuit is when voltage in phase with resulting current is applied to an RLC circuit network. At resonance, the equivalent network impedance is purely resistive since supply voltage and current in phase.

As illustrated in the diagram given below, below resonant frequency, the impedance of the circuit is dominated by capacitive reactive and the current in the circuit leads voltage. On the other hand, above resonant frequency, inductive reactance dominates the circuit and the current in the circuit lags the voltage.

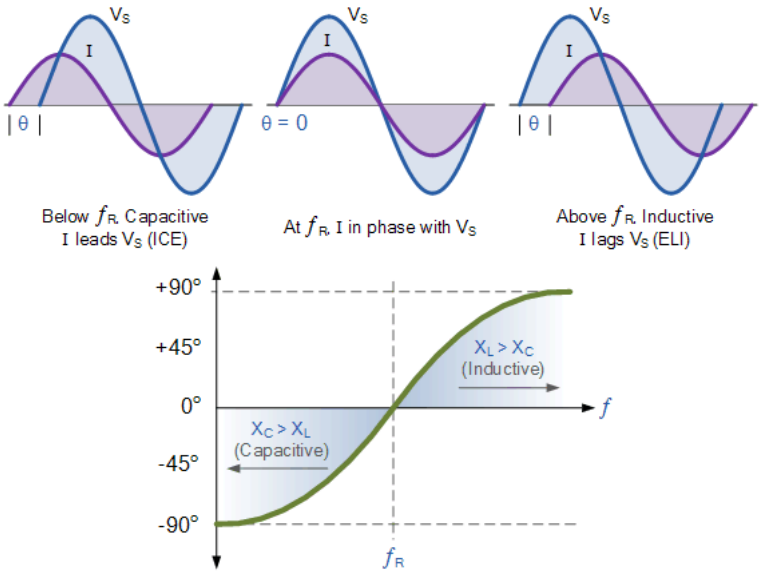


Figure 3: Resonant in Second-Order Circuit

The power factor (pf) of resonant network is unity (e.g. $\text{pf} = \cos\phi = \cos 0 = 1$). This means that at resonant the impedance of the second-order RLC circuit is dominated by the resistance, with the capacitive and inductive reactance canceled off each other as illustrated in the figure below.

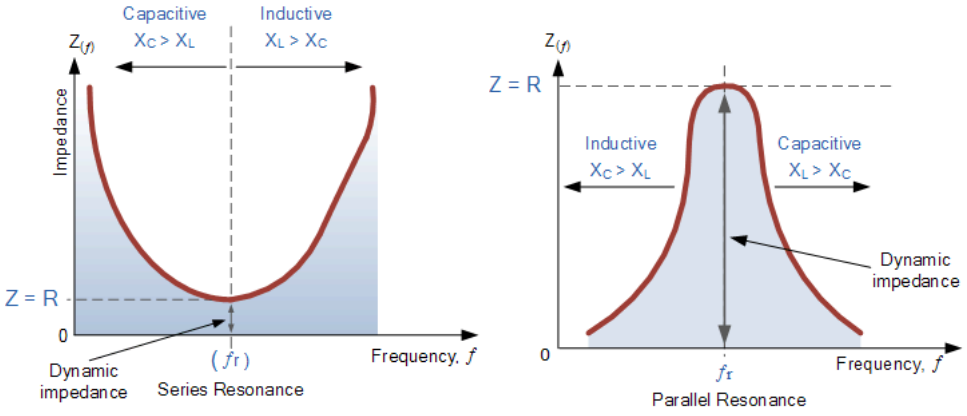


Figure 4: Resistance and reactance in resonant second-order circuit

2.2. Types of Resonant Second-Order Circuit

Two types of resonant circuit e.g. series (minimum input impedance) and parallel (maximum input impedance) as shown in the figure below.

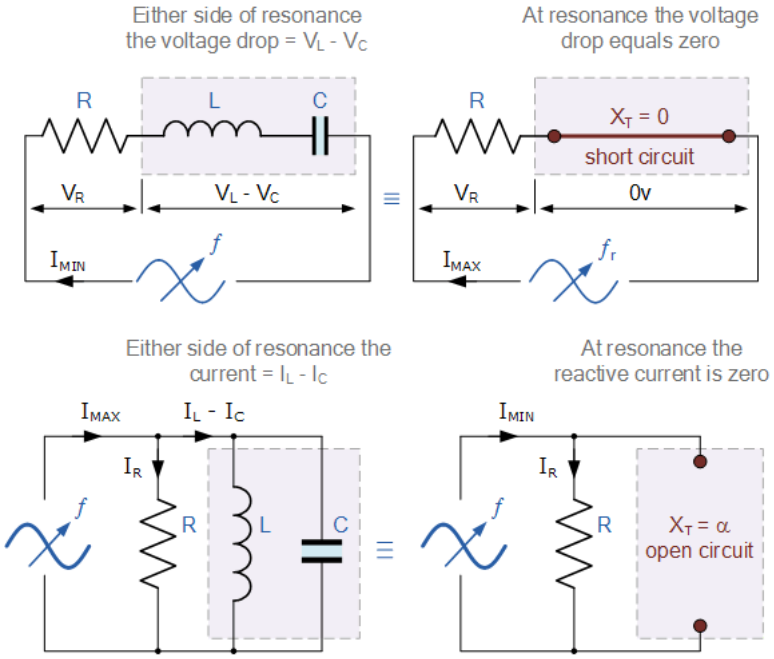


Figure 5: Series (top) and parallel (bottom) resonant second-order circuits

2.3. Applications of Resonant Second-Order Circuit

There are many applications of resonant second-order circuit. Two of main uses of resonant second-order circuit are generation of various signals at specific frequency (oscillation) and sorting out of signals e.g. selective block and pass of these signals at given operating frequency (filtering).

For this reason, the resonance in the second-order circuit is a great benefit for areas dealing with signal generation (oscillator application) and selection (filter application) such as radio, television and communication applications.

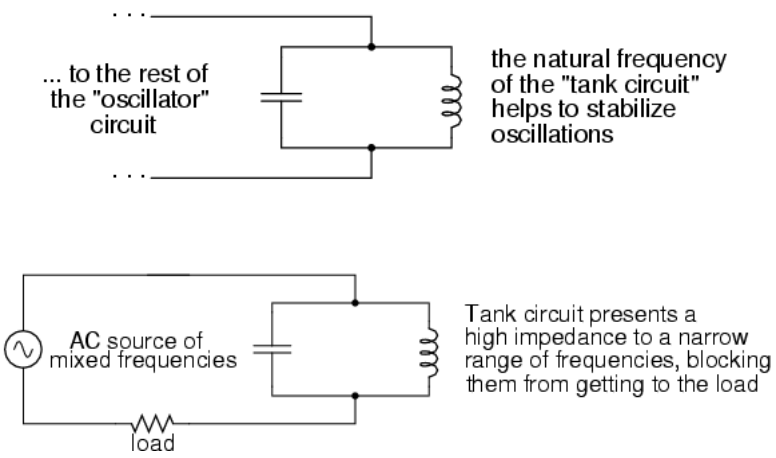


Figure 6: Applications of resonant second-order circuits e.g. oscillator (top) and filter (bottom) applications

3. Parameters of Series RLC Circuit

For a given series resonant RLC circuit shown below, perform the following tasks:

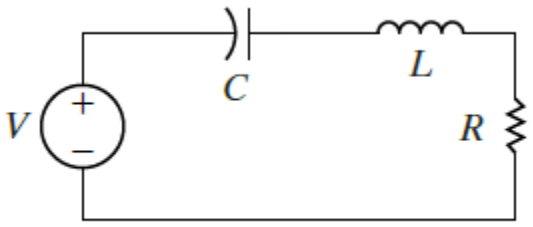


Figure 7: Second order series RLC circuits

3.1. Transfer Function Equation of Series RLC Circuit

For a series resonant RLC circuit as shown above, its equivalent circuit is as shown in the figure below.

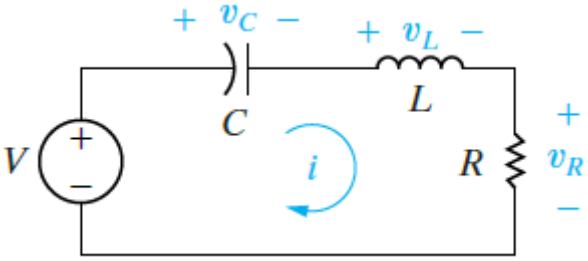


Figure 8: Equivalent circuit of series RLC circuit

Notice from the circuit given above that the input voltage is V ($V_{in} = V$) and the output voltage is taken across the resistor ($V_{out} = V_R$).

Applying KVL around the closed loop in the circuit above, the input voltage is:

$$V_{in} = V_C + V_L + V_R = i(R + Z_L + Z_C) \qquad (Eq. 27)$$

Entering each individual voltage in the circuit, the equation above becomes:

$$V_{in}(\omega) = i(\omega)[R + j\omega L + (1/j\omega C)] \qquad (Eq. 28)$$

Since the output voltage is taken across the resistor, it is thus:

$$V_{out}(\omega) = i(\omega)R \qquad (Eq. 29)$$

Rearranging the equations (28) and (29) given above, the transfer function equation for the series RLC circuit is:

$$H(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{R}{R + j\omega L + (1/j\omega C)} \qquad (Eq. 30)$$

Rearrange the equation above, thus:

$$H(\omega) = \frac{j\omega RC}{(1 - \omega^2 LC) + j\omega RC} \qquad (Eq. 31)$$

3.2. Resonant Frequency of Series RLC Circuit

For resonant frequency of the series RLC circuit, the resonant is happening when $1 - \omega^2 LC$.

$$1 - \omega^2 LC \qquad (Eq. 32)$$

Rearranging the equation given above, it becomes:

$$\omega^2 = \frac{1}{LC} \qquad (Eq. 33)$$

The resonant frequency (ω_0) of the series RLC circuit is:

$$\omega_0 = \omega = \frac{1}{\sqrt{LC}} \qquad (Eq. 34)$$

For series RLC circuit, the dependence of $H(\omega)$ on frequency is shown on the figure below with L = 47 mH, C = 47 µF and various values of *R*.

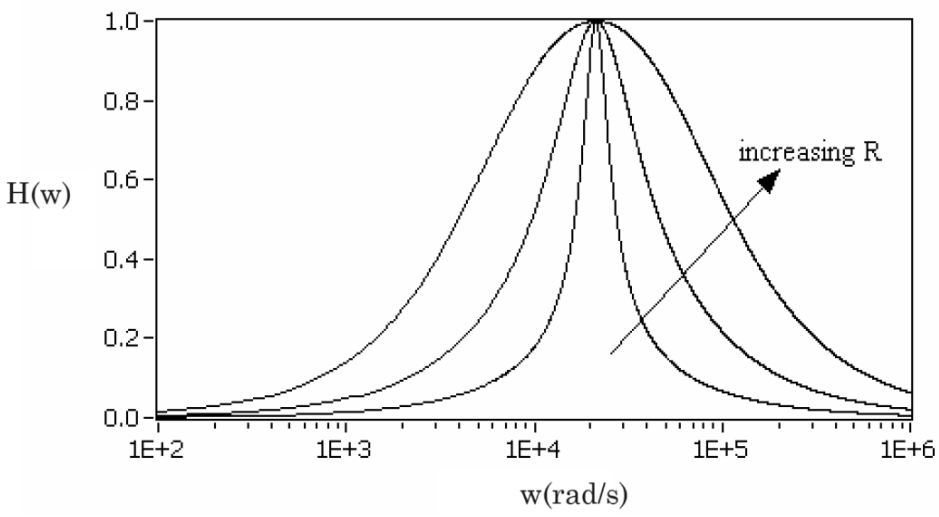
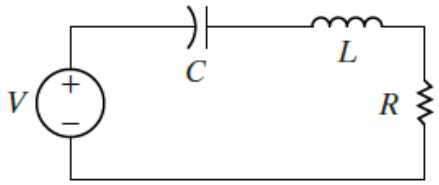


Figure 9: Resonant frequency in second-order series RLC circuit

Example 4: Resonant Frequency of Series RLC Circuits

A coil having a resistance of 10 Ω and an inductance of 75 mH is connected in series with a 40 μF capacitor across a 220 V AC supply.



Determine:

a. Resonant frequency. [4 marks]

b. Current that flows at the resonance. [2 marks]

Solution

a. Resonant frequency is calculated from:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Thus

$$f_c = \frac{1}{2\pi\sqrt{LC}}$$

Entering values of component into the equation above:

$$f_r = \frac{1}{2\pi\sqrt{(75 \times 10^{-3})(40 \times 10^{-6})}} \quad = 91.93 \text{ Hz}$$

b. Current at resonance is calculated from:

$$I = \frac{V}{R} = \frac{220}{10} = 20 \text{ A}$$

3.3. Cut-Off Frequency of Series RLC Circuit

Knowing that the transfer function equation for the series RLC circuit is as shown below

$$H(\omega) = \frac{j\omega RC}{(1 - \omega^2 LC) + j\omega RC} \qquad (Eq. 35)$$

The magnitude of the equation above is:

$$|H(\omega)| = \frac{\omega RC}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}} \qquad (Eq. 36)$$

Suffice to say that the cut-off frequencies are at which the gain is $1/\sqrt{2}$ of the peak value.

$$\frac{1}{\sqrt{2}} = \frac{\omega RC}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}} \qquad (Eq. 37)$$

Rearranging the equation above, it becomes:

$$\omega^4 L^2 C^2 - (R^2 C^2 + 2LC)\omega^2 + 1 \qquad (Eq. 38)$$

We could solve this equation to solve for the cut-off frequencies of the resonant circuit. However, it requires factorisation of a fourth order polynomial, which is messy. The cut-off frequencies ($\omega_{1,2}$) of the series RLC circuit are:

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \qquad (Eq. 39a)$$

And

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \qquad (Eq. 39b)$$

Notice that the two cut-off frequencies are not located symmetrically around the resonant frequency ($\omega_0 = 1/\sqrt{LC}$). However, if R is small, then the $(R/2L)^2$ term will be relatively small and we can make an approximation that the cut-off frequencies are located symmetrically around the resonance peak at:

$$\omega_{1,2} = \frac{1}{\sqrt{LC}} \pm \frac{R}{2L} \qquad (Eq. 40)$$

By referring to the cut-off frequencies, the resonant frequency is actual a geometric mean of ω_1 and ω_2.

$$\omega_0 = \sqrt{\omega_1 \omega_2} \qquad (Eq. 41)$$

3.4. Bandwidth of Series RLC Circuit

The bandwidth (B) of the series RLC circuit is determined from:

$$B_s = \omega_2 - \omega_1 \qquad (Eq.42)$$

The cut-off frequencies ($\omega_{1,2}$) of the series RLC circuit are:

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \qquad (Eq.43a)$$

And

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \qquad (Eq.43b)$$

Thus

$$B_s = \left(\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right) - \left(-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right) = \frac{R}{L} \qquad (Eq.44)$$

The bandwidth (B) of the series RLC circuit is as shown in the figure below.

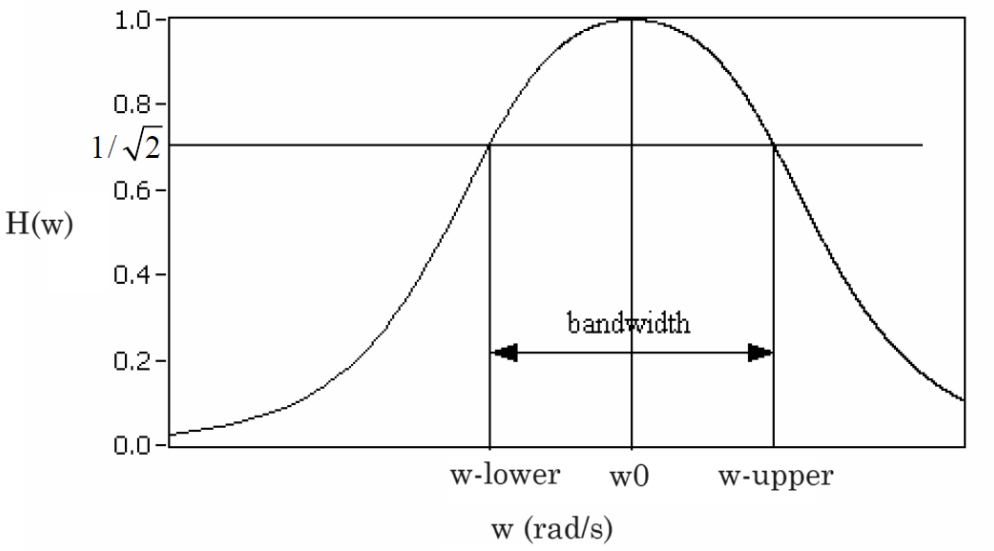
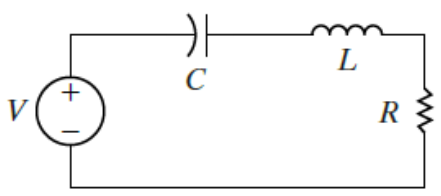


Figure 10: Bandwidth of second-order series RLC circuit.

Example 5: Bandwidth of Series RLC Circuit

A filter in the form of a series RLC circuit is designed to operate at a resonant frequency of 20 kHz. Included within the filter is a 20 mH inductance and 8 Ω resistance.



a. Determine the bandwidth of the filter. [2 marks]

b. If we wish to double the bandwidth, determine the value of the inductor while the resistance is the same. [2 marks]

c. Would change the value of capacitor change the bandwidth? [2 marks]

Solution

a. The bandwidth of the series RLC filter is:

$$B_s = \frac{R}{L} = \frac{8}{20 \times 10^{-3}} = 400 \text{ Hz}$$

b. The value of the inductor when the bandwidth is doubled is:

$$L = \frac{R}{B_s} = \frac{8}{800 \text{ Hz}} = 10 \times 10^{-3} \text{ H}$$

c. The value of capacitor would not affect the bandwidth. It influences the cut-off frequencies and transient response.

3.5. Q-Factor of Series RLC Circuit

The Q-factor (Q) of the series RLC circuit is calculated from:

$$Q = \frac{\omega_0}{B} \qquad (Eq. 45)$$

The resonant frequency (ω_0) of the series RLC circuit is:

$$\omega_0 = \omega = \frac{1}{\sqrt{LC}} \qquad (Eq. 46)$$

Bandwidth of series RLC circuit is:

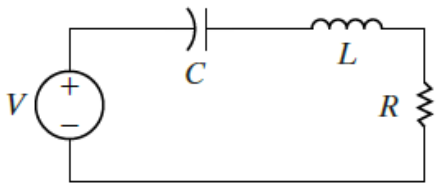
$$B = \frac{R}{L} \qquad (Eq. 47)$$

Thus

$$Q = \frac{\left(\frac{1}{\sqrt{LC}}\right)}{\left(\frac{R}{L}\right)} = \frac{1}{R}\sqrt{\frac{L}{C}} \qquad (Eq. 48)$$

Example 6: Q-Factor of Series RLC Circuit

A series RLC circuit comprises a 10 Ω resistance, a 5 μF capacitor, and a variable inductance L. The supply voltage is 20∠0° V at a frequency of 318.3 Hz. The inductance is adjusted until the voltage across the 10 Ω resistance is a maximum.



Determine for this condition:

a. The value of the inductance L. [2 marks]

b. The potential difference across each component. [16 marks]

c. The Q-factor. [2 marks]

Solution

a. The maximum voltage across the resistance occurs at resonance when the current is a maximum. At resonance, $X_L = X_C$, thus $\omega_r L = 1/(\omega_r C)$, from which:

$$L = \frac{1}{\omega_r^2 C} = \frac{1}{2\pi(318.3)^2(5\times10^{-6})} = 0.05 \text{ H}$$

b. The potential differences across each component are:

Current at resonance:

$$I_r = \frac{V}{R} = \frac{20\angle0°}{10\angle0°} = 2\angle0° \text{ A}$$

Potential difference across resistance:

$$V_R = I_r R = (2\angle0°)(10) = 20\angle0° \text{ V}$$

Reactance of the inductor and capacitor are:

$$X_L = 2\pi\omega L = 2\pi(318.3)(0.05) = 100\ \Omega$$

$$X_C = \frac{2\pi}{j\omega C} = \frac{2\pi}{j(318.3)(0.05)} = 100\angle-90°\ \Omega$$

Potential differences across the inductor and capacitor are:

$$V_L = (2\angle0°)(100\angle90°) = 200\angle90°$$

$$V_C = IX_C = (2\angle0°)(100\angle-90°) = 200\angle-90°$$

c. The Q-factor of the series RLC circuit is:

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{10}\sqrt{\frac{0.05}{5\times10^{-9}}} = 316.2$$

4. Parameters of Parallel RLC Circuit

For a given parallel resonant RLC circuit as shown below, perform the following tasks:

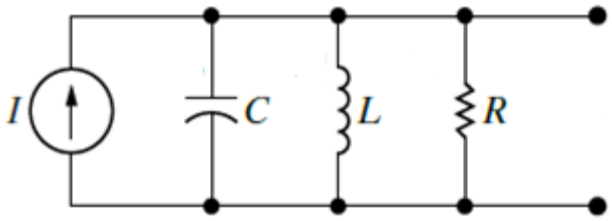


Figure 11: Second-order parallel RLC circuit

4.1. Transfer Function Equation of Parallel RLC Circuit

For a parallel resonant RLC circuit as shown above, its equivalent circuit is as shown in the figure below.

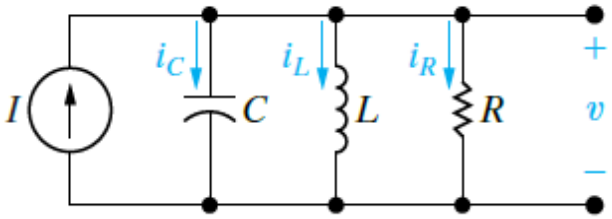


Figure 12: Equivalent circuit of the second-order parallel RLC circuit

Applying current division rule to the circuit given above, the currents in the circuit are determined from:

$$I_R = I_S\left(\frac{Y_R}{Y_R + Y_C + Y_L}\right) = I_S\left(\frac{\frac{1}{Z_R}}{\frac{1}{Z_R} + \frac{1}{Z_C} + \frac{1}{Z_L}}\right) \qquad (Eq.49)$$

Rearrange the equation above, thus:

$$\frac{I_R(\omega)}{I_S(\omega)} = \frac{\frac{1}{R}}{\frac{1}{R} + \frac{1}{(1/j\omega C)} + \frac{1}{j\omega L}} \qquad (Eq.50)$$

The transfer function equation for the parallel RLC circuit is:

$$H(\omega) = \frac{I_R(\omega)}{I_S(\omega)} = \frac{j\omega L}{(R - \omega^2 LCR) + j\omega L} \qquad (Eq.51)$$

4.2. Resonant Frequency of Parallel RLC Circuit

Like in the series RLC circuit case, the resonant frequency of the parallel RLC circuit occurs when $1 - \omega^2 LC$. Thus, it is also:

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad (Eq.52)$$

For parallel RLC circuit, the dependence of $H(\omega)$ on frequency is shown on the figure below with L = 47mH, C = 47µF and various values of R.

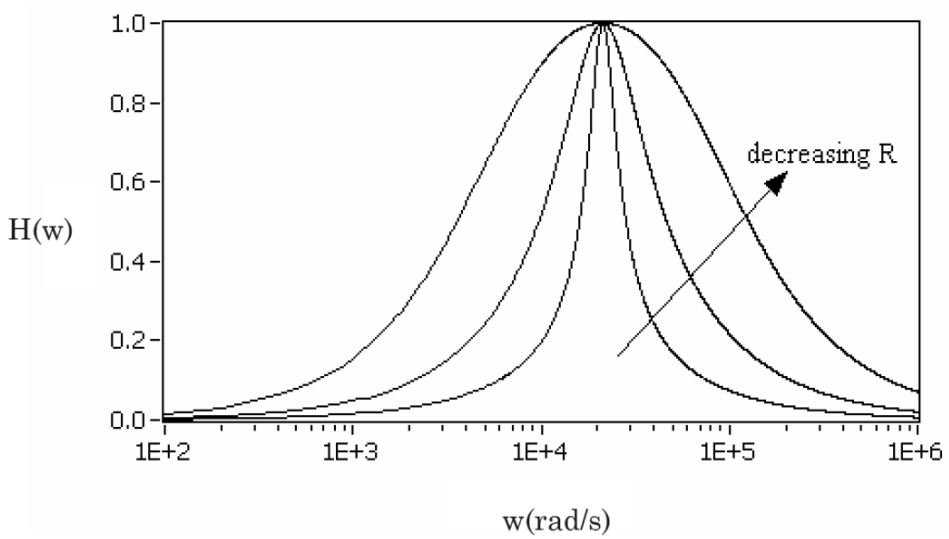


Figure 13: Resonant frequency of the second-order parallel RLC circuit

4.3. Cut-Off Frequency of Parallel RLC Circuit

The magnitude of the transfer function for the parallel RLC circuit is:

$$|H(\omega)| = \frac{\omega L}{\sqrt{(R - \omega^2 LCR)^2 + (\omega L)^2}} \qquad (Eq.53)$$

The cut-off frequencies are at which the gain is $1/\sqrt{2}$ of the peak value.

$$\frac{1}{\sqrt{2}} = \frac{\omega L}{\sqrt{(R - \omega^2 LCR)^2 + (\omega L)^2}} \qquad (Eq.54)$$

Like in the series-RLC circuit case, the equation above is also solved as roots of a fourth order polynomial equation. The cut-off frequencies of the parallel RLC circuit are:

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \qquad (Eq.55a)$$

And

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \qquad (Eq.55b)$$

4.4. Bandwidth of Parallel RLC Circuit

As the cut-off frequencies of the parallel RLC circuit are:

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \qquad (Eq.56a)$$

And

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \qquad (Eq.56b)$$

The bandwidth of the parallel RLC circuit is:

$$B = \omega_2 - \omega_1 = \frac{1}{RC} \qquad (Eq.57)$$

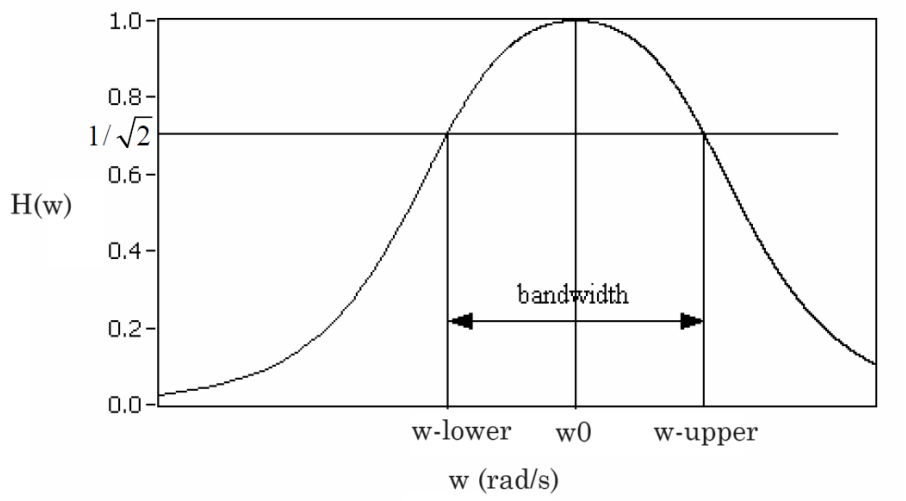


Figure 14: Bandwidth of the second-order parallel RLC circuit

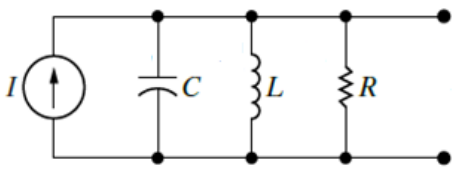
4.5. Q-Factor of Parallel RLC Circuit

Knowing $\omega = 1/\sqrt{LC}$ and $B_p = 1/RC$, the Q-factor of the parallel RLC circuit is:

$$Q = \frac{\omega_0}{B} = \frac{\left(\frac{1}{\sqrt{LC}}\right)}{\left(\frac{1}{RC}\right)} = R\sqrt{\frac{C}{L}} \qquad (Eq.58)$$

Example 7: Resonant Frequency and Q-Factor of Parallel RLC Circuit

A coil of inductance 5 mH and resistance 10 Ω is connected in parallel with a 250 nF capacitor across a 50 A variable-frequency supply.



Determine:

a. The resonant frequency. [2 marks]

b. The circuit Q-factor at resonance. [2 marks]

Solution

a. The resonant frequency of parallel RLC circuit is:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Thus

$$f_c = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(5\times10^{-3})(250\times10^{-9})}} = 4.5 \text{ Hz}$$

b. The Q-factor of the parallel RLC circuit at resonance is:

$$Q = R\sqrt{\frac{C}{L}} = 10\sqrt{\frac{250\times10^{-9}}{5\times10^{-3}}} = 0.0707$$

5. Comparison of Responses and Circuit Parameters of Series and Parallel RLC Circuits

The equations for the transient responses of the RLC circuits are outlined in the following tables.

Type of Circuit	**Natural Response Equation**	**Coefficient Equation**
Overdamped	$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$	$x(0) = A_1 + A_2$ $\frac{dx}{dt(0)} = A_1 s_1 + A_2 s_2$
Underdamped	$x(t) = (B_1 \cos\omega_d t + B_2 \sin\omega_d t)e^{-\alpha t}$	$x(0) = B_1$ $\frac{dx}{dt(0)} = -\alpha B_1 + \omega_d B_2$ Where: $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Critically damped	$x(t) = (D_1 t + D_2)e^{-\alpha t}$	$x(0) = D_2$ $\frac{dx}{dt(0)} = D_1 - \alpha D_2$

Table 1: Transient responses of second-order RLC circuits

Based on the results obtained in the sections above, we compare the characteristics of series RLC circuit with parallel RLC circuit. The comparison of the characteristics of the series RLC circuit vs. the parallel RLC circuit is listed in the table below.

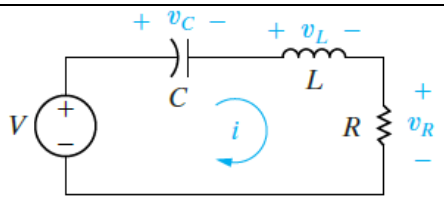
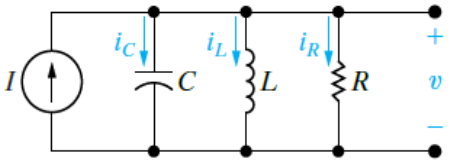
Parameter	**Series**	**Parallel**
Circuit diagram		
Transfer function	$\frac{V_R(\omega)}{V(\omega)} = \frac{j\omega RC}{(1-\omega^2 LC)+j\omega RC}$	$\frac{I_R(\omega)}{I(\omega)} = \frac{j\omega L}{(R-\omega^2 LCR)+j\omega L}$
Resonant frequency	$\omega_0 = \frac{1}{\sqrt{LC}}$	$\omega_0 = \frac{1}{\sqrt{LC}}$
Cut-off frequencies	$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{\omega_0^2}}$ $\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{\omega_0^2}}$	$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{\omega_0^2}}$ $\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{\omega_0^2}}$
Bandwidth	$B_s = \omega_2 - \omega_1 = \frac{R}{L}$	$B_p = \omega_2 - \omega_1 = \frac{1}{RC}$
Q-factor	$Q = \frac{\omega_0}{B_s} = \frac{\omega_o L}{R} = \frac{1}{\omega_o RC}$	$Q = \frac{\omega_0}{B_p} = \omega_o RC = \frac{R}{\omega_o L}$

Table 2: Parameter comparison of series and parallel second-order RLC circuits