XMUT203 Analogue Circuits and Systems

Note 10a: Introduction to Laplace Transform

Topics:

- Introduction to Laplace transform.
- Forward and reverse (functional) transforms of Laplace transform.
- Properties (operational) transforms of Laplace transform.
- Applying Laplace transform.
- Inverse Laplace transforms.
- Partial fraction expansions.
- Poles and zeros of $F(s)$.
- Initial- and final-value theorems.
- Examples of application of Laplace transform.

1. Introduction to Laplace Transform

Laplace transform is a mathematical transformation technique put together by Pierre Simon. It is used for converting the solving of integro-integral equation from integration and/or differentiation in the time domain to simpler algebraic manipulation.





Figure 1: Pierre Simon, Marquis de Laplace (left) and Oliver Heaviside (right)

To help with conversion from the time domain to the intermediate domain (i.e. s-domain) and vice versa, partial fraction expansion method, formulated by Olivier Heaviside, that is used to perform the tasks.

1.1. Mathematical Transform

When analysing circuit, we have learnt to apply a few transforms already:

- Logarithms are used to change a multiplication or division problem into a simpler addition or subtraction problem. Antilogs are used to carry out the inverse process.
- The phasor converts a sinusoidal signal into a complex number for easier algebraic computation of circuit values. After determining the phasor value of a signal, we transform it back to its time-domain expression.

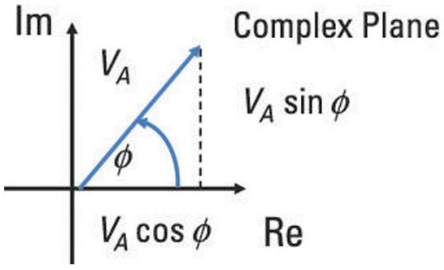


Figure 2: Illustration of a phasor transformation

Both of these examples point out the essential feature of mathematical transforms: They are designed to create a new domain to make the mathematical manipulations easier. After finding the unknown in the new domain, we inverse-transform it back to the original domain.

1.2. Circuit and Laplace Transform

Consider the transient behavior of circuits whose describing equations consist of more than a single node-voltage or mesh current differential equation e.g. multiple-node and multiple-mesh circuits that are described by sets of linear differential equations.

Determine the transient response of circuits whose signal sources vary in ways more complicated than the simple DC level jumps.

Use the Laplace transform to introduce the concept of the transfer function as a tool for analysing the steady-state sinusoidal response of a circuit when the frequency of the sinusoidal source is varied.

Relate, in a systematic fashion, the time-domain behavior of a circuit to its frequency-domain behavior.

2. Laplace Transform

The Laplace transform of a function is given by the expression:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st} dt \qquad (Eq. 1)$$

The Laplace transform of $f(t)$ is also denoted as:

$$F(s) = \mathcal{L}\{f(t)\} \qquad (Eq. 2)$$

This notation emphasizes that when the integral in the equation has been evaluated, the resulting expression is a function of s.

In our applications, t represents the time domain, and, because the exponent of e in the integral of the equation above must be dimensionless, s must have the dimension of reciprocal time, or frequency.

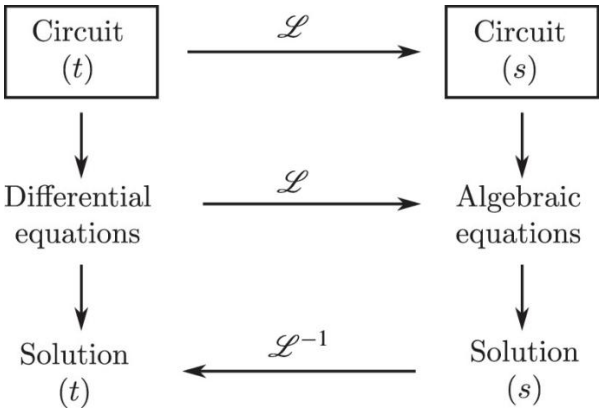


Figure 3: Inter-domain transformation using Laplace-transform

The Laplace transform transforms the problem from the time domain to the frequency domain.

After obtaining the frequency-domain expression for the unknown, we could inverse-transform it back to the time domain.

In circuit analysis, we use the Laplace transform to transform a set of integro-differential equations from the time domain to a set of algebraic equations in the frequency domain.

We therefore simplify the solution for an unknown quantity to the manipulation of a set of algebraic equations.

Knowing that a Laplace transform is given as:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st} dt \qquad (Eq. 3)$$

The integral is improper because the upper limit is infinite. Does the integral converge? In linear circuit analysis, we only excite circuits with sources that have Laplace transforms.

The lower limit on the integral is zero, the Laplace transform ignores $f(t)$ for negative values of t. e.g. behaviour of $f(t)$ only for positive values of t (one-sided, or unilateral).

2.1. Laplace Transform Characteristics

If $f(t)$ is continuous or has a finite discontinuity at origin as given below. Will Laplace transform consider the value at $t = 0$?

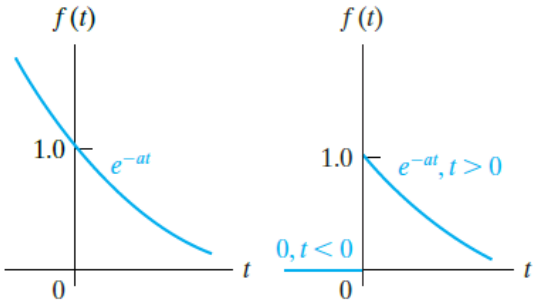


Figure 4: Discontinuity of a function at the origin ($t = 0$)

Thus, generally, we use 0^- as the lower limit in Laplace transform, so that the integration from 0^- to 0^+ is zero.

The only exception is when the discontinuity at the origin is an impulse function.

As a result, the continuous and discontinuous functions have the same unilateral Laplace transform because there is no impulse function at the origin.

The one-sided Laplace transform ignores $f(t)$ for $t < 0^-$. What happens prior to 0^- is accounted for by the initial conditions.

Thus, we use the Laplace transform to predict the response to a disturbance that occurs after initial conditions have been established.

Example 1: Laplace Transform

With regard to Laplace transform, answer the following questions:

a. Describe the definition of Laplace transform. [2 marks]

b. Describe three characteristics of the Laplace transform. [3 marks]

2.2. Analysis of Laplace Transform

We divide our coverage about the Laplace transforms into: two types: functional transforms and (operational) transforms properties.

A functional transform is the Laplace transform of a specific function, such as $\sin \omega t$, t, e^{-at} and so on.

An (operational) transform characteristic defines a general mathematical property of the Laplace transform, such as finding the transform of the derivative of $f(t)$.

Before considering functional and operational transforms, however, we need to introduce the step and impulse functions.

3. Functional Transforms

A functional transform is simply the Laplace transform of a specified function of t.

Because we are limiting our introduction to the unilateral, or one-sided, Laplace transform, we define all functions to be zero for $t < 0^-$.

3.1. Impulse Function

First, we derived one functional transform pair before, where we showed that the Laplace transform of the unit impulse function equals 1.

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)\, e^{-st} dt = \int_{0^-}^{\infty} \delta(t)\, dt = 1 \qquad (Eq.4)$$

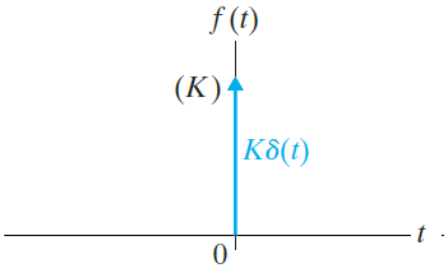


Figure 5: Impulse function

3.2. Step Function

From before, a second illustration of the functional transform is the unit step function of the figure below, where:

$$\mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} f(t)\, e^{-st} dt = \int_{0^+}^{\infty} 1\, e^{-st} dt = \frac{e^{-st}}{-s}\Big|_{0^+}^{\infty} = \frac{1}{s} \qquad (Eq.5)$$

Equation above shows that the Laplace transform of the unit step function is $1/s$.

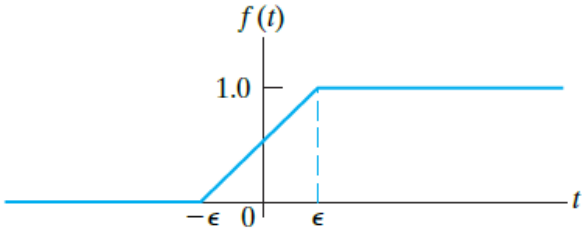


Figure 6: Unit step function as $\epsilon \to 0$ at origin ($t = 0$)

3.3. Exponential Function

The Laplace transform of the decaying exponential function shown in the figure below is:

$$\mathcal{L}\{e^{-at}\} = \int_{0^+}^{\infty} e^{-at}\, e^{-st} dt$$

$$= \int_{0^+}^{\infty} e^{-(a+s)t}$$

$$= \frac{1}{s+a} \qquad (Eq. 6)$$

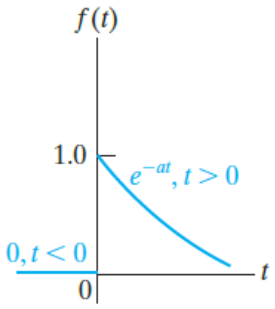


Figure 7: Decaying exponential function

In deriving the last two equations, we used the fact that integration across the discontinuity at the origin is zero.

3.4. Sinusoidal Function

A third illustration of finding a functional transform is a sinusoidal function shown in the figure below.

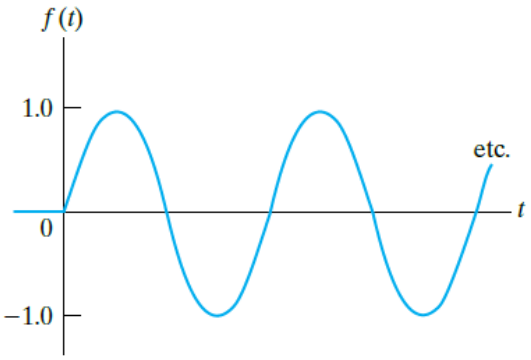


Figure 8: Sinusoidal input function

The expression for $f(t)$ for $t > 0^-$ is $\sin \omega t$, hence its Laplace transform is:

$$\mathcal{L}\{\sin \omega t\} = \int_{0^-}^{\infty} (\sin \omega t)\, e^{-st} dt$$

$$= \int_{0^-}^{\infty} \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right) e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} \left(\frac{e^{-(s-j\omega)t} - e^{-(s+j\omega)t}}{2j} \right) dt$$

$$= \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega} \right) = \frac{\omega}{s^2+\omega^2} \qquad (Eq.7)$$

3.5. Common Mathematical Functions

Table next gives an abbreviated list of Laplace transform pairs. It includes the functions of most interest in an introductory course on circuit applications.

Function	$\boldsymbol{f(t)(t > 0^-)}$	$\boldsymbol{F(s)}$
Impulse	$\delta(t)$	1
Step	$u(t)$	$\frac{1}{s}$
Ramp	t	$\frac{1}{s^2}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Sine	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
Damped ramp	te^{-at}	$\frac{1}{(s+a)^2}$
Damped sine	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
Damped cosine	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Table 1: Operational transforms of Laplace transform

Example 2: Functional Transforms

There are two types of transformations of the functions in Laplace transform e.g. functional and operational transforms.

a. Describe what is functional Laplace transform. [2 marks]

b. Describe functional Laplace transforms of the following functions:

i. Impulse function. [2 marks]

ii. Step function. [2 marks]

iii. Exponential function. [2 marks]

iv. Sinusoidal function. [2 marks]

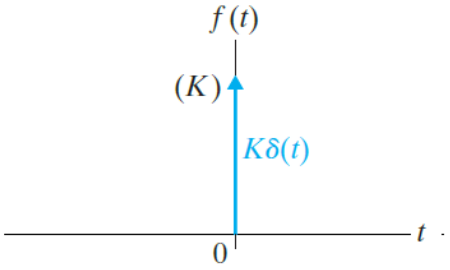
Solution

a. A functional transform is simply the Laplace transform of a specified function of t. Because we are limiting our introduction to the unilateral, or one-sided, Laplace transform, we define all functions to be zero for $t < 0^-$.

b. The functional Laplace transforms of the listed functions are described and derived as follow:

i. The functional Laplace transform of an impulse function equals 1. We use the sifting property of the impulse function to find its Laplace transform:

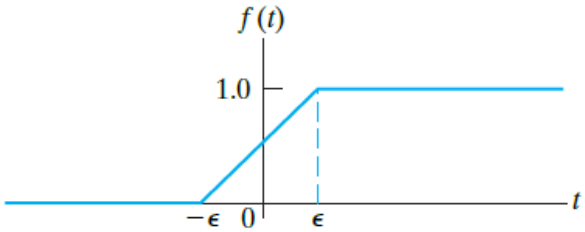
$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt = \int_{0^-}^{\infty} \delta(t)\,dt = 1$$



ii. For a given unit step function, its Laplace transform is derived as follow:

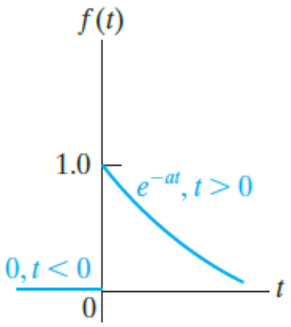
$$\mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} f(t)\,e^{-st}dt = \int_{0^+}^{\infty} 1e^{-st}\,dt = \frac{e^{-st}}{-s}\Big|_{0^+}^{\infty} = \frac{1}{s}$$

Equation above shows that the Laplace transform of the unit step function is 1/s. In deriving the equation above, we used the fact that integration across the discontinuity at the origin is zero.



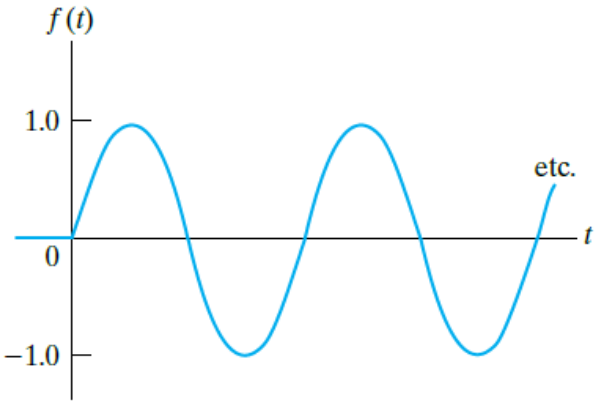
iii. The Laplace transform of the decaying exponential function shown in the figure below is:

$$\mathcal{L}\{e^{-at}\} = \int_{0^+}^{\infty} e^{-at}\, e^{-st} dt = \int_{0^+}^{\infty} e^{-(a+s)t}\, dt = \frac{1}{s+a}$$



Like in the step function, in deriving the equation above, we used the fact that integration across the discontinuity at the origin is zero.

iv. The functional transform of a sinusoidal function shown in the figure below is derived as follow.



The expression for $f(t)$ when $t > 0^-$ is $\sin \omega t$, hence the Laplace transform is:

$$\mathcal{L}\{\sin \omega t\} = \int_{0^-}^{\infty} \sin \omega t\, e^{-st} dt$$

$$= \int_{0^-}^{\infty} \left(\frac{e^{j\omega t} - e^{j\omega t}}{2j} \right) e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} \left(\frac{e^{-(s-j\omega)t} - e^{-(s+j\omega)t}}{2j} \right) dt$$

$$= \frac{1}{2j}\left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right)$$

$$= \frac{\omega}{s^2 + \omega^2}$$

4. Introduction to Operational Laplace Transform

Operational transforms indicate how mathematical operations performed on either or are converted into the opposite domain. The operations of primary interest are:

a. Multiplication by a constant.

b. Addition (subtraction).

c. Differentiation.

d. Integration.

e. Translation in the time domain.

f. Translation in the frequency domain.

g. Scale changing.

4.1. Multiplication by a Constant

From the defining integral, if:

$$\mathcal{L}\{f(t)\} = F(s) \qquad (Eq.8)$$

Then

$$\mathcal{L}\{Kf(t)\} = KF(s) \qquad (Eq.9)$$

Thus, multiplication of $f(t)$ by a constant corresponds to multiplying $F(s)$ by the same constant.

4.2. Addition (Subtraction)

Addition (subtraction) in the time domain translates into addition (subtraction) in the frequency domain. Thus, if:

$$\mathcal{L}\{f_1(t)\} = F_1(s) \qquad (Eq.10a)$$

$$\mathcal{L}\{f_2(t)\} = F_2(s) \qquad (Eq.10b)$$

$$\mathcal{L}\{f_3(t)\} = F_3(s) \qquad (Eq.10c)$$

then

$$\mathcal{L}\{f_1(t) + f_2(t) + f_3(t)\} = F_1(s) + F_2(s) + F_3(s) \qquad (Eq.11)$$

which is derived by simply substituting the algebraic sum of time-domain functions into the defining integral.

4.3. Differentiation

Differentiation in the time domain corresponds to multiplying $F(s)$ by s and then subtracting the initial value of $f(t)$ - that is, $f(0^-)$ - from this product:

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-) \qquad (Eq.12)$$

which is obtained directly from the definition of the Laplace transform, or:

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty}\left[\frac{df(t)}{dt}\right]e^{-st}dt \qquad (Eq.13)$$

We evaluate the integral given in equation (13) above by integrating by parts.

Letting $e^{-st}f(t)$ and $dv = [df(t)/dt]dt$ yields:

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = e^{-st}f(t)|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)\,(-se^{-st})dt \qquad (Eq.14)$$

Because we are assuming that $f(t)$ is Laplace transformable, the evaluation of $e^{-st}f(t)$ at $t = \infty$ is zero.

Therefore, the right-hand side of equation above reduces to:

$$-f(0^-) + s\int_{0^-}^{\infty} f(t)\,e^{-st}dt = sF(s) - f(0^-) \qquad (Eq.15)$$

Differentiation property is important because it states that differentiation in the time domain reduces to an algebraic operation in the s domain.

We determine the Laplace transform of higher-order derivatives by using the differentiation property as the starting point.

We find the Laplace transform of the n-th derivative by successively applying the preceding process, which leads to the general result:

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt}$$

$$-s^{n-3}\frac{d^2 f(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}} \qquad (Eq.16)$$

4.4. Integration

Integration in the time domain corresponds to dividing by s in the s domain.

As before, we establish the relationship by the defining integral:

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)dx\right\} = \int_{0^-}^{\infty}\left[\int_{0^-}^{t} f(x)dx\right]e^{-st}dt \qquad (Eq.17)$$

We evaluate the integral on the right-hand side of equation (17) given above by integrating by parts, first letting:

$$u = \int_{0^-}^{t} f(x)dx \qquad (Eq.18)$$

And

$$dv = e^{-st}dt \qquad (Eq.19)$$

Then

$$du = f(t)dt \qquad (Eq.20)$$

Thus

$$v = -\frac{e^{-st}}{s} \qquad (Eq.21)$$

The integration-by-parts formula yields:

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)dx\right\} = -\frac{e^{-st}}{s}\int_{0^-}^{t} f(x)dx\Big|_{0^-}^{\infty} + \int_{0^-}^{\infty}\frac{e^{-st}}{s}f(t)dt \qquad (Eq.22)$$

The first term on the right-hand side of equation (2) is zero at both the upper and lower limits.

The evaluation at the lower limit obviously is zero, whereas the evaluation at the upper limit is zero because we are assuming that $f(t)$ has a Laplace transform.

The second term on the right-hand side of equation (22) is $F(s)/s$; therefore:

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)dx\right\} = \frac{F(s)}{s} \qquad (Eq.23)$$

which reveals that the operation of integration in the time domain is transformed to the algebraic operation of multiplying by $1/s$ in the s-domain.

Knowing the given equations:

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt}$$
$$-s^{n-3}\frac{d^2 f(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}} \qquad (Eq.24)$$

And

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)dx\right\} = \frac{F(s)}{s} \qquad (Eq.25)$$

Differentiation and integration properties prove that the Laplace transform translates a set of integro-differential equations into a set of algebraic equations.

4.5. Translation in the Time Domain

If we start with any function $f(t)u(t)$, we can represent the same function, translated in time by the constant a, as $f(t-a)u(t-a)$.

Note that throughout we multiply any arbitrary function by the unit step function to ensure that the resulting function is defined for all positive time.

Translation in the time domain corresponds to multiplication by an exponential in the frequency domain. Thus:

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s), \qquad a > 0 \qquad (Eq.26)$$

For example, knowing that:

$$\mathcal{L}\{tu(t)\} = \frac{1}{s^2} \qquad (Eq.27)$$

Eq. 1 permits writing the Laplace transform of $(t-a)u(t-a)$ directly:

$$\mathcal{L}\{f(t-a)u(t-a)\} = \frac{e^{-as}}{s^2} \qquad (Eq.28)$$

The proof of equation (28) follows from the defining integral:

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_{0^-}^{\infty} u(t-a)f(t-a)\, e^{-st}dt$$

$$= \int_{a}^{\infty} f(t-a)e^{-st}dt \qquad (Eq.29)$$

In writing equation (29), we took advantage of $u(t-a)$ for $t > a$. Now, we change the variable of integration.

Specifically, we let $x = t - a$. Then, $x = 0$ when $t = a$, $x = \infty$ when $t = \infty$ and $dx = dt$.

Thus, we write the integral in equation (29) as:

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_{0}^{\infty} f(x)e^{-s(x+a)}dx$$

$$= e^{-sa}\int_{0}^{\infty} f(x)e^{-sx}dx$$

$$= e^{-as}F(s) \qquad (Eq.30)$$

4.6. Translation in the Frequency Domain

Translation in the frequency domain corresponds to multiplication by an exponential in the time domain:

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a) \qquad (Eq.31)$$

which follows from the defining integral. The derivation of equation (31) is shown below**.**

$$\mathcal{L}\{e^{-at}f(t)\} = \int_{0^-}^{\infty} [e^{-at}f(t)]\, e^{-st}dt$$

$$= \int_{0^-}^{\infty} f(t)\, e^{-(s+a)t}dt$$

$$= F(s+a) \qquad (Eq.32)$$

4.7. Scale Changing

The scale-change property gives the relationship between and when the time variable is multiplied by a positive constant:

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \quad a > 0 \qquad (Eq.33)$$

The derivation of equation (33) is shown below.

$$\mathcal{L}\{f(at)\} = \int_{0^-}^{\infty} f(at)\, e^{-st} dt \qquad (Eq.34)$$

Let $u = at$ or $t = u/a$, $du = adt$ or $t = du/a$, $u = 0^-$ when $t = 0^-$, and $u = \infty$ when $t = \infty$.

$$\int_{0^-}^{\infty} f(at)\, e^{-st} dt = \int_{0^-}^{\infty} f(u)\, e^{-s\left(\frac{u}{a}\right)} \frac{du}{a} \qquad (Eq.35)$$

Rearrange the equation

$$\int_{0^-}^{\infty} f(u)\, e^{-s\left(\frac{u}{a}\right)} \frac{du}{a} = \frac{1}{a}\int_{0^-}^{\infty} f(u)\, e^{-\left(\frac{s}{a}\right)u} du \qquad (Eq.36)$$

Since

$$\int_{0^-}^{\infty} f(u)\, e^{-\left(\frac{s}{a}\right)u} du = F\left(\frac{s}{a}\right) \qquad (Eq.37)$$

Therefore, perform substitution of the variables and the above equation becomes:

$$\mathcal{L}\{f(at)\} = \int_{0^-}^{\infty} f(u)\, e^{-s\left(\frac{u}{a}\right)} \frac{du}{a}$$

$$= \frac{1}{a}F\left(\frac{s}{a}\right) \qquad (Eq.38)$$

The scale-change property is particularly useful in experimental work, especially where time-scale changes are made to facilitate building a model of a system.

Example 3: Operational Laplace Transforms 1

With regard to operational transforms, attempt the following tasks:

a. Describe operational transforms. [2 marks]

b. Prove the following operational transforms:

   i. Multiplication by a constant. [4 marks]

$$\mathcal{L}\{Kf(t)\} = KF(s)$$

   ii. Addition (subtraction). [4 marks]

$$\mathcal{L}\{f_1(t) + f_2(t) + f_3(t)\} = F_1(s) + F_2(s) + F_3(s)$$

iii. Differentiation. [4 marks]

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$$

iv. Integration. [4 marks]

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)dx\right\} = \frac{F(s)}{s}$$

v. Translation in the time domain. [4 marks]

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-sa}F(s), \qquad a > 0$$

vi. Translation in the frequency domain. [4 marks]

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

vii. Scale changing. [4 marks]

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F(s/a), \quad a > 0$$

Example 4: Operational Laplace Transforms 2

Some of the properties (operational transforms) of Laplace transform could be used to solve problems in the Laplace transform of function.

a. Use the differentiation property of Laplace transform to prove that: [12 marks]

$$\mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$$

b. Use to translation in the frequency domain property of Laplace transform to prove that: [8 marks]

$$\mathcal{L}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2 + \omega^2}$$

c. Use scale changing property of Laplace transform to prove that: [8 marks]

$$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2 + \omega^2}$$

Solution

a. We can determine the Laplace transform of second-order derivatives by using the differentiation property of Laplace transform as the starting point.

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$$

For example, to find the Laplace transform of the second derivative of $f(t)$, we first let:

$$g(t) = \frac{df(t)}{dt} \qquad (Eq.4)$$

Now we use equation 1 to write:

$$G(s) = sF(s) - f(0^-) \qquad (Eq.5)$$

But, because:

$$\frac{dg(t)}{dt} = \frac{d^2f(t)}{dt^2}$$

we write:

$$\mathcal{L}\left\{\frac{dg(t)}{dt}\right\} = \mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = sG(s) - g(0^-) \qquad (Eq.6)$$

Combining Eqs. 4, 5, and 6 gives:

$$\mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt} \qquad (Eq.7)$$

b. We may use the translation in the frequency domain property of Laplace transform to prove:

$$\mathcal{L}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2+\omega^2}$$

Translation in the frequency domain property of Laplace transform is:

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

Thus, knowing that:

$$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$$

We can deduce that:

$$\mathcal{L}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2+\omega^2}$$

c. We can use the scale changing property of Laplace transform to formulate prove:

$$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$$

Scale changing property of Laplace transform is:

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F(s/a), \quad a > 0$$

Thus, knowing that:

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$$

We deduce that:

$$\mathcal{L}\{\cos \omega t\} = \left(\frac{1}{\omega}\right)\frac{s/\omega}{(s/\omega)^2+1} = \frac{s}{s^2+\omega^2}$$

5. Inverse Laplace Transform

In general, we need to find the inverse transform of a function that has the form

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0} \qquad (Eq.39)$$

The coefficients a and b are real constants, and the exponents m and n are positive integers.

The ratio $N(s)/D(s)$ is called a proper rational function if $m > n$ and an improper rational function if $m \leq n$.

Only a proper rational function can be expanded as a sum of partial fractions.

Converting the equation into its equation with the roots.

$$F(s) = \frac{(s^n + z_n)(s^n + z_{n-1}) \ldots (s + \mathrm{z})}{(s^n + p_n)(s^n + p_{n-1}) \ldots (s + p)} \qquad (Eq.40)$$

Applying partial fraction, we could expand the equation given above.

$$F(s) = \frac{K_n}{(s^n + p_n)} + \frac{K_{n-1}}{(s^n + p_{n-1})} + \cdots \frac{K_{nth}}{(s + p)} \qquad (Eq.41)$$

By referring to the table of Laplace transform pairs, the inverse Laplace transform could be found.

In the end, we can determine the time-domain equation of the circuit ($f(t)$).

6. Partial fraction Expansions

Partial fraction expansion helps to simplify complex forms of Laplace transform equation into several standard functions that we can find its inverse from the Laplace transform table pairs.

a. Proper Rational Functions.

   i. Distinct Real Roots of Denominator.

   ii. Distinct Complex Roots of Denominator.

   iii. Repeated Real Roots of Denominator.

   iv. Repeated Complex Roots of Denominator.

b. Improper Rational Functions.

6.1. Proper Rational Functions

A proper rational function is expanded into a sum of partial fractions by writing a term or a series of terms for each root of $D(s)$. Thus, $D(s)$ must be in factored form before we can make a partial fraction expansion. For each distinct root of $D(s)$, a single term appears in the sum of partial fractions. For each multiple root of $D(s)$ of multiplicity r, the expansion contains r terms.

For example, in the rational function:

$$\frac{s+z}{s(s+p_1)(s+p_2)^2} \qquad (Eq.42)$$

the denominator has four roots. Two of these roots are distinct—namely, at $s=0$ and $s=-p_1$. A multiple root of multiplicity 2 occurs at $s=-p_2$.

Thus, the partial fraction expansion of this function takes the form:

$$\frac{s+z}{s(s+p_1)(s+p_2)^2} \equiv \frac{K_1}{s} + \frac{K_2}{s+p_1} + \frac{K_3}{(s+p_2)^2} + \frac{K_4}{s+p_2} \qquad (Eq.43)$$

The key to the partial fraction technique for finding inverse transforms lies in recognizing the corresponding to each term in the sum of partial fractions.

From Laplace transform pair Table, you should be able to verify that:

$$\mathcal{L}^{-1}\left\{\frac{s+z}{s(s+p_1)(s+p_2)^2}\right\} = (K_1 + K_2e^{-p_1t} + K_3te^{-p_2t} + K_4e^{-p_2t})u(t) \qquad (Eq.44)$$

All that remains is to establish a technique for determining the coefficients (K_1, K_2, K_3,…) generated by making a partial fraction expansion.

There are four general forms this problem can take.

Specifically, the roots of $D(s)$ are either (1) real and distinct; (2) complex and distinct; (3) real and repeated; or (4) complex and repeated.

Before we consider each situation in turn, a few general comments are in order.

We used the identity sign (≡) in Eq. 1 to emphasize that expanding a rational function into a sum of partial fractions establishes an identical equation.

Thus, both sides of the equation must be the same for all values of the variable s.

Also, the identity relationship must hold when both sides are subjected to the same mathematical operation.

These characteristics are pertinent to determining the coefficients, as we will see.

Be sure to verify that the rational function is proper. This check is important because nothing in the procedure for finding the various K's will alert you to nonsense results if the rational function is improper.

We present a procedure for checking the Ks, but you can avoid wasted effort by forming the habit of asking yourself, "Is a proper rational function?"

6.1.1. Distinct Real Roots of Functions

We first consider determining the coefficients in a partial fraction expansion when all the roots of $D(s)$ are real and distinct.

To find a K associated with a term that arises because of a distinct root of $D(s)$, we multiply both sides of the identity by a factor equal to the denominator beneath the desired K.

Then, when we evaluate both sides of the identity at the root corresponding to the multiplying factor, the right-hand side is always the desired K, and the left-hand side is always its numerical value.

For example,

$$F(s) = \frac{K(s+z_1)(s+z_2)}{s(s+p_1)(s+p_2)} = \frac{K_1}{s} + \frac{K_2}{s+p_1} + \frac{K_3}{s+p_2} \qquad (Eq.45)$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{K(s+z_1)(s+z_2)}{s(s+p_1)(s+p_2)}\right\} = (K_1 + K_2 e^{-p_1 t} - K_3 e^{-p_2 t})u(t) \qquad (Eq.46)$$

6.1.2. Distinct Complex Roots of Functions

The only difference between finding the coefficients associated with distinct complex roots and finding those associated with distinct real roots is that the algebra in the former involves complex numbers.

For example, a function $F(s)$ with distinct complex roots as shown below:

$$F(s) = \frac{K(s+z)}{(s+p)(s^2+xs+y)} = \frac{K_1}{s+p} + \frac{K_2\angle\theta}{(s+a-jb)} + \frac{K_3\angle\theta}{(s+a+jb)} \qquad (Eq.47)$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{K(s+z)}{(s+p)(s^2+xs+y)}\right\} = \left(K_1 e^{-pt} + K_2 e^{-j\theta} e^{-(a-jb)t} + K_3 e^{j\theta} e^{-(a+jb)t}\right)u(t) \qquad (Eq.48)$$

Where:

$$x = 2a \qquad \text{and} \qquad y = \sqrt{a^2+b^2} \qquad (Eq.49)$$

Because distinct complex roots appear frequently in lumped-parameter linear circuit analysis, we need to summarize these results with a new transform pair.

Whenever $D(s)$ contains distinct complex roots—that is, factors of the form $(s+\alpha-j\beta)(s+\alpha+j\beta)$ - a pair of terms of the form:

$$\frac{K}{(s+\alpha-j\beta)} + \frac{K^*}{(s+\alpha+j\beta)} \qquad (Eq.50)$$

appears in the partial fraction expansion, where the partial fraction coefficient is, in general, a complex number.

In polar form,

$$K = |K|e^{j0} = |K|\angle\theta^\circ \qquad (Eq.51)$$

where $|K|$ denotes the magnitude of the complex coefficient. Then:

$$K^* = |K|e^{-j\theta} = |K|\angle -\theta^\circ \qquad (Eq.52)$$

The complex conjugate pair in equation (5) always inverse-transforms as:

$$\mathcal{L}^{-1}\left\{\frac{K}{(s+\alpha-j\beta)}+\frac{K^*}{(s+\alpha+j\beta)}\right\} = [2|K|e^{-\alpha t}\cos(\beta t+\theta)]u(t) \qquad (Eq.53)$$

In applying equation (53), it is important to note that K is defined as the coefficient associated with the denominator term $s+\alpha-j\beta$ and K^* is defined as the coefficient associated with the denominator $s+\alpha+j\beta$.

6.1.3. Repeated Real Roots of Functions

To find the coefficients associated with the terms generated by a multiple root of multiplicity r, we multiply both sides of the identity by the multiple roots raised to its rth power.

We find the K appearing over the factor raised to the rth power by evaluating both sides of the identity at the multiple roots.

To find the remaining $(r-1)$ coefficients, we differentiate both sides of the identity $(r-1)$ times.

At the end of each differentiation, we evaluate both sides of the identity at the multiple roots.

The right-hand side is always the desired K, and the left-hand side is always its numerical value.

For example, a function $F(s)$ with repeated real-roots as shown below:

$$F(s) = \frac{K(s+z)}{s(s+p)^3} = \frac{K_1}{s}+\frac{K_2}{(s+p)^3}+\frac{K_3}{(s+p)^2}+\frac{K_4}{s+p} \qquad (Eq.54)$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{K(s+z)}{s(s+p)^3}\right\} = (K_1+K_2t^2e^{-pt}+K_3te^{-pt}+K_4e^{-pt})u(t) \qquad (Eq.55)$$

6.1.4. Repeated Complex Roots of Functions

We handle repeated complex roots in the same way that we did the repeated real roots.

The only difference is that the algebra involves complex numbers.

Recall that complex roots always appear in conjugate pairs and that the coefficients associated with a conjugate pair are also conjugates, so that only half the K's need to be evaluated.

For example, a function $F(s)$ with repeated complex roots as shown below:

$$F(s) = \frac{K}{(s+a-jb)^2(s+a+jb)^2}$$

$$= \frac{K_1}{(s+a-jb)^2} + \frac{K_2}{s+a-jb} + \frac{K_1^*}{(s+a+jb)^2} + \frac{K_2^*}{s+a+jb} \qquad (Eq.56)$$

Thus, we now group the partial fraction expansion by conjugate terms to obtain:

$$F(s) = \frac{K_1}{(s+a-jb)^2} + \frac{K_2\angle\theta}{s+a-jb} + \frac{K_1^*}{(s+a+jb)^2} + \frac{K_2^*\angle\theta}{s+a+jb} \qquad (Eq.57)$$

We now write the inverse transform of $F(s)$:

$$f(t) = [2K_1 te^{-at}\cos(bt) + 2K_2 e^{-at}\cos(bt-\theta)]u(t) \qquad (Eq.58)$$

Note that if $F(s)$ has a real root a of multiplicity r in its denominator, the term in a partial fraction expansion is of the form:

$$\frac{K}{(s+a)^r} \qquad (Eq.58)$$

The inverse transform of this term is:

$$\mathcal{L}^{-1}\left\{\frac{K}{(s+a)^r}\right\} = \left[\frac{Kt^{r-1}e^{-at}}{(r-1)!}\right]u(t) \qquad (Eq.59)$$

If $F(s)$ has a complex root of $\alpha + j\beta$ of multiplicity r in its denominator, the term in partial fraction expansion is the conjugate pair:

$$\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}$$

The inverse transform of this pair is:

$$\mathcal{L}^{-1}\left\{\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}\right\} = \left[\frac{2|K|t^{r-1}}{(r-1)!}e^{-\alpha t}\cos(\beta t+\theta)\right]u(t) \qquad (Eq.60)$$

Equations (59) and (60) are the key to being able to inverse-transform any partial fraction expansion by inspection.

One further note regarding these two equations: In most circuit analysis problems, r is seldom greater than 2.

Therefore, the inverse transform of a rational function can be handled with four transform pairs.

6.2. Improper Rational Functions

We conclude the discussion of partial fraction expansions by returning to an observation made at the beginning of this section, namely, that improper rational functions pose no serious problem in finding inverse transforms.

An improper rational function can always be expanded into a polynomial plus a proper rational function.

The polynomial is then inverse-transformed into impulse functions and derivatives of impulse functions.

The proper rational function is inverse-transformed by the techniques outlined in this section.

Example 5: Inverse Laplace Transforms and Partial Fraction Expansions

For a given inverse Laplace transform, attempt the following tasks:

a. Describe the role of partial fraction expansion in the inverse Laplace transform. [2 marks]

b. Describe and solve the following partial fraction expansions:

i. Proper rational functions. [6 marks]

$$D_1(s) = \frac{s+6}{s(s+3)(s+1)^2}$$

ii. Distinct real roots of functions. [20 marks]

$$D_2(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)}$$

iii. Distinct complex roots of functions. [30 marks]

$$D_3(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$

iv. Repeated real roots of functions. [24 marks]

$$D_4(s) = \frac{100(s+25)}{s(s+5)^3}$$

v. Repeated complex roots of functions. [24 marks]

$$D_5(s) = \frac{768}{(s^2+6s+25)^2}$$

vi. Improper rational functions of functions. [10 marks]

$$D_6(s) = \frac{s^4+13s^3+66s^2+200s+300}{s^2+9s+20}$$

Solution

a. In general, we need to find the inverse transform of a function that has the form

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}$$

The coefficients a and b are real constants, and the exponents m and n are positive integers. The ratio $N(s)/D(s)$ is called a proper rational function if $m > n$ and an improper rational function if $m \leq n$. Only a proper rational function can be expanded as a sum of partial fractions.

Converting the equation into its equation with the roots.

$$F(s) = \frac{(s^n + z_n)(s^n + z_{n-1}) \dots (s + \mathrm{z})}{(s^n + p_n)(s^n + p_{n-1}) \dots (s + p)}$$

Applying partial fraction, we could expand the equation given above.

$$F(s) = \frac{K_n}{(s^n + p_n)} + \frac{K_{n-1}}{(s^n + p_{n-1})} + \cdots \frac{K_{nth}}{(s + p)}$$

By referring to the table of Laplace transform, the inverse Laplace transform could be found.

In the end, we can determine the time-domain equation of the Laplace transform pairs from the Laplace transform table.

Partial fraction expansion helps to simplify complex forms of Laplace transform equation into several standard functions that we can find its inverse Laplace transform pair from the Laplace transform table.

b. Description and illustration of the following partial fraction expansion:

i. Proper rational functions - A proper rational function is expanded into a sum of partial fractions by writing a term or a series of terms for each root of $D_1(s)$. Thus, $D_1(s)$ must be in factored form before we can make a partial fraction expansion. For each distinct root of $D_1(s)$, a single term appears in the sum of partial fractions. For each multiple root of $D_1(s)$ of multiplicity r, the expansion contains r terms.

For example, in the rational function:

$$D_1(s) = \frac{s + 6}{s(s + 3)(s + 1)^2}$$

the denominator has four roots. Two of these roots are distinct—namely, at $s = 0$ and $s = -3$. A multiple root of multiplicity 2 occurs at $s = -1$. Thus, the partial fraction expansion of this function takes the form:

$$\frac{s + 6}{s(s + 3)(s + 1)^2} \equiv \frac{K_1}{s} + \frac{K_2}{s + 3} + \frac{K_3}{(s + 1)^2} + \frac{K_4}{s + 1} \qquad (Eq. 1)$$

The key to the partial fraction technique for finding inverse transforms lies in recognizing the corresponding to each term in the sum of partial fractions. From Laplace transform pair Table, you should be able to verify that:

$$\mathcal{L}^{-1}\left\{\frac{s + 6}{s(s + 3)(s + 1)^2}\right\} = (K_1 + K_2 e^{-3t} + K_3 t e^{-t} + K_4 e^{-t})u(t) \qquad (Eq. 2)$$

All that remains is to establish a technique for determining the coefficients (K_1, K_2, K_3,…) generated by making a partial fraction expansion. There are four general forms this problem can take. Specifically, the roots of $D_1(s)$ are either (1) real and distinct; (2) complex and distinct; (3) real and repeated; or (4) complex and repeated. Before we consider each situation in turn, a few general comments are in order.

We used the identity sign (≡) in Eq. 1 to emphasize that expanding a rational function into a sum of partial fractions establishes an identical equation. Thus, both sides of the equation

must be the same for all values of the variable s. Also, the identity relationship must hold when both sides are subjected to the same mathematical operation. These characteristics are pertinent to determining the coefficients, as we will see.

Be sure to verify that the rational function is proper. This check is important because nothing in the procedure for finding the various K's will alert you to nonsense results if the rational function is improper. We present a procedure for checking the K's, but you can avoid wasted effort by forming the habit of asking yourself, "Is a proper rational function?"

ii. Distinct real roots of function - We first consider determining the coefficients in a partial fraction expansion when all the roots of $D_2(s)$ are real and distinct. To find a K associated with a term that arises because of a distinct root of $D_2(s)$, we multiply both sides of the identity by a factor equal to the denominator beneath the desired K. Then when we evaluate both sides of the identity at the root corresponding to the multiplying factor, the right-hand side is always the desired K, and the left-hand side is always its numerical value.

For example,

$$D_2(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} = \frac{K_1}{s} + \frac{K_2}{s+8} + \frac{K_3}{s+6} \qquad (Eq.1)$$

To find the value of K_1, we multiply both sides by s and then evaluate both sides at $s = 0$:

$$\frac{96(s+5)(s+12)}{(s+8)(s+6)}\bigg|_{s=0} = K_1 + \frac{K_2 s}{s+8}\bigg|_{s=0} + \frac{K_3 s}{s+6}\bigg|_{s=0}$$

or

$$\frac{96(5)(12)}{8(6)} = K_1 = 120 \qquad (Eq.2)$$

To find the value of K_2, we multiply both sides by $s+8$ and then evaluate both sides at $s = -8$:

$$\frac{96(s+5)(s+12)}{s(s+6)}\bigg|_{s=-8} = \frac{K_1(s+8)}{s}\bigg|_{s=-8} + K_2 + \frac{K_3(s+8)}{s+6}\bigg|_{s=-8}$$

or

$$\frac{96(-3)(4)}{(-8)(-2)} = K_2 = -72 \qquad (Eq.3)$$

Then, K_3 is:

$$\frac{96(s+5)(s+12)}{s(s+8)}\bigg|_{s=-6} = \frac{96(-1)(6)}{(-6)(2)} = K_3 = 48 \qquad (Eq.4)$$

From Eq. 1 and the K values obtained,

$$\frac{96(s+5)(s+12)}{s(s+8)(s+6)} = \frac{120}{s} + \frac{48}{s+8} + \frac{72}{s+6} \qquad (Eq.5)$$

At this point, testing the result to protect against computational errors is a good idea. As we already mentioned, a partial fraction expansion creates an identity; Thus, both sides of Eq. 5 must be the same for all s values.

The choice of test values is completely open; hence we choose values that are easy to verify. For example, in Eq. 5, testing at either -5 or -12 is attractive because in both cases the left-hand side reduces to zero. Choosing -5 yields:

$$\frac{120}{-5}+\frac{48}{1}-\frac{72}{3}=-24+48-24=0$$

whereas testing -12 gives:

$$\frac{120}{-12}+\frac{48}{-6}-\frac{72}{-4}=-10-8+18=0$$

Now confident that the numerical values of the various Ks are correct, we proceed to find the inverse transform:

$$\mathcal{L}^{-1}\left\{\frac{96(s+5)(s+12)}{s(s+8)(s+6)}\right\}=(120+48e^{-6t}-72e^{-8t})u(t)$$

iii. Distinct complex roots of functions - The only difference between finding the coefficients associated with distinct complex roots and finding those associated with distinct real roots is that the algebra in the former involves complex numbers.

We illustrate by expanding the rational function:

$$D_3(s)=\frac{100(s+3)}{(s+6)(s^2+6s+25)} \qquad (Eq.1)$$

We begin by noting that $D_3(s)$ is a proper rational function. Next, we must find the roots of the quadratic term $s^2+6s+25$:

$$s^2+6s+25=(s+3-j4)(s+3+j4) \quad (Eq.2)$$

With the denominator in factored form, we proceed as before:

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)}=\frac{K_1}{s+6}+\frac{K_2}{(s+3-j4)}+\frac{K_3}{(s+3+j4)} \qquad (Eq.3)$$

To find K_1, K_2, and K_3we use the same process as before:

$$K_1=\left.\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right|s=-6$$

$$=\frac{100(-3)}{25}=-12 \qquad (Eq.4)$$

$$K_2=\left.\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right|s=-3+j4$$

$$=\frac{100(j4)}{(3+j4)(j8)}$$

$$=6-j8=10e^{-j53.13^\circ} \qquad (Eq.5)$$

$$K_2 = \left.\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right|_{s=-3+j4}$$

$$= \frac{100(-j4)}{(3-j4)(-j8)}$$

$$= 6 + j8 = 10e^{j53.13^\circ} \qquad (Eq.6)$$

Then

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{-12}{s+6} + \frac{10\angle -53.13^\circ}{(s+3-j4)} + \frac{10\angle 53.13^\circ}{(s+3+j4)} \qquad (Eq.7)$$

Again, we need to make some observations. First, in physically realizable circuits, complex roots always appear in conjugate pairs. Second, the coefficients associated with these conjugate pairs are themselves conjugates.

Note, for example, that K_3 (Eq. 6) is the conjugate of K_3 (Eq. 5). Thus, for complex conjugate roots, you actually need to calculate only half the coefficients.

Before inverse-transforming Eq. 7, we check the partial fraction expansion numerically. Testing at -3 is attractive because the left-hand side reduces to zero at this value:

$$D_3(s) = \frac{-12}{3} + \frac{10\angle -53.13^\circ}{-j4} + \frac{10\angle 53.13^\circ}{j4}$$

$$= -4 + 2.5\angle 36.87^\circ + 2.5\angle -36.87^\circ$$

$$= -4 + 2.0 + j1.5 + 2.0 - j1.5 = 0$$

We now proceed to inverse-transform Eq. 7:

$$\mathcal{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\}$$

$$= \left(-12e^{-6t} + 10e^{-j53.13^\circ}e^{-(3-j4)t} + +10e^{j53.13^\circ}e^{-(3+j4)t}\right)u(t) \quad (Eq.8)$$

In general, having the function in the time domain contain imaginary components is undesirable.

Fortunately, because the terms involving imaginary components always come in conjugate pairs, we can eliminate the imaginary components simply by adding the pairs:

$$10e^{-j53.13^\circ}e^{-(3-j4)t} + +10e^{j53.13^\circ}e^{-(3+j4)t}$$

$$= 10e^{-3t}\left(e^{j(4t-j53.13^\circ)} + e^{-j(4t-53.13^\circ)}\right)$$

$$= 20e^{-3t}\cos(4t - 53.13^\circ) \qquad (Eq.9)$$

which enables us to simplify Eq. 8:

$$\mathcal{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\}$$

$$= (-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^\circ))u(t) \quad (Eq.10)$$

Because distinct complex roots appear frequently in lumped-parameter linear circuit analysis, we need to summarize these results with a new transform pair. Whenever $D(s)$

contains distinct complex roots—that is, factors of the form $(s+\alpha-j\beta)(s+\alpha+j\beta)$ - a pair of terms of the form:

$$\frac{K}{(s+\alpha-j\beta)}+\frac{K^*}{(s+\alpha+j\beta)} \qquad (Eq.11)$$

appears in the partial fraction expansion, where the partial fraction coefficient is, in general, a complex number. In polar form,

$$K=|K|e^{j0}=|K|\angle\theta° \qquad (Eq.12)$$

where $|K|$ denotes the magnitude of the complex coefficient. Then:

$$K^*=|K|e^{-j\theta}=|K|\angle-\theta° \qquad (Eq.13)$$

The complex conjugate pair in Eq. 11 always inverse-transforms as:

$$\mathcal{L}^{-1}\left\{\frac{K}{(s+\alpha-j\beta)}+\frac{K^*}{(s+\alpha+j\beta)}\right\}=[2|K|e^{-\alpha t}\cos(\beta t+\theta)]u(t) \qquad (Eq.14)$$

In applying Eq. 14 it is important to note that K is defined as the coefficient associated with the denominator term $s+\alpha-j\beta$ and K^* is defined as the coefficient associated with the denominator $s+\alpha+j\beta$.

iv. Repeated real roots of functions - To find the coefficients associated with the terms generated by a multiple root of multiplicity r, we multiply both sides of the identity by the multiple roots raised to its rth power. We find the K appearing over the factor raised to the rth power by evaluating both sides of the identity at the multiple roots. To find the remaining $(r-1)$ coefficients, we differentiate both sides of the identity $(r-1)$ times. At the end of each differentiation, we evaluate both sides of the identity at the multiple roots. The right-hand side is always the desired K, and the left-hand side is always its numerical value.

For example,

$$D_4(s)=\frac{100(s+25)}{s(s+5)^3}=\frac{K_1}{s}+\frac{K_2}{(s+5)^3}+\frac{K_3}{(s+5)^2}+\frac{K_4}{s+5} \qquad (Eq.1)$$

We find K_1 as previously described; that is,

$$K_1=\left|\frac{100(s+25)}{(s+5)^3}\right|_{s=0}=\frac{100(25)}{125}=20 \qquad (Eq.2)$$

To find K_2, we multiply both sides by $(s+5)^3$ and then evaluate both sides at $s=-5$:

$$\left|\frac{100(s+25)}{s}\right|_{s=-5}$$

$$=\left|\frac{K_1(s+5)^3}{s}\right|_{s=-5}+K_2+K_3(s+5)|_{s=-5}+K_4(s+5)^2|_{s=-5} \qquad (Eq.2)$$

Thus

$$\frac{100(20)}{(-5)}=K_1\times0+K_2+K_3\times0+K_4\times0=K_2=-400 \qquad (Eq.3)$$

To find K_3, we first must multiply both sides of Eq. 1 by $(s+5)^3$. Next, we differentiate both sides once with respect to s and then evaluate at $s=-5$:

$$\frac{d}{ds}\left[\frac{100(s+5)}{s}\right]s=-5 = \frac{d}{ds}\left[\frac{K_1(s+5)^3}{s}\right]_{s=-5} + \frac{d}{ds}[K_2]_{s=-5} + \frac{d}{ds}[K_3(s+5)]_{s=-5} + \frac{d}{ds}[K_4(s+5)^2]_{s=-5} \quad (Eq.4)$$

Knowing that differential of a function with U/V form is given as:

$$\frac{d}{ds}\left[\frac{U(s)}{V(s)}\right] = \frac{U'(s)V(s) - U(s)V'(s)}{V^2(s)}$$

Thus

$$\frac{d}{ds}\left[\frac{100(s+5)}{s}\right]_{s=-5} = 100\left[\frac{s-(s+25)}{s^2}\right]_{s=-5} = K_3 = -100 \quad (Eq.5)$$

To find K_4, we first multiply both sides of Eq. 1 by $(s+5)^3$. Next, we differentiate both sides twice with respect to s and then evaluate both sides at $s=-5$. After simplifying the first derivative, the second derivative becomes:

$$100\frac{d}{ds}\left[-\frac{25}{s^2}\right]_{s=-5} = K_1\frac{d}{ds}\left[\frac{(s+5)^2(2s-5)}{s^2}\right]_{s=-5} + 0 + \frac{d}{ds}[K_3]_{s=-5} + \frac{d}{ds}[2K_4(s+5)]_{s=-5}$$

or

$$-40 = 2K_4 \quad (Eq.6)$$

Solving Eq. 6 for K_4 gives:

$$K_4 = -20 \quad (Eq.7)$$

Then

$$\frac{100(s+25)}{s(s+5)^3} = \frac{20}{s} - \frac{400}{(s+5)^3} - \frac{100}{(s+5)^2} - \frac{20}{s+5} \quad (Eq.8)$$

At this point, we can check our expansion by testing both sides of Eq. 8 at $s=-25$. Noting both sides of Eq. 8 equal zero when $s=-25$ gives us confidence in the correctness of the partial fraction expansion. The inverse transform of Eq. 8 yields:

$$\mathcal{L}^{-1}\left\{\frac{100(s+25)}{s(s+5)^3}\right\} = (20 - 200t^2e^{-5t} - 100te^{-5t} - 20e^{-5t})u(t)$$

v. Repeated complex roots of functions - We handle repeated complex roots in the same way that we did the repeated real roots; the only difference is that the algebra involves complex numbers. Recall that complex roots always appear in conjugate pairs and that the

coefficients associated with a conjugate pair are also conjugates, so that only half the K's need to be evaluated.

For example,

$$D_5(s) = \frac{768}{(s^2 + 6s + 25)^2} \qquad (Eq.1)$$

After factoring the denominator polynomial, we write:

$$D_5(s) = \frac{768}{(s+3-j4)^2(s+3+j4)^2}$$

$$= \frac{K_1}{(s+3-j4)^2} + \frac{K_2}{s+3-j4} + \frac{K_1^*}{(s+3+j4)^2} + \frac{K_2^*}{s+3+j4} \qquad (Eq.2)$$

Now, we need to evaluate only K_1 and K_2 because K_1^* and K_2^* are conjugate values.

The value of K_1 is:

$$K_1 = \frac{768}{(s+3+j4)^2}\Big| s = -3+j4 = \frac{768}{(j8)^2} = -12 \qquad (Eq.3)$$

The value of K_2 is:

$$K_2 = \frac{d}{ds}\left[\frac{768}{(s+3+j4)^2}\right] s = -3+j4$$

$$= \frac{2(768)}{(s+3+j4)^3}\Big| s = -3+j4$$

$$= \frac{2(768)}{(j8)^3}$$

$$= -j3 = 3\angle -90° \qquad (Eq.4)$$

From Eqs. 3 and 4,

$$K_1^* = -12 \qquad (Eq.5)$$

And

$$K_2^* = j3 = 3\angle 90° \qquad (Eq.6)$$

We now group the partial fraction expansion by conjugate terms to obtain:

$$D_5(s) = \frac{-12}{(s+3-j4)^2} + \frac{3\angle -90°}{s+3-j4} + \frac{-12}{(s+3+j4)^2} + \frac{3\angle 90°}{s+3+j4} \qquad (Eq.7)$$

We now write the inverse transform of $F(s)$:

$$d_5(t) = [-24te^{-3t}\cos 4t + 6e^{-3t}\cos(4t - 90°)]u(t) \qquad (Eq.8)$$

Note that if $D_5(s)$ has a real root a of multiplicity r in its denominator, the term in a partial fraction expansion is of the form:

$$\frac{K}{(s+a)^r}$$

The inverse transform of this term is:

$$\mathcal{L}^{-1}\left\{\frac{K}{(s+a)^r}\right\} = \left[\frac{Kt^{r-1}e^{-at}}{(r-1)!}\right]u(t) \qquad (Eq.9)$$

If $D_5(s)$ has a complex root of $\alpha + j\beta$ of multiplicity r in its denominator, the term in partial fraction expansion is the conjugate pair:

$$\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}$$

The inverse transform of this pair is:

$$\mathcal{L}^{-1}\left\{\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}\right\} = \left[\frac{2|K|t^{r-1}}{(r-1)!}e^{-\alpha t}\cos(\beta t+\theta)\right]u(t) \qquad (Eq.10)$$

Equations 9 and 10 are the key to being able to inverse-transform any partial fraction expansion by inspection. One further note regarding these two equations: In most circuit analysis problems, r is seldom greater than 2. Therefore, the inverse transform of a rational function can be handled with four transform pairs. Table 2 lists these pairs.

vi. Improper rational functions - We conclude the discussion of partial fraction expansions by returning to an observation made at the beginning of this section, namely, that improper rational functions pose no serious problem in finding inverse transforms.

An improper rational function can always be expanded into a polynomial plus a proper rational function. The polynomial is then inverse-transformed into impulse functions and derivatives of impulse functions. The proper rational function is inverse-transformed by the techniques outlined in this section.

To illustrate the procedure, we use the function:

$$D_6(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20} \qquad (Eq.1)$$

Dividing the denominator into the numerator until the remainder is a proper rational function gives:

$$D_6(s) = s^2 + 4s + 10 + \left(\frac{30s + 100}{s^2 + 9s + 20}\right) \qquad (Eq.2)$$

where the term $30s + 100/(s^2 + 9s + 20)$ is the remainder.

Next, we expand the proper rational function into a sum of partial fractions:

$$\frac{30s + 100}{s^2 + 9s + 20} = \frac{30s + 100}{(s+4)(s+5)} = -\frac{20}{s+4} + \frac{50}{s+5} \qquad (Eq.3)$$

Substituting Eq. 3 into Eq. 2 yields:

$$D_6(s) = s^2 + 4s + 10 - \frac{20}{s+4} + \frac{50}{s+5} \qquad (Eq.4)$$

Now we can inverse-transform Eq. 4 by inspection. Hence:

$$d_6(t) = \frac{d^2\delta(t)}{dt^2} + 4\frac{d\delta(t)}{dt} + 10\delta(t) - (20e^{-4t} - 50e^{-5t})u(t)$$

7. Poles and Zeros

Given an inverse Laplace transform function that has the form:

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0} \qquad (Eq.61)$$

The rational function of the equation above also may be expressed as the ratio of two factored polynomials.

In other words, we may write $F(s)$as:

$$F(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_n)}{(s+p_1)(s+p_2)\ldots(s+p_m)} \qquad (Eq.62)$$

Where K is the constant a_n/b_m.

For example, we may also write the function:

$$F(s) = \frac{8s^2 + 120s + 400}{2s^4 + 20s^3 + 70s^2 + 100s + 48} \qquad (Eq.63)$$

as

$$F(s) = \frac{8(s^2 + 15s + 50)}{2(s^4 + 10s^3 + 35s^2 + 50s + 24)}$$

$$= \frac{4(s+5)(s+10)}{(s+1)(s+2)(s+3)(s+4)} \qquad (Eq.64)$$

The roots of the denominator polynomial, that is, $-p_1$, $-p_2$, $-p_3$, …, $-p_m$ are called the poles of $F(s)$. They are the values of s at which $F(s)$ becomes infinitely large. In the function described by equation (64), the poles of $F(s)$ are -1, -2, -3, and -4.

The roots of the numerator polynomial, that is, $-z_1$, $-z_2$, $-z_3$, …, $-z_n$ are called the zeros of $F(s)$. They are the values of s at which $F(s)$ becomes zero. In the function described by the equation 2, the zeros of are -5 and -10.

In what follows, you may find that being able to visualize the poles and zeros of $F(s)$ as points on a complex s-plane is helpful.

A complex plane is needed because the roots of the polynomials may be complex.

In the complex s plane, we use the horizontal axis to plot the real values of s and the vertical axis to plot the imaginary values of s.

Note that the poles and zeros for equation given below are located in the finite s-plane.

$F(s)$ can also have either an rth-order pole or an rth-order zero at infinity.

$$F(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_n)}{(s+p_1)(s+p_2)\ldots(s+p_m)} \qquad (Eq.65)$$

For example, the function described by equation below has a second-order zero at infinity, because for large values of s the function reduces to $4/s^2$ and $F(s) = 0$ when $s = \infty$.

$$F(s) = \frac{4(s+5)(s+10)}{(s+1)(s+2)(s+3)(s+4)}\Big| \; s \to \infty \cong \frac{4}{s^2} \qquad (Eq.66)$$

In this case, we are interested in the poles and zeros located in the finite s-plane.

Therefore, when we refer to the poles and zeros of a rational function of s, we are referring to the finite poles and zeros. For the given system as described by the transfer function, the poles and zeros of the system are:

- Zeros: two real zeros at s_{z1} = -5 and s_{z2} = -10.
- Poles: four real poles at s_{p1} = -1, s_{p2} = -2, s_{p3} = -3, and s_{p4} = -4.

The following diagram shows the location of the poles and zeros in the s-plane diagram for a system with the following transfer function:

$$D(s) = \frac{(s+z_1)}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)} \qquad (Eq.67)$$

Poles:

- P_1 – pole at origin.
- P_2 – simple (real) pole.
- P_3 and P_4 – complex pair poles.

Zeros:

- Z_1 – simple (real) zero.

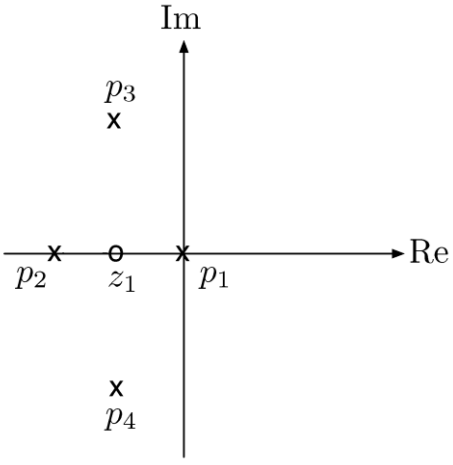


Figure 9: Poles and zeros in the s-plane diagram

The following diagram shows the location of poles and zeros in the s-diagram and their relevant transient responses.

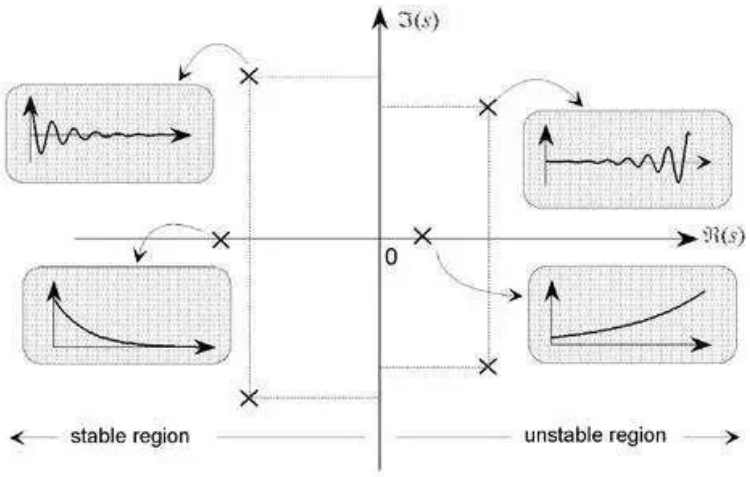


Figure 10: Poles and zeros in the s-plane diagram and their relevant transient responses

As shown in the figure above, when the pole exists on the left-hand side, the system is found to be stable and when the pole is on the right-hand side, the system is unstable. Any real pole will produce in an exponential decaying transient response, whereas complex pole will produce oscillatory transient response.

Example 6: Poles and Zeros

For given inverse Laplace transform of a given system, attempt the following tasks:

a. Describe the poles and zeros. [2 marks]

b. Determine the poles and zeros of the following system. [6 marks]

$$F_1(s) = \frac{8s^2 + 120s + 400}{2s^4 + 20s^3 + 70s^2 + 100s + 48}$$

c. For plotting the poles and zeros, consider the function: [4 marks]

$$F_2(s) = \frac{10(s+5)(s+3-j4)(s+3+j4)}{s(s+10)(s+6-j8)(s+6+j8)}$$

Solution

a. The pole is any root(s) in the denominator of the transfer function equation polynomial of the system and the zero is any root(s) in the numerator of the transfer function equation polynomial of the system.

b. Simplifying the transfer function equation of the given system

$$F_1(s) = \frac{8(s^2 + 15s + 50)}{2(s^4 + 10s^3 + 35s^2 + 50s + 24)} = \frac{4(s^2 + 15s + 50)}{(s^4 + 10s^3 + 35s^2 + 50s + 24)}$$

Then, factorising the transfer function equation of the given system, it is now:

$$F_1(s) = \frac{4(s+5)(s+10)}{(s^2+5s+6)(s^2+5s+4)} = \frac{4(s+5)(s+10)}{(s+2)(s+3)(s+1)(s+4)}$$

Thus

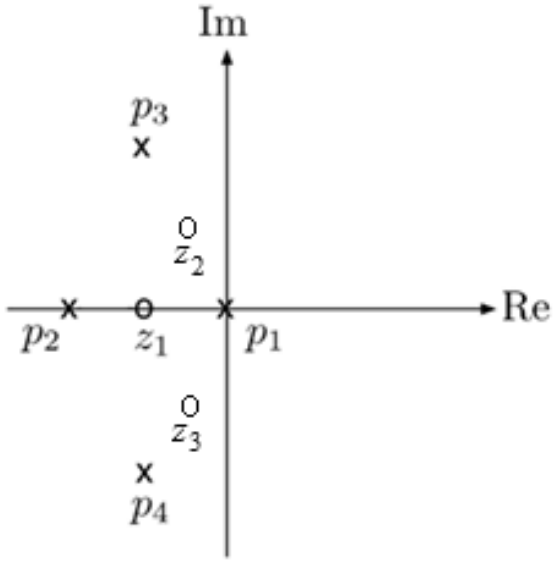
Thus, the poles and zeros of the given $F_1(s)$ system are:

- Zeros: s_{z1} = -5 and s_{z2} = -10.
- Poles: s_{p1} = -1, s_{p2} = -2, s_{p3} = -3, and s_{p4} = 4.

c. For the given system, its poles and zeros are:

- Zeros: s_{z1} = -5, $s_{z2} = -3 + j4$, and $s_{z3} = -3 - j4$.
- Poles: s_{p1} = 0, s_{p2} = -10, $s_{p3} = -6 + j8$, and $s_{p4} = -6 - j8$.

The plot of the poles and zeros of the given $F_2(s)$ system is as shown in the figure below.



8. Initial-Value and Final-Value Theorems

The initial- and final-value theorems are useful because they enable us to determine from $F(s)$ the behaviour of $f(t)$ at 0 and ∞.

Hence, we can check the initial and final values of $f(t)$ to see if they conform with known circuit behaviour, before actually finding the inverse transform of $F(s)$.

8.1. Initial-Value Theorem

The initial-value theorem states that:

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \qquad (Eq.68)$$

The initial-value theorem is based on the assumption that $f(t)$ contains no impulse functions.

In the equation for the final value theorem, we must add the restriction that the theorem is valid only if the poles of $F(s)$, except for a first-order pole at the origin, lie in the left half of the s-plane.

To prove the equation for the initial value theorem, we start with the operational transform of the first derivative:

$$\mathcal{L}^{-1}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-) = \int_0^\infty \left(\frac{df}{dt}\right) e^{-st} dt \qquad (Eq. 69)$$

Now we take the limit as $s \to \infty$:

$$\lim_{s\to\infty}[sF(s) - f(0^-)] = \lim_{s\to\infty} \int_0^\infty \left(\frac{df}{dt}\right) e^{-st} dt \qquad (Eq. 70)$$

Observe that the right-hand side of the Eq. 2 given above may be written as:

$$\lim_{s\to\infty} \left(\int_{0^-}^{0^+} \left(\frac{df}{dt}\right) e^{0} dt + \int_{0^+}^\infty \left(\frac{df}{dt}\right) e^{-st} dt \right) \qquad (Eq. 71)$$

As $s \to \infty$, $(df/dt)e^{-st} \to 0$; hence the second integral vanishes in the limit.

The first integral reduces to $f(0^+) - f(0^-)$ which is independent of s.

Thus, the right-hand side of equation (70) becomes:

$$\lim_{s\to\infty} \int_{0^-}^{0^+} \left(\frac{df}{dt}\right) e^{-st} dt = f(0^+) - f(0^-) \qquad (Eq. 72)$$

Because $f(0^-)$ is independent of s, the left-hand side of equation (70) may be written:

$$\lim_{s\to\infty}[sF(s) - f(0^-)] = \lim_{s\to\infty}[sF(s)] - f(0^-) \qquad (Eq. 73)$$

From equations (71) and (72),

$$\lim_{s\to\infty}[sF(s)] = f(0^+) = \lim_{t\to 0^-} f(t) \qquad (Eq. 74)$$

which completes the proof of the initial-value theorem.

8.2. Final-Value Theorem

The final-value theorem states that:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \qquad (Eq. 75)$$

The proof of the final-value theorem also starts with the operational transform of the first derivative.

$$\mathcal{L}^{-1}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-) = \int_0^\infty \left(\frac{df}{dt}\right) e^{-st} dt \qquad (Eq. 76)$$

Here, we take the limit as $s \to 0$:

$$\lim_{s\to 0}[sF(s) - f(0^-)] = \lim_{s\to 0} \left(\int_{0^-}^\infty \left(\frac{df}{dt}\right) e^{-st}\, dt \right) \qquad (Eq. 77)$$

The integration is with respect to t and the limit operation is with respect to s, so the right-hand side of equation (77) reduces to:

$$\lim_{s\to 0}\left(\int_{0^-}^{\infty}\left(\frac{df}{dt}\right)e^{-st}\,dt\right)=\int_{0^-}^{\infty}\left(\frac{df}{dt}\right)dt \qquad (Eq.78)$$

Because the upper limit on the integral is infinite, this integral may also be written as a limit process:

$$\int_{0^-}^{\infty}\left(\frac{df}{dt}\right)dt=\lim_{t\to\infty}\int_{0^-}^{t}\left(\frac{df}{dy}\right)dy \qquad (Eq.79)$$

where we use y as the symbol of integration to avoid confusion with the upper limit on the integral.

Carrying out the integration process yields proof of the derivation:

$$\lim_{t\to\infty}[f(t)-f(0^-)]=\lim_{t\to\infty}|f(t)|-f\,(0^-) \qquad (Eq.80)$$

Substituting equation (80) into equation (7) gives:

$$\lim_{s\to 0}[sF(s)]-f(0^-)=\lim_{t\to\infty}|f(t)|-f\,(0^-) \qquad (Eq.81)$$

Because $f(0^-)$ cancels, equation (81) reduces to the final-value theorem, namely,

$$\lim_{s\to 0}sF(s)=\lim_{t\to\infty}f(t) \qquad (Eq.82)$$

The final-value theorem is useful only if $f(\infty)$ exists. This condition is true only if all the poles of $F(s)$, except for a simple pole at the origin, lie in the left half of the s-plane.

Example 7: Initial-Value and Final-Value Theorems 1

For a given Laplace transform, attempt the following tasks.

a. Why are initial and final values theorems useful? [2 marks]

b. Describe and prove initial-value theorem. [6 marks]

$$\lim_{t\to 0}f(t)=\lim_{s\to\infty}sF(s)$$

c. Describe and prove final-value theorem. [6 marks]

$$\lim_{t\to\infty}f(t)=\lim_{s\to 0}sF(s)$$

Solution

a. The initial value theorem is useful for solving the equation for determining initial transient response of the given system. On the other hand, the final value theorem is useful for solving the equation for determining steady-state response of the given system.

b. The initial value theorems are Laplace transform techniques that find the initial values of a function (as $t \to 0$) without needing to compute the inverse Laplace transform. The initial value of $f(t)$ is found by calculating $\lim_{s\to\infty} sF(s)$.

To prove the equation for the initial value theorem, we start with the operational transform of the first derivative:

$$\mathcal{L}^{-1}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-) = \int_0^\infty \left(\frac{df}{dt}\right) e^{-st} dt \qquad (Eq.\,1)$$

Now we take the limit as $s \to \infty$:

$$\lim_{s\to\infty}[sF(s) - f(0^-)] = \lim_{s\to\infty} \int_0^\infty \left(\frac{df}{dt}\right) e^{-st} dt \qquad (Eq.\,2)$$

Observe that the right-hand side of the Eq. 2 given above may be written as:

$$\lim_{s\to\infty} \left(\int_{0^-}^{0^+} \left(\frac{df}{dt}\right) e^{0} dt + \int_{0^+}^{\infty} \left(\frac{df}{dt}\right) e^{-st} dt \right)$$

As $s \to \infty$, $(df/dt)e^{-st} \to 0$; hence the second integral vanishes in the limit.

The first integral reduces to $f(0^+) - f(0^-)$ which is independent of s.

Thus, the right-hand side of Eq. 2 becomes:

$$\lim_{s\to\infty} \int_{0^-}^{0^+} \left(\frac{df}{dt}\right) e^{-st} dt = f(0^+) - f(0^-) \qquad (Eq.\,3)$$

Because $f(0^-)$ is independent of s, the left-hand side of Eq. 2 may be written:

$$\lim_{s\to\infty}[sF(s) - f(0^-)] = \lim_{s\to\infty}[sF(s)] - f(0^-) \qquad (Eq.\,4)$$

From Eqs. 3 and 4,

$$\lim_{s\to\infty}[sF(s)] = f(0^+) = \lim_{t\to 0^-} f(t) \qquad (Eq.\,5)$$

c. The final value theorems are Laplace transform techniques that find the final values of a function (as $t \to \infty$) without needing to compute the inverse Laplace transform. The final value of $f(t)$ is found by calculating $\lim_{s\to 0} sF(s)$, provided the final value is stable (i.e. all poles of $sF(s)$ are in the left-half of the s-plane for the final value theorem).

The proof of the final-value theorem also starts with the operational transform of the first derivative.

$$\mathcal{L}^{-1}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-) = \int_0^\infty \left(\frac{df}{dt}\right) e^{-st} dt \qquad (Eq.\,1)$$

Here, we take the limit as $s \to 0$:

$$\lim_{s\to 0}[sF(s) - f(0^-)] = \lim_{s\to 0} \left(\int_{0^-}^{\infty} \left(\frac{df}{dt}\right) e^{-st}\, dt \right) \qquad (Eq.\,2)$$

The integration is with respect to t and the limit operation is with respect to s, so the right-hand side of Eq. 2 reduces to:

$$\lim_{s\to 0}\left(\int_{0^-}^{\infty}\left(\frac{df}{dt}\right)e^{-st}\,dt\right)=\int_{0^-}^{\infty}\left(\frac{df}{dt}\right)dt \qquad (Eq.3)$$

Because the upper limit on the integral is infinite, this integral may also be written as a limit process:

$$\int_{0^-}^{\infty}\left(\frac{df}{dt}\right)dt=\lim_{t\to\infty}\int_{0^-}^{t}\left(\frac{df}{dy}\right)dy \qquad (Eq.4)$$

where we use y as the symbol of integration to avoid confusion with the upper limit on the integral.

Carrying out the integration process yields proof of the derivation:

$$\lim_{t\to\infty}[f(t)-f(0^-)]=\lim_{t\to\infty}|f(t)|-f\,(0^-) \qquad (Eq.5)$$

Substituting Eq. 5 into Eq. 2 gives:

$$\lim_{s\to 0}[sF(s)]-f(0^-)=\lim_{t\to\infty}|f(t)|-f\,(0^-) \qquad (Eq.6)$$

Because $f(0^-)$ cancels, Eq. 6 reduces to the final-value theorem, namely,

$$\lim_{s\to 0}sF(s)=\lim_{t\to\infty}f(t)$$

Example 8: Initial-Value and Final-Value Theorems 2

Apply the initial and final value theorems to the following transform pair.

a. Transform pair 1. [6 marks]

$$\mathcal{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\}=(-12e^{-6t}+20e^{-3t}\cos(4t-53.13°))u(t)$$

b. Transform pair 2. [6 marks]

$$V(s)=\frac{\left(\frac{I_{dc}}{C}\right)}{s^2+\left(\frac{1}{RC}\right)s+\left(\frac{1}{LC}\right)}$$

Solution

a. For the given transform pair, we can find the solution by applying the initial-value and final value theorems.

$$\mathcal{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\}=(-12e^{-6t}+20e^{-3t}\cos(4t-53.13°))u(t)$$

The initial-value theorem gives:

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{100s^2\left[1+\frac{3}{s}\right]}{s^3\left[1+\frac{6}{s}\right]\left[1+\frac{6}{s}+\frac{25}{s^2}\right]} = 0$$

The final-value theorem gives:

$$\lim_{t\to 0} f(t) = [-12 + 20\cos(-53.13°)](1) = -12 + 12 = 0$$

In applying the theorems to the given equation, we already had the time-domain expression and were merely testing our understanding.

But, the real value of the initial- and final-value theorems lies in being able to test the s-domain expressions before working out the inverse transform.

b. We can use the initial-value and final value theorems for solving the given equation for $V(s)$.

$$V(s) = \frac{\left(\frac{I_{dc}}{C}\right)}{s^2 + \left(\frac{1}{RC}\right)s + \left(\frac{1}{LC}\right)}$$

Although we cannot calculate $v(t)$ until the circuit parameters are specified, we can check to see if $V(s)$ predicts the correct values of $v(0^+)$ and $v(\infty)$. We know from the statement of the problem that generated $V(s)$ that $v(0^+)$ is zero. We also know that $v(\infty)$ must be zero because the ideal inductor is a perfect short circuit across the dc current source.

Finally, we know that the poles of $V(s)$ must lie in the left half of the s-plane because R, L, and C are positive constants. Hence the poles of $sV(s)$ also lie in the left half of the s-plane.

Applying the initial-value theorem yields:

$$\lim_{s\to\infty} sV(s) = \lim_{s\to\infty} \frac{s\left(\frac{I_{dc}}{C}\right)}{s^2\left[1+\frac{1}{RCs}+\frac{1}{LCs^2}\right]} = 0$$

Applying the final-value theorem gives:

$$\lim_{s\to 0} sV(s) = \lim_{s\to 0} \frac{s\left(\frac{I_{dc}}{C}\right)}{s^2 + \frac{1}{RC} + \frac{1}{LC}} = 0$$

The derived expression for $V(s)$ correctly predicts the initial and final values of $v(t)$.

Appendix

Table 1: Selected functional transform pairs of Laplace transform

Type of Function	$\boldsymbol{f(t)(t>0^-)}$	$\boldsymbol{F(s)}$
Impulse	$\delta(t)$	1
Step	$u(t)$	$\frac{1}{s}$
Ramp	t	$\frac{1}{s^2}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Sine	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
Damped ramp	te^{-at}	$\frac{1}{(s+a)^2}$
Damped sine	$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
Damped cosine	$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Table 2: Selected operational transform pairs of Laplace transform

$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$	**Operation**
$Kf(t)$	$KF(s)$	Multiplication by a constant
$f_1(t)+f_2(t)+f_3(t)$	$F_1(s)+F_2(s)+F_3(s)$	Addition/subtraction
$\frac{df(t)}{dt}$	$sF(s)-f(0^-)$	First derivative (time)
$\frac{d^2f(t)}{dt^2}$	$s^2F(s)-sf(0^-)-\frac{df(0^-)}{dt}$	Second derivative (time)
$\frac{d^nf(t)}{dt^n}$	$s^nF(s)-s^{n-1}f(0^-)-s^{n-2}\frac{df(0^-)}{dt}-s^{n-3}\frac{d^2f(0^-)}{dt^2}-\cdots-\frac{d^{n-1}f(0^-)}{dt^{n-1}}$	nth derivative (time)
$\int_0^t f(x)\,dx$	$\frac{F(s)}{s}$	Time integral

$f(t-a)u(t-a),\ a>0$	$e^{-as}F(s)$	Translation in time
$f(at),\quad a>0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	Scale changing
$tf(t)$	$-\frac{dF(s)}{ds}$	First derivative (s)
$t^n f(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$	nth derivative (s)
$\frac{f(t)}{t}$	$\int_s^\infty F(u)\,du$	s integral

Table 3: Pair transformations of Laplace transform

#Pair	$\boldsymbol{F(s)}$	$\boldsymbol{f(t)}$	**Description**
1	$\frac{K}{s+a}$	$Ke^{-at}u(t)$	Distinct real
2	$\frac{K}{(s+a)^2}$	$Kte^{-at}u(t)$	Repeated real
3	$\frac{K}{s+\alpha-j\beta}+\frac{K^*}{s+\alpha+j\beta}$	$2\lvert K\rvert e^{-at}\cos(\beta t+\theta)\,u(t)$	Distinct complex
4	$\frac{K}{(s+\alpha-j\beta)^2}+\frac{K^*}{(s+\alpha+j\beta)^2}$	$2t\lvert K\rvert e^{-at}\cos(\beta t+\theta)\,u(t)$	Repeated complex