XMUT203 Analogue Circuits and Systems

Note 7b: Circuit Analysis with Laplace Transform

1. Introduction to Circuit Analysis with Laplace Transform

The Laplace transform has two characteristics that make it an attractive tool in circuit analysis:

a. First, it transforms a set of linear constant-coefficient differential equations into a set of linear polynomial equations, which are easier to manipulate.

b. Second, it automatically introduces into the polynomial equations the initial values of the current and voltage variables. Thus, initial conditions are an inherent part of the transform process.

   This contrasts with the classical approach to the solution of differential equations, in which initial conditions are considered when the unknown coefficients are evaluated.

We begin this topic by showing how we can skip the step of writing time-domain integro-differential equations and transforming them into the s-domain.

The s-domain descriptions of circuit input and output lead us to the concept of the transfer function. The transfer function for a particular circuit is the ratio of the Laplace transform of its output to the Laplace transform of its input.

We will develop the s-domain circuit models for resistors, inductors, and capacitors so that we can write s-domain equations for all circuits directly. See also topics related to the Ohm's and Kirchhoff's laws in the context of the s-domain. After establishing these fundamentals, we apply the Laplace transform method to a variety of circuit problems.

Analytical and simplification techniques first introduced with resistive circuits—such as mesh-current and node-voltage methods and source transformations—can be used in the s-domain as well.

After solving for the circuit response in the s-domain, we inverse transform back to the time domain, using partial fraction expansion (as demonstrated in the preceding chapter).

As before, checking the final time-domain equations in terms of the initial conditions and final values is an important step in the solution process.

Then, we continue with a look at the role of partial fraction expansion and the convolution integral in employing the transfer function in circuit analysis. We will then conclude with a look at the impulse function in circuit analysis.

2. Circuit Elements in the s-Domain

The procedure for developing an s-domain equivalent circuit for each circuit element is simple.

1. Write the time-domain equation that relates the terminal voltage to the terminal current.
2. Take the Laplace transform of the time-domain equation. This step generates an algebraic relationship between the s-domain current and voltage.

   Note that the dimension of a transformed voltage is Volt-Seconds, and the dimension of a transformed current is Ampere-Seconds. A voltage-to-current ratio in the s-domain carries the dimension of Volts per Ampere. An impedance in the s-domain is measured in Ohms, and an admittance is measured in Siemens.
3. Construct a circuit model that satisfies the relationship between the s-domain current and voltage. Use the passive sign convention in all the derivations.

2.1. Resistor in s-Domain

The s-domain equivalent circuit of a resistor is simply a resistance of R ohms that carries a current of I ampere-seconds and has a terminal voltage of V volt-seconds. From Ohm's law

$$v(t) = Ri(t) \qquad (Eq.1)$$

Because R is a constant, the Laplace transform of the equation above is:

$$V(s) = RI(s) \qquad (Eq.2)$$

Figure below shows the time- and frequency-domain circuits of the resistor.

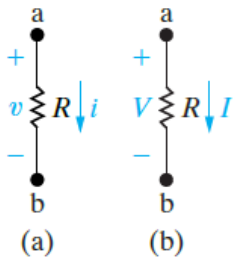


Figure 1: Resistance element (a) time domain and (b) frequency domain

Note that going from the time domain to the frequency domain does not change the resistance element.

2.2. Inductor in s-Domain

Figure below shows an inductor carrying an initial current of amperes.

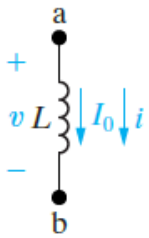


Figure 2: An inductor of L carrying an initial current of I_0

The time-domain equation that relates the terminal voltage to the terminal current is:

$$v(t) = L\frac{di}{dt} \qquad (Eq.3)$$

The Laplace transform of the equation above gives:

$$V(s) = L[sI(s) - i(0^-)] = sLI(s) - LI_0 \qquad (Eq.4)$$

Two different circuit configurations satisfy the equation above.

The first consists of an impedance of sL ohms in series with an independent voltage source of LI_0volt-seconds, as shown in the figure below. Note that the polarity marks on the voltage source LI_0 agree with the minus sign in the equation above.

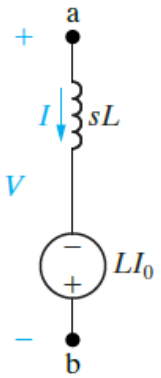


Figure 3: Series equivalent circuit for an inductor of L carrying an initial current of I_0

Note also that LI_0 carries its own algebraic sign; that is, if the initial value of $i(t)$ is opposite to the reference direction for $i(t)$, then I_0 has a negative value.

The second s-domain equivalent circuit that satisfies Eq. 4 consists of an impedance of $sL\ \Omega$ in parallel with an independent current source of I_0/s ampere-seconds, as shown in the figure below.

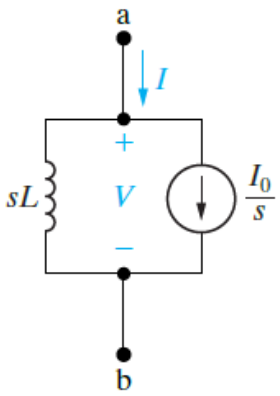


Figure 4: Parallel equivalent circuit for an inductor of L carrying an initial current of I_0

We can derive the alternative equivalent circuit shown in the figure above in several ways.

One way is simply to solve Eq. 4 for the current $I(s)$ and then construct the circuit to satisfy the resulting equation. Thus:

$$I(s) = \frac{V(s) + LI_0}{sL} = \frac{V(s)}{sL} + \frac{I_0}{s} \qquad (Eq. 5)$$

Two other ways are:

1. Find the Norton equivalent of the circuit shown in Fig. 3
2. Start with the inductor current as a function of the inductor voltage and then find the Laplace transform of the resulting integral equation.

If the initial energy stored in the inductor is zero, that is, if the s-domain equivalent circuit of the inductor reduces to an inductor with an impedance of sL ohms. Figure below shows this circuit.

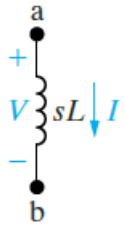


Figure 5: The s-domain circuit for an inductor when the initial current is zero

2.3. Capacitor in s-Domain

An initially charged capacitor also has two s-domain equivalent circuits.

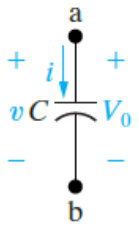


Figure 6: A capacitor of C initially charged to V_0

Figure 6 shows a capacitor initially charged to volts. The terminal current is:

$$i(t) = C\frac{dv}{dt} \qquad (Eq.6)$$

Transforming Eq. 6 yields:

$$I(s) = C[sV(s) - v(0^-)] \qquad (Eq.7a)$$

or

$$I(s) = sCV(s) - CV_0 \qquad (Eq.7b)$$

This indicates that the s-domain current $I(s)$ is the sum of two branch currents. One branch consists of an admittance of sC Siemens, and the second branch consists of an independent current source of CV_0 ampere-seconds. Figure below shows this parallel equivalent circuit.

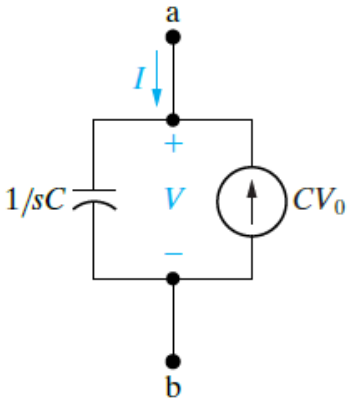


Figure 7: Parallel equivalent circuit for a capacitor initially charged to V_0

We derive the series equivalent circuit for the charged capacitor by solving equation (7b) for $V(s)$:

$$V(s) = \left(\frac{1}{sC}\right)I(s) + \frac{V_0}{s} \qquad (Eq.8)$$

Figure below shows the circuit that satisfies the equation given above.

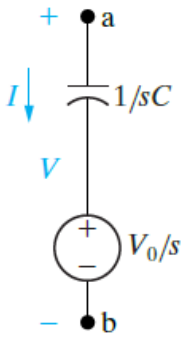


Figure 8: Series equivalent circuit for a capacitor initially charged to V_0

In the equivalent circuits shown in Figs. 7 and 8, V_0 carries its own algebraic sign. In other words, if the polarity of V_0 is opposite to the reference polarity for $v(t)$, V_0 is a negative quantity. If the initial voltage on the capacitor is zero, both equivalent circuits reduce to an impedance of $1/sC\ \Omega$, as shown in the figure below.

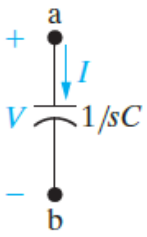


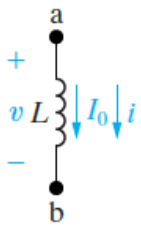
Figure 9: The s-domain circuit for a capacitor when the initial voltage is zero

In this topic, an important first problem-solving step will be to choose between the parallel or series equivalents when inductors and capacitors are present. With a little forethought and some experience, the correct choice will often be quite evident. The equivalent circuits are summarised in Table 13.1 listed in the Appendix.

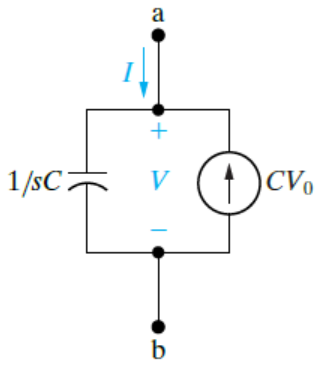
Example 1: s-domain Equivalent Circuit of Circuit Elements

For transforming circuit elements to their equivalent s-domain circuit, attempt these tasks:

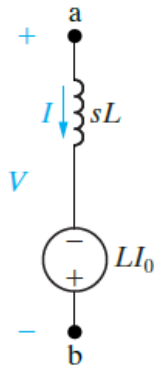
a. Derive the s-domain equivalent circuit shown in the figure below by expressing the inductor current $i(t)$ as a function of the terminal voltage and then finding the Laplace transform of this time-domain integral equation. [4 marks]



b. Find the Thévenin equivalent of the circuit shown in the figure below. [4 marks]



c. Find the Norton equivalent of the circuit shown in the figure below. [4 marks]



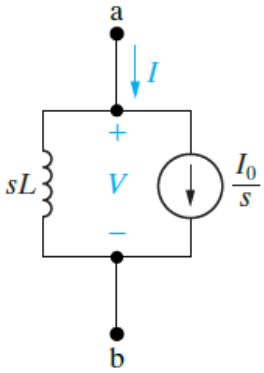
Solution

a. The current in the circuit is:

$$i = \int_0^t v\, d\tau + I_0$$

Therefore

$$I(s) = \frac{1}{L}\left(\frac{V}{s}\right) + \frac{I_o}{s} = \frac{V(s)}{sL} + \frac{I_0}{s}$$



b. The Thevenin equivalent voltage of the circuit is:

$$V_{sc(a-b)}(s) = V_{Th}(s) = CV_0\left(\frac{1}{sC}\right) = \frac{V_o}{s}$$

The Thevenin equivalent resistance of the circuit is:

$$Z_{Th}(s) = \frac{1}{sC}$$

c. The Norton equivalent current of the circuit is:

$$I_{sc(a-b)}(s) = I_N = \frac{-LI_0}{sL} = \frac{-I_0}{s}$$

The Norton equivalent resistance of the circuit is:

$$Z_N(s) = sL$$

Therefore, the Norton equivalent is the same as the circuit in the given figure.

3. Circuit Analysis in the s-Domain

Before illustrating how to use the s-domain equivalent circuits in analysis, we need to lay some groundwork.

First, we know that if no energy is stored in the inductor or capacitor, the relationship between the terminal voltage and current for each passive element takes the form:

$$V(s) = Z(s)I(s) \qquad (Eq.9)$$

Where: Z refers to the s-domain impedance of the element.

Thus, a resistor has an impedance of Ω, an inductor has an impedance of $sL\ \Omega$, and a capacitor has an impedance of $1/sC\ \Omega$. The relationship contained in the equation above is also contained in Figs. 1(b), 5, and 9. Equation 9 is sometimes referred to as Ohm's law for the s-domain.

The reciprocal of the impedance is admittance. Therefore, the s-domain admittance of a resistor is $1/R$ Siemens, an inductor has an admittance of $1/sL$ Siemens, and a capacitor has an admittance of sC Siemens.

The rules for combining impedances and admittances in the s-domain are the same as those for frequency-domain circuits. Thus, series-parallel simplifications and Δ-to-Y conversions also are applicable to s-domain analysis.

In addition, Kirchhoff's laws apply to s-domain currents and voltages. Their applicability stems from the operational transform stating that the Laplace transform of a sum of time-domain functions is the sum of the transforms of the individual functions (see the table of operational Laplace transform).

Because the algebraic sum of the currents at a node is zero in the time domain, the algebraic sum of the transformed currents is also zero. A similar statement holds for the algebraic sum of the transformed voltages around a closed path. The s-domain version of Kirchhoff's laws is:

$$alg \sum I(s) = 0 \qquad (Eq.10)$$

And

$$alg \sum V(s) = 0 \qquad (Eq.11)$$

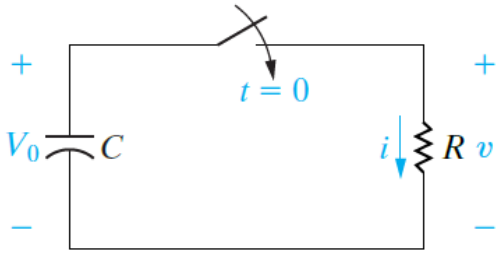
Because the voltage and current at the terminals of a passive element are related by an algebraic equation and because Kirchhoff's laws still hold, all the techniques of circuit analysis developed for pure resistive networks may be used in s-domain analysis.

Thus, node voltages, mesh currents, source transformations, and Thévenin-Norton equivalents are all valid techniques, even when energy is stored initially in the inductors and capacitors.

Initially stored energy requires that we modify equation (9) by simply adding independent sources either in series or parallel with the element impedances. The addition of these sources is governed by Kirchhoff's laws.

Example 2: Natural Response of an RC Circuit in s-domain

We first revisit the natural response of an RC circuit given below via Laplace transform techniques. The capacitor is initially charged to V_0 volts, and we are interested in the time-domain expressions for i and v.



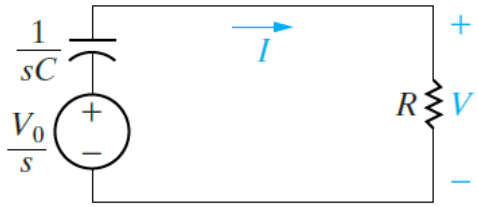
a. Determine the time-domain expressions for i and v by finding i first. [8 marks]

b. Determine the time-domain expressions for i and v without finding i first. [6 marks]

Solution

a. We start by finding i. In transferring the circuit to the s-domain, we have a choice of two equivalent circuits for the charged capacitor.

Because we are interested in the current, the series-equivalent circuit is more attractive; it results in a single-mesh circuit in the frequency domain. Thus, we construct the s-domain circuit as shown in the figure below.



Summing the voltages around the mesh generates the expression

$$\frac{V_0}{s} = \left(\frac{1}{sC}\right) I + RI$$

Solving the equation above for I yields:

$$I = \frac{cV_0}{RCs + 1} = \frac{(V_0/R)}{s + (1/RC)}$$

Note that the expression for I is a proper rational function of s and can be inverse transformed by inspection:

$$i = \left(\frac{V_0}{R}\right) e^{-t/RC} u(t)$$

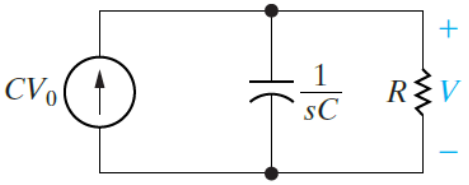
which is equivalent to the expression for the current derived by the classical methods.

After we have found i, the easiest way to determine is simply to apply Ohm's law; that is, from the circuit, it is:

$$v = Ri = V_0 e^{-t/RC} u(t)$$

b. We now illustrate a way to find from the circuit without first finding i. In this alternative approach, we return to the original circuit and transfer it to the s-domain using the parallel equivalent circuit for the charged capacitor.

Using the parallel equivalent circuit is attractive now because we can describe the resulting circuit in terms of a single node voltage. The figure below shows the new s-domain equivalent circuit.



The node-voltage equation that describes the new circuit is:

$$\frac{V}{R} + sCV = CV_0$$

Solving the equation above for V gives:

$$V = \frac{V_0}{s + (1/RC)}$$

Inverse-transforming the equation above leads to the same expression for given by the equation in part (a), namely,

$$v = V_0 e^{-t/RC} = V_0 e^{-t/\tau} u(t)$$

4. Transfer Function and Laplace Transforms

The transfer function is defined as the s-domain ratio of the Laplace transform of the output (response) to the Laplace transform of the input (source).

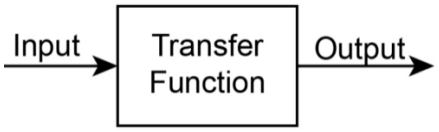


Figure 10: Transfer function equation

In computing the transfer function, we restrict our attention to circuits where all initial conditions are zero. If a circuit has multiple independent sources, we can find the transfer function for each source and use superposition to find the response to all sources. The transfer function is:

$$H(s) = \frac{Y(s)}{X(s)} \qquad (Eq.12)$$

Where: $Y(s)$ is the Laplace transform of the output signal, and $X(s)$ is the Laplace transform of the input signal.

Note that the transfer function depends on what is defined as the output signal.

Consider, for example, the series circuit shown in the figure below.

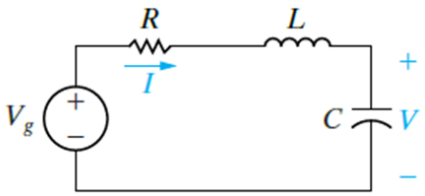


Figure 11: A series RLC circuit

Sketch the equivalent circuit after application of the Laplace transform

Considering the current is defined as the response signal of the circuit, derive the transfer function equation of the circuit $H(s) = I(s)/V_g(s)$.

Taking the Laplace transform of the series circuit, its s-domain equivalent circuit is as shown in the figure below.

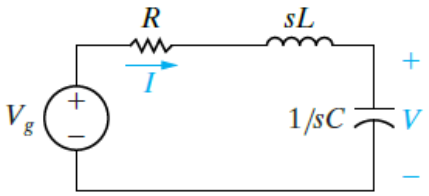


Figure 12: Equivalent s-domain circuit of the series RLC circuit

If the current is defined as the response signal of the circuit, the transfer function of the circuit is:

$$H(s) = \frac{I(s)}{V_g(s)} = \frac{1}{R + sL + 1/sC} = \frac{sC}{s^2LC + RCs + 1} \qquad (Eq. 13)$$

In deriving the equation above, we recognized that $I(s)$ corresponds to the output $Y(s)$ and $V_g(s)$ corresponds to the input $X(s)$.

If the voltage across the capacitor is defined as the output signal of the circuit shown in the figure above, the transfer function is:

$$H(s) = \frac{V(s)}{V_g(s)} = \frac{1/sC}{R + sL + 1/sC} = \frac{1}{s^2LC + RCs + 1} \qquad (Eq. 14)$$

Thus, because circuits may have multiple sources and because the definition of the output signal of interest can vary, a single circuit can generate many transfer functions.

Remember that when multiple sources are involved, no single transfer function can represent the total output—transfer functions associated with each source must be combined using superposition to yield the total response.

5. The Location of Poles and Zeros of $H(s)$

For linear lumped-parameter circuits, $H(s)$ is always a rational function of s.

$$H(s) = \frac{(s + z_1)(s + z_2)\ldots(s + z_n)}{(s + p_1)(s + p_2)\ldots(s + p_n)} \qquad (Eq. 15)$$

Complex poles and zeros always appear in conjugate pairs. The poles of $H(s)$ must lie in the left half of the s-plane if the response to a bounded source (one whose values lie within some finite bounds) is to be bounded. The zeros of $H(s)$ may lie in either the right half or the left half of the s plane.

Given a circuit with the following transfer function:

$$H(s) = \frac{(s - 3 + j4)(s - 3 - j4)(s + 5)}{s(s + 10)(s - 6 + j8)(s - 6 - j8)} \qquad (Eq. 16)$$

Thus, the poles and zeros in s-plane diagram are:

- Zeros:

  $s = -5$, $s = -3 + j4$, and $s = -3 - j4$.

- Poles:

$s = 0$, $s = -10$, $s = -6 + j8$, and $s = -6 - j8$.

The following diagram shows the locations of these poles and zeros in the s-plane diagram.

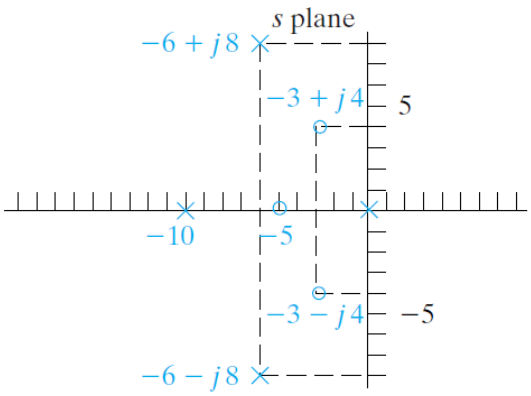


Figure 13: Poles and zeros in the s-plane diagram

With these general characteristics in mind, we next discuss the role that $H(s)$ plays in determining the response function. We begin with the partial fraction expansion technique for finding $y(t)$.

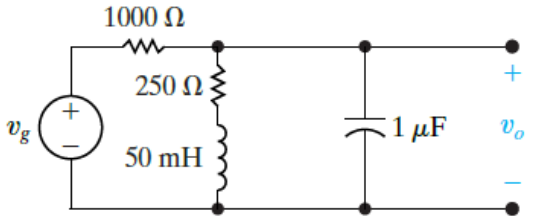
Example 3: Analysing Transfer Function and Poles and Zeros of a Circuit

The voltage source v_g drives the circuit shown in the figure below. The response signal is the voltage across the capacitor, v_o.

a. Sketch the equivalent circuit after applying Laplace transform and calculate the numerical expression for the transfer function. [6 marks]

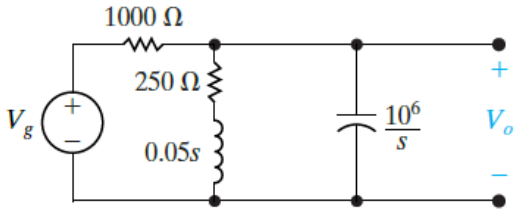
b. Calculate the numerical values for the poles and zeros of the transfer function. [6 marks]

c. Sketch the pole-zero diagram of the circuit. Analyse the characteristics of the circuit. [4 marks]



Solution

a. The first step in finding the transfer function is to construct the s-domain equivalent circuit, as shown in the figure below.



By definition, the transfer function is the ratio of V_o/V_g, which can be computed from a single node-voltage equation.

Summing the currents away from the upper node generates:

$$\frac{(V_o - V_g)}{1000} + \frac{V_o}{(250 + 0.05s)} = \frac{V_o s}{10^6} = 0$$

Solving for V_o yields:

$$V_o = \frac{1000(s + 5000)V_g}{s^2 + 6000s + 25 \times 10^6}$$

Hence the transfer function is:

$$H(s) = \frac{V_o}{V_g} = \frac{1000(s + 5000)}{s^2 + 6000s + 25 \times 10^6}$$

b. The poles of $H(s)$ are the roots of the denominator polynomial. Therefore

$$p_1 = -3000 - j4000$$

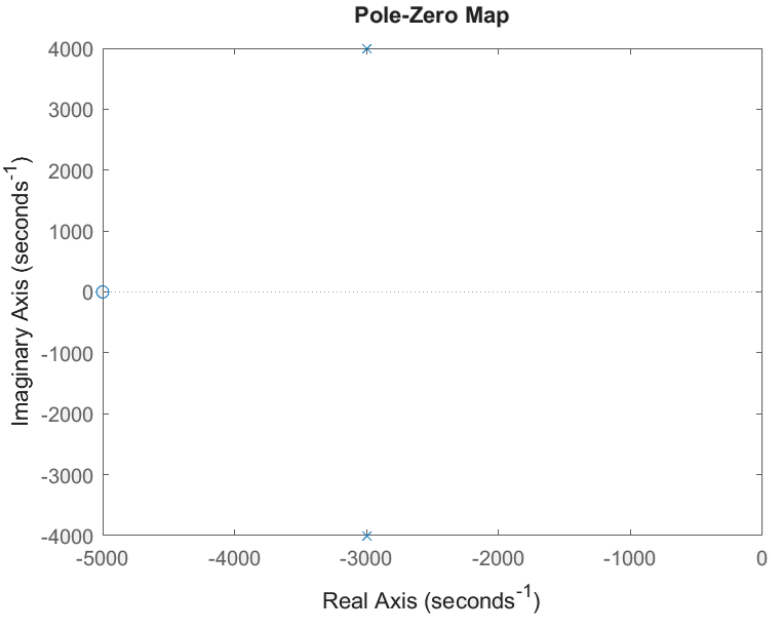
And

$$p_2 = -3000 + j4000$$

The zeros of $H(s)$ are the roots of the numerator polynomial; thus $H(s)$ has a zero at:

$$z_1 = -5000$$

c. The poles and zero diagram of the circuit is as shown in the figure below.



MATLAB Code:

```
s=tf('s');
G=(1000*(s+5000))/(s^2+6000*s+25e6);
pzmap(G);
```

With a pair of complex poles and a real zero, transient response of the circuit is underdamped, and its frequency response features a peak on its cut-off frequencies.

6. Initial Conditions and Their Laplace Transforms

With circuit that has initial conditions, these could be solved using Laplace transforms. There are typically two cases of the circuit with initial conditions.

The first case is the circuit with some leftover energy in the capacitor that exists as electric fields across the plates in the capacitor. This energy will be responsible for the initial voltage that presents in the capacitor. Notice the second term in the equation below.

$$I(s) = C[sV(s) - v(0^-)] = sCV(s) - CV_0 \qquad (Eq.17)$$

Or

$$V(s) = \left(\frac{1}{sC}\right) I(s) + \frac{V_0}{s} \qquad (Eq.18)$$

The second case is the energy left behind the inductor that exists as magnetic flux in the inductor. This causes the initial current that flows across the inductor. Notice the second term in the equation below.

$$V(s) = L[sI(s) - i(0^-)] = sLI(s) - LI_0 \qquad (Eq.19)$$

Or

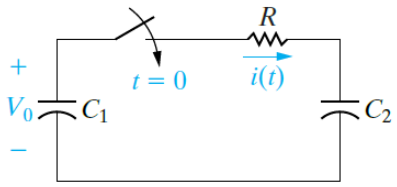
$$I(s) = \frac{V(s) + LI_0}{sL} = \frac{V(s)}{sL} + \frac{I_0}{s} \qquad (Eq.20)$$

To simplify the calculation for circuit analysis, we will be mostly dealing with zero initial conditions of the circuit in the course.

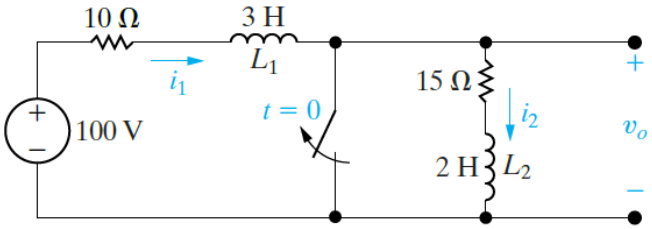
Example 4: Application of Laplace Transform in Circuit 1

For the two RC and RL circuits with initial conditions given below, derive the relevant time domain equations of the circuits.

a. In the circuit shown in the figure below, the capacitor C_1 is charged to an initial voltage of V_0 at the time the switch is closed. The initial charge on C_2 is zero. Find the expression for $i(t)$.
[6 marks]



b. The circuit shown in the figure below illustrates a second switching operation that produces an impulsive response. Determine the time-domain expression for v_o after the switch has been opened. Given that the current in the 3 H inductor L_1 at $t = 0^-$ is 10 A, and the current in the 2 H inductor L_2 at $t = 0^-$ is zero. [14 marks]

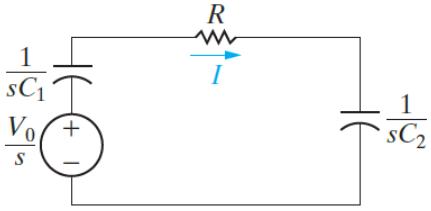


Solution

a. Notice that with the capacitor C_1 is charged to an initial voltage of V_0 at the time the switch is closed and the initial charge on C_2 is zero.

$$V_{C1}(0) = \left(\frac{1}{sC_1}\right) I(s) + \frac{V_0}{s} \qquad (Eq.1)$$

Thus, the figure below shows the s-domain equivalent circuit.



From the equivalent circuit given above, the current in the circuit is:

$$I = \frac{V}{Z_T} = \frac{(V_0/s)}{R + (1/sC_1) + (1/sC_2)} = \frac{(V_0/R)}{s + (1/RC_e)} \quad (Eq.2)$$

Where: $C_e = C_1C_2/(C_1 + C_2)$.

We inverse-transform the equation given above by inspection to obtain:

$$i = \left[\left(\frac{V_0}{R}\right)e^{-\frac{t}{RC}}\right]u(t) \quad (Eq.3)$$

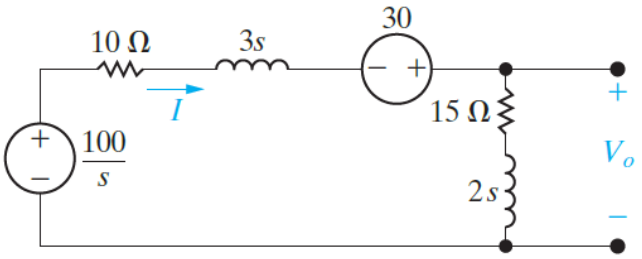
b. Note that opening the switch forces an instantaneous change in the current of L_2 which causes v_o to contain an impulsive component.

$$V(s) = sLI(s) - LI_0 \quad (Eq.4)$$

Thus

$$V(s) = s(3\text{ H})I(s) - (3\text{ H})(10\text{ A}) = 3sI(s) - 30 \quad (Eq.5)$$

As a result, the figure below shows the s-domain equivalent with the switch open.



In deriving this circuit, we recognised that the current in the 3 H inductor at $t = 0^-$ is 10 A, and the current in the 2 H inductor at $t = 0^-$ is zero.

Using the initial conditions at $t = 0^-$ is a direct consequence of our using 0^- as the lower limit on the defining integral of the Laplace transform. We derive the expression for V_o from a single node-voltage equation. Summing the currents away from the node between the 15 Ω resistor and the 30 V source gives:

$$\frac{V_o}{2s + 15} + \frac{V_o - [(100/s) + 30]}{3s + 10} = 0 \quad (Eq.6)$$

Solving for V_o yields:

$$V_o = \frac{40(s + 7.5)}{s(s + 5)} + \frac{12(s + 7.5)}{(s + 5)} \quad (Eq.7)$$

We anticipate that v_o will contain an impulse term because the second term on the right-hand side of Eq. 7 is an improper rational function. We can express this improper fraction as a constant plus a rational function by simply dividing the denominator into the numerator; that is,

$$\frac{12(s + 7.5)}{(s + 5)} = 12 + \frac{30}{(s + 5)} \quad (Eq.8)$$

Combining Eq. 8 with the partial fraction expansion of the first term on the right-hand side of Eq. 7 gives:

$$V_o = \frac{60}{s} - \frac{20}{(s+5)} + 12 + \frac{30}{(s+5)} = 12 + \frac{60}{s} + \frac{10}{(s+5)} \quad (Eq.9)$$

from which:

$$v_o = 12\delta(t) + (60 + 10e^{-5t})u(t) \quad \text{V} \quad (Eq.10)$$

7. Partial Fraction Expansions and Their Uses

From equation (12), we can write the circuit output as the product of the transfer function and the driving function:

$$Y(s) = H(s)X(s) \qquad (Eq.21)$$

We have already noted that $H(s)$ is a rational function of s. Reference to Table 1 in the Appendix shows that $X(s)$ also is a rational function of s for the excitation functions of most interest in circuit analysis.

Expanding the right-hand side of the equation above into a sum of partial fractions produces a term for each pole of $H(s)$ and $X(s)$.

Remember from introduction to Laplace transform topic that poles are the roots of the denominator polynomial; zeros are the roots of the numerator polynomial.

The terms generated by the poles of $H(s)$ give rise to the transient component of the total response, whereas the terms generated by the poles of give rise to the steady-state component of the response.

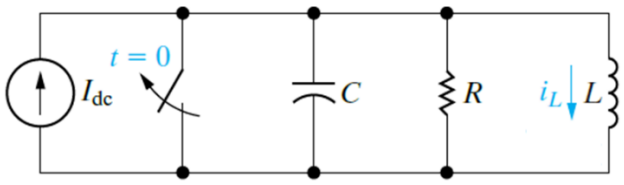
By steady-state response, we mean the response that exists after the transient components have become negligible.

Roots	$\boldsymbol{F(s)}$	$\boldsymbol{f(t)}$
Distinct real	$\frac{K}{s+a}$	$Ke^{-at}u(t)$
Repeated real	$\frac{K}{(s+a)^2}$	$Kte^{-at}u(t)$
Distinct complex	$\frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta}$	$2\lvert K\rvert e^{-\alpha t}\cos(\beta t+\theta)\,u(t)$
Repeated complex	$\frac{K}{(s+\alpha-j\beta)^2} + \frac{K^*}{(s+\alpha+j\beta)^2}$	$2t\lvert K\rvert e^{-\alpha t}\cos(\beta t+\theta)\,u(t)$

Table 1: Useful pairs of forward and inverse Laplace transform

Example 5: Application of Laplace Transform in Circuit 2

In this example, we analyse the parallel RLC circuit, shown in the figure below, that we analysed before in the topic on the second-order RLC circuit. The goal is to find the expression for i_L after the constant current source is switched across the parallel elements. The initial energy stored in the circuit is zero.



a. Construct the equivalent circuit in the s-domain. [4 marks]

b. Derive the equation for $I_L(s)$ in the circuit using Laplace transform. [8 marks]

c. If I_{dc} = 24 mA, C = 25 nF, R = 625 Ω, and L = 25 mH, determine the equation for the $I_L(s)$. [4 marks]

d. Sketch the poles and zeros diagram of the circuit on the s-plane. [4 marks]

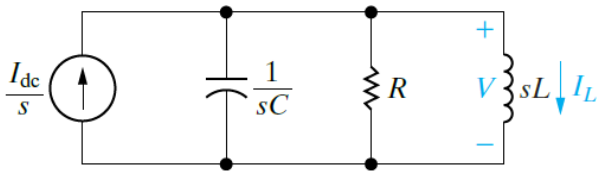
e. Using final value theorem in Laplace transform, establish the steady-state condition for the $i_L(\infty)$. [4 marks]

f. Apply partial fraction expansion and determine the expression for the $I_L(s)$. [8 marks]

g. Taking inverse Laplace transform, determine the equation for $i_L(t)$. [2 marks]

Solution

a. We begin by constructing the s-domain equivalent circuit as shown in the figure below.



Note how easily an independent source can be transformed from the time domain to the frequency domain. We transform the source to the s-domain simply by determining the Laplace transform of its time-domain function.

Here, opening the switch results in a step change in the current applied to the circuit. Therefore, the s-domain current source is $\mathcal{L}\{I_{dc}u(t)\}$, or I_{dc}/s.

b. To find I_L, we first solve for V and then use the following equation to establish the s-domain expression for I_L.

$$I_L = \frac{V}{sL}$$

Summing the currents away from the top node generates the expression:

$$sCV + \frac{V}{R} + \frac{V}{sL} = \frac{I_{dc}}{s}$$

Solving the equation above for V gives:

$$V = \frac{(I_{dc}/C)}{s^2 + (1/RC)s + (1/LC)}$$

Substituting the equation above into the previous equation for I_L for V gives:

$$I_L = \frac{(I_{dc}/LC)}{s[s^2 + (1/RC)s + (1/LC)]}$$

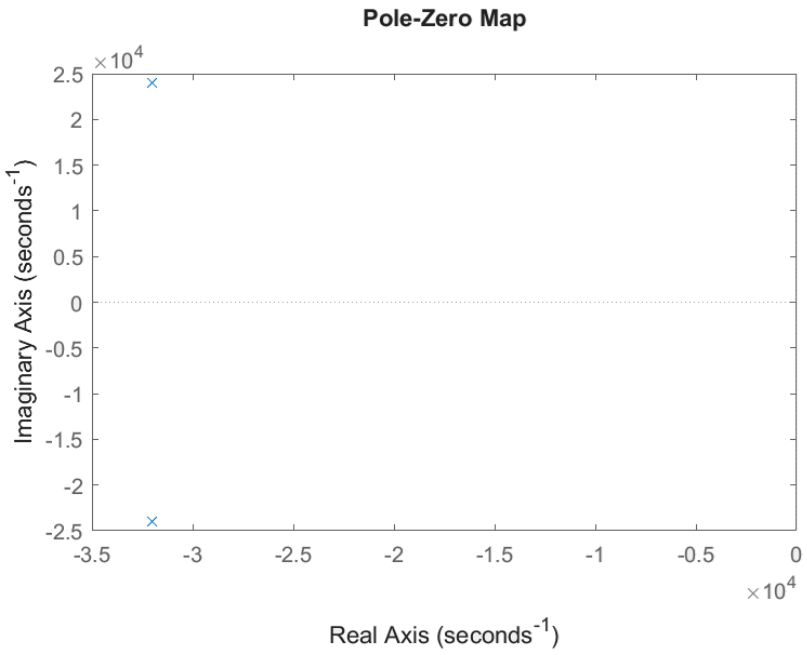
c. Substituting the numerical values of R, L, C, and into the equation above yields:

$$I_L = \frac{384 \times 10^5}{s(s^2 + 64{,}000s + 16 \times 10^8)}$$

Before expanding the equation above into a sum of partial fractions, we factorise the quadratic term in the denominator:

$$I_L = \frac{384 \times 10^5}{s(s + 32{,}000 - j24{,}000)(s + 32{,}000 + j24{,}000)}$$

d. Using the equation from part (c), the following diagram shows the pole-zero diagram of the circuit in the s-plane.



All the poles of I_L, except for the first-order pole at the origin, lie in the left half of the s-plane, so the theorem is applicable.

e. Now, we can test the s-domain expression for by checking to see whether the final-value theorem predicts the correct value for i_L at $t = \infty$.

We know from the behaviour of the circuit that after the switch has been open for a long time, the inductor will short-circuit the current source. Therefore, the final value of I_L must be 24 mA. The limit of sI_L as $s \to 0$ is:

$$\lim_{s\to 0} sI_L = s\left[\frac{384\times 10^5}{s(s+32{,}000-j24{,}000)(s+32{,}000+j24{,}000)}\right] = \frac{384\times 10^5}{16\times 10^8}$$

Currents in the s domain carry the dimension of ampere-seconds, so the dimension of sI_L will be amperes. Thus, our s-domain expression checks out.

f. We now proceed with the partial fraction expansion of the above equation:

$$I_L = \frac{K_1}{s} + \frac{K_2}{(s+32{,}000-j24{,}000)} + \frac{K_2^*}{(s+32{,}000+j24{,}000)}$$

The partial fraction coefficients are:

$$K_1 = \left.\frac{384\times 10^5}{(s+32{,}000-j24{,}000)(s+32{,}000+j24{,}000)}\right|_{s\to 0} = \frac{384\times 10^5}{16\times 10^8} = 24\times 10^{-3}$$

And

$$K_2 = \left.\frac{384\times 10^5}{s(s+32{,}000+j24{,}000)}\right|_{s\to -32{,}000+j24{,}000} = \frac{384\times 10^5}{(-32{,}000+j24{,}000)(j48{,}000)}$$

$$= 20\times 10^{-3}\angle 126.87°$$

Substituting the numerical values of and into the equation for I_L, it becomes:

$$I_L = \frac{24\times 10^{-3}}{s} + \frac{20\times 10^{-3}\angle 126.87°}{(s+32{,}000-j24{,}000)} + \frac{20\times 10^{-3}\angle -126.87°}{(s+32{,}000+j24{,}000)}$$

g. Inverse-transforming the resulting expression yields:

$$i_L = [24 + (40)e^{32{,}000t}\cos(24{,}000t + 126.87°)]\times 10^{-3}u(t) \quad \text{A}$$

8. Observations on the Use of $H(s)$ in Circuit Analysis

Example 5 clearly shows how the transfer function relates to the response of a circuit through a partial fraction expansion. However, the example raises questions about the practicality of driving a circuit with an increasing ramp voltage that generates an increasing ramp response.

Eventually the circuit components will fail under the stress of excessive voltage, and when that happens our linear model is no longer valid.

The ramp response is of interest in practical applications where the ramp function increases to a maximum value over a finite time interval. If the time taken to reach this maximum value is long

compared with the time constants of the circuit, the solution assuming an unbounded ramp is valid for this finite time interval.

We make two additional observations regarding equation (15).

First, let's look at the response of the circuit due to a delayed input. If the input is delayed by a second,

$$\mathcal{L}\{x(t-a)u(t-a)\} = e^{-as}X(s) \qquad (Eq.22)$$

and, from equation (15), the response becomes:

$$Y(s) = H(s)X(s)e^{-as} \qquad (Eq.23)$$

If $y(t) = \mathcal{L}^{-1}\{H(s)Y(s)\}$ then, from equation (17),

$$y(t-a)u(t-a) = \mathcal{L}^{-1}\{H(s)Y(s)e^{-as}\} \qquad (Eq.24)$$

Therefore, delaying the input by a seconds, simply delays the response function by a seconds. A circuit that exhibits this characteristic is said to be time invariant.

Second, if a unit impulse source drives the circuit, the response of the circuit equals the inverse transform of the transfer function. Thus, if

$$x(t) = \delta(t) \qquad (Eq.25)$$

Then

$$X(s) = 1 \qquad (Eq.26)$$

And

$$Y(s) = H(s) \qquad (Eq.27)$$

Hence, from the equation given above,

$$y(t) = h(t) \qquad (Eq.28)$$

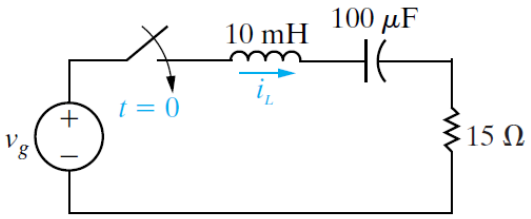
Where the inverse transform of the transfer function equals the unit impulse response of the circuit.

Note that this is also the natural response of the circuit because the application of an impulsive source is equivalent to instantaneously storing energy in the circuit. The subsequent release of this stored energy gives rise to the natural response.

Actually, the unit impulse response of a circuit, contains enough information to compute the response to any source that drives the circuit. The convolution integral is used to extract the response of a circuit to an arbitrary source.

Example 6: Application of Laplace Transform in Circuit 3

Consider the RLC circuit shown below, comprised of E12 standard value components and powered by a 60 Hz sinusoidal source. The 10 mH inductor has a current rating of 40 mA. The amplitude of the sinusoidal source has been chosen so that this rating is met in the steady state.



a. Using KVL, derive the equation for the circuit. [2 marks]

b. Derive the equation for $I_L(s)$ by applying Laplace transform. [4 marks]

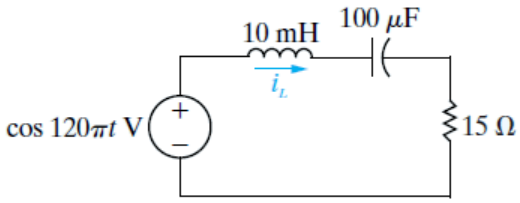
c. Using partial fraction expansion, determine the equation for $I_L(s)$ in the circuit. [6 marks]

d. Applying inverse Laplace transform, determine the equation for $i_L(t)$. Based on the result, evaluate the characteristics of the circuit. [4 marks]

e. Calculate the current $i_L(t)$ at 0^+ and 1 ms. Determine whether or not the current rating for i_L is exceeded when the source is first switched on and both the transient and steady-state components of the inductor current are active. [6 marks]

Solution

a. The figure below shows the equivalent circuit of the circuit given in the figure above with the switch closed and the chosen sinusoidal source.

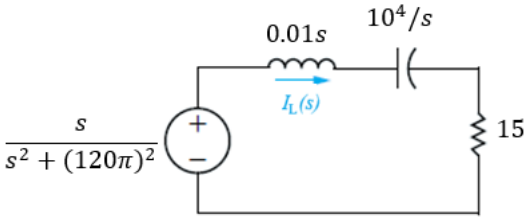


We use the Laplace methods to determine the complete response of the inductor current, $i_L(t)$. To begin, use KVL to sum the voltages drops around the circuit, in the clockwise direction:

$$15i_L(t) + 0.01\left[\frac{di_L(t)}{dt}\right] + \frac{1}{100 \times 10^{-6}}\int_0^t i_L(x)\,dx = \cos 120\pi t \qquad (Eq.1)$$

b. Now, we take the Laplace transform of Eq. 1, using Laplace transform pairs tables:

$$15i_L(s) + 0.01si_L(s) + 10^4\left[\frac{i_L(s)}{s}\right] = \frac{s}{s^2 + (120\pi)^2} \qquad (Eq.2)$$



Next, rearrange the terms in Eq. 2 to get an expression for $i_L(s)$:

$$i_L(s) = \frac{100s^2}{[s^2 + 1500s + 10^6][s^2 + (120\pi)^2]} \qquad (Eq.3)$$

c. Note that the expression for $i_L(s)$ has two complex conjugate pairs of poles, so the partial fraction expansion of $i_L(s)$ will have four terms:

$$i_L(s) = \frac{K_1}{(s + 750 - j661.44)} + \frac{K_1^*}{(s + 750 + j661.44)} + \frac{K_2}{(s - j120\pi)} + \frac{K_2^*}{(s + j120\pi)} \qquad (Eq.4)$$

Determine the values of K_1 and K_2:

$$K_1 = \frac{100s^2}{[s + 750s + j661.44][s^2 + (120\pi)^2]}\Bigg| s = -750 + j661.44$$

$$= 0.07357\angle -97.89° \qquad (Eq.5)$$

And

$$K_2 = \frac{100s^2}{[s^2 + 1500s + 10^6][s + 120\pi]}\Bigg| s = j120\pi$$

$$= 0.018345\angle 56.61° \qquad (Eq.6)$$

d. Finally, we can use Laplace transform inverse table to calculate the inverse Laplace transform of Eq. 4 to give $i_L(t)$:

$$i_L(t) = 147.14e^{750t}\cos(661.44t - 97.89°) + 36.69\cos(120\pi t - 56.61°)\ \text{mA} \qquad (Eq.7)$$

The first term of Eq. 7 is the transient response, which will decay to essentially zero in about 7 ms.

The second term of Eq. 7 is the steady-state response, which has the same frequency as the 60 Hz sinusoidal source and will persist so long as this source is connected in the circuit.

Note that the amplitude of the steady-state response is 36.69 mA, which is less than the 40 mA current rating of the inductor.

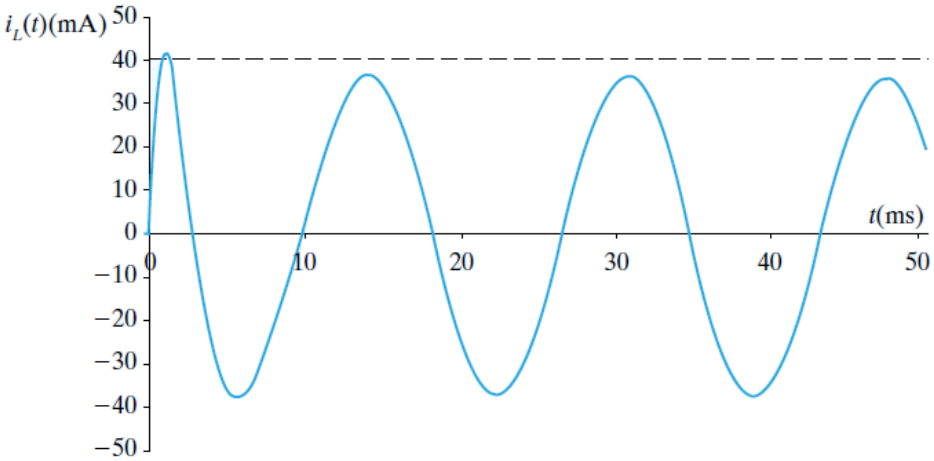
But, the transient response has an initial amplitude of 147.14 mA, far greater than the 40 mA current rating.

e. Calculate the value of the inductor current at 0^+ s:

$$i_L(0) = 147.14\ (1)\cos(-97.89°) + 36.69\cos(56.61°) = -6.21\ \text{mA}$$

Clearly, the transient part of the response does not cause the inductor current to exceed its rating initially.

But, we need a plot of the complete response to determine whether or not the current rating is ever exceeded, as shown in the figure below.



The plot suggests we check the value of the inductor current at 1 ms:

$$i_L(0.001) = 147.14\, e^{-0.75}\cos(-59.82°) + 36.69\cos(78.21°) = 42.6 \text{ mA}$$

Thus, the current rating is exceeded in the inductor, at least momentarily.

If we determine that we never want to exceed the current rating, we should reduce the magnitude of the sinusoidal source.

This example illustrates the importance of considering the complete response of a circuit to a sinusoidal input, even if we are satisfied with the steady-state response.

Appendix

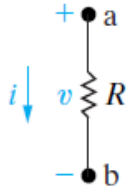
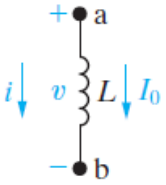
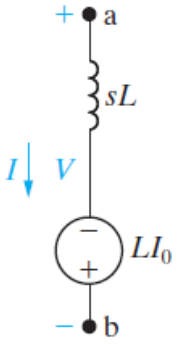
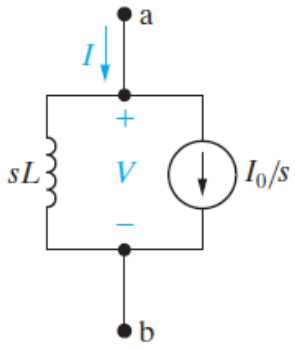
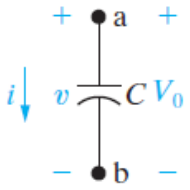
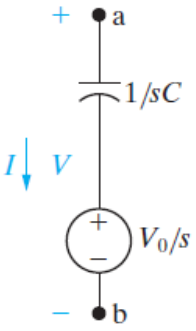
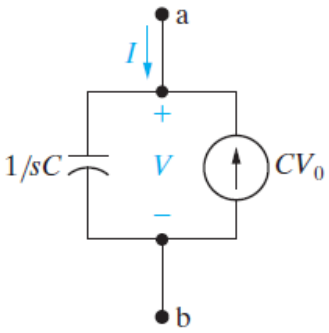
Time Domain	Frequency Domain	
		
$v = Ri$	$V = RI$	
		
$v = L\left(\frac{di}{dt}\right)$ $i = \frac{1}{L}\int_0^t v\,dx + I_o$	$V = sLI - LI_0$	$I = \frac{V}{sL} + \frac{I_0}{s}$
		
$i = C\left(\frac{dv}{dt}\right)$ $v = \frac{1}{C}\int_0^t i\,dx + V_o$	$v = \frac{1}{sC} + \frac{V_0}{s}$	$I = sCV - CV_0$

Table 1: Summary of the s-domain equivalent circuits