XMUT203 Analogue Circuits and Systems

Exam Formulae (Second Half)

A. RLC Circuits

Parameters	**Equations**
Characteristic equation	$s^2 + 2\alpha s + \omega_0^2 = 0$ Where: $\alpha = \frac{1}{2RC}$ (parallel circuit) $\alpha = \frac{R}{2L}$ (series circuit) $\omega_0^2 = \frac{1}{LC}$ (both parallel and series circuit) $\omega_d = \sqrt{\omega_0^2 - 1}$ (damped frequency)
Roots of the equation	$s_{1,2} = \alpha \pm \sqrt{\alpha^2 - \omega_0^2}$
Response of circuit	$\alpha^2 > \omega_0^2$ (overdamped) $\alpha^2 < \omega_0^2$ (underdamped) $\alpha^2 = \omega_0^2$ (critically damped)
Natural response	$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ (overdamped) $x(t) = (B_1 cos\ \omega_d t + B_2 sin\ \omega_d t) e^{-\alpha t}$ (underdamped) $x(t) = (D_1 t + D_2) e^{\alpha t}$ (critically damped)
Step response	$x(t) = X_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}$ (overdamped) $x(t) = X_f + (B_1' cos\ \omega_d t + B_2' sin\ \omega_d t) e^{-\alpha t}$ (underdamped) $x(t) = X_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}$ (critically damped)

B. Resonant RLC Circuits

Parameters	**Series**	**Parallel**
Resonant Frequency (ω_0)	$\frac{1}{\sqrt{LC}}$	$\frac{1}{\sqrt{LC}}$
Quality Factor (Q)	$\frac{1}{R}\sqrt{\frac{L}{C}}$	$R\sqrt{\frac{C}{L}}$

Bandwidth (B)	$\frac{\omega_0}{Q}$	$\frac{\omega_0}{Q}$
Cut-off Frequencies ($\omega_{1,2}$)	$\omega_0\sqrt{1+\left(\frac{1}{2Q}\right)^2} \pm \frac{\omega_0}{2Q}$	$\omega_0\sqrt{1+\left(\frac{1}{2Q}\right)^2} \pm \frac{\omega_0}{2Q}$
For Cut-off Frequencies ($\omega_{1,2}$) with $Q \geq 10$	$\omega_0 \pm \frac{\omega_0}{2Q}$	$\omega_0 \pm \frac{\omega_0}{2Q}$

C. Laplace Transform

Signal	**Laplace Transform**	**ROC**
$\delta(t)$	1	All s
$\delta^{(n)}(t)$	s^n	All s
$u(t)$	$\frac{1}{s}$	Re(s)>0
$e^{-at}u(t)$	$\frac{1}{s+a}$	Re(s)>-Re(a)
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	Re(s)>-Re(a)
$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	Re(s)>-Re(a)
$\sin(\omega t)\,u(t)$	$\frac{\omega}{s^2+\omega^2}$	Re(s)>0
$\cos(\omega t)\,u(t)$	$\frac{s}{s^2+\omega^2}$	Re(s)>0
$e^{-at}\sin(\omega t)\,u(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	Re(s)>-Re(a)
$e^{-at}\cos(\omega t)\,u(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$	Re(s)>-Re(a)

D. Properties of Laplace Transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\,e^{-st}dt \qquad \text{and} \qquad \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)\,e^{st}ds$$

Property	$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
t-scaling	$f(ct)$	$\left(\frac{1}{\lvert c\rvert}\right)F\left(\frac{s}{c}\right)$
t-shifting	$f(t-t_0)u(t-t_0)$	$e^{-st_0}F(s)$

s-shifting	$e^{s_0 t} f(t)$	$F(s - s_0)$
Differentiation in t	$\frac{df(t)}{dt}$	$sF(s) - f(0)$
	$\frac{d^2 f(t)}{dt^2}$	$s^2 F(s) - sf(0) - \frac{df(0)}{dt}$
Integration in t	$\int_0^t f(\tau)\, d\tau$	$\left(\frac{1}{s}\right) F(s)$
Differentiation in s	$tf(t)$	$-\frac{dF(s)}{ds}$
Integration in s	$\frac{f(t)}{t}$	$\int_s^\infty F(\tilde{s}) d\tilde{s}$
Convolution	$f(t) * g(t)$	$F(s) \cdot G(s)$
	$f(t) \cdot g(t)$	$\frac{1}{2\pi j}[F(s) * G(s)]$
Periodicity	$f(t)$	$F_1(s) \times \frac{1}{1 - e^{-sp}}$ (For $f_1(t)$ one cycle of $f(t)$ with period p)
Initial Value Theorem	$f(0^+) = \lim_{s \to \infty} sF(s)$	
Final Value Theorem	$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$	

Note: For a, b, t_0, $s_0 \in R$, $c \in R_{++}$).

E. Partial Fractions Expansion

If a partial fraction expansion of $Y(s)$ includes terms:

$$\frac{A_m}{(s-a)^m} + \frac{A_{m-1}}{(s-a)^{m-1}} + \cdots + \frac{A_1}{s-a}$$

Then, the coefficients of factors having multiplicity $m > 1$ are given by the following expressions (where $k \neq m$).

$$A_m = \lim_{s \to a} (s-1)^m Y(s)$$

$$A_k = \left[\frac{1}{(m-k)!}\right] \lim_{s \to a} \left(\frac{d^{m-k}}{ds^{m-k}}\right) (s-1)^m Y(s)$$

F. Trigonometric Identities

$\sin(\theta \pm \phi) = \sin\theta \cos\phi \pm \cos\theta \sin\phi$	$\begin{cases} \sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta) \\ \sin\left(\theta - \frac{\pi}{2}\right) = -\cos(\theta) \end{cases}$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$	$\begin{cases}\cos\left(\theta + \frac{\pi}{2}\right) = -\sin(\theta) \\ \cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)\end{cases}$
$\sin(2\theta) = 2\sin\theta\cos\theta$	
$\cos(2\theta) = \cos^2\theta - \sin^2\theta$ $= 2\cos^2(\theta) - 1$ $= 1 - 2\sin(\theta)$	

G. Modelling of Electrical Systems

	Voltage	**Current**
Resistors	$v(t) = i(t)R$	$i(t) = \frac{v(t)}{R}$
Capacitors	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\left[\frac{dv(t)}{dt}\right]$
Inductors	$v(t) = L\left[\frac{di(t)}{dt}\right]$	$i(t) = \frac{1}{L}\int_0^t v(\tau)\,d\tau$

H. Electrical Power

The following lists some selected formulae for electrical power.

Parameter	**Formulae**
Complex power	$S = P + jQ = V_{RMS}(I_{RMS})^* = \lvert V_{RMS}\rvert\lvert I_{RMS}\rvert\angle(\theta_v - \theta_i)$
Apparent power	$S = \lvert S\rvert = \lvert V_{RMS}\rvert\lvert I_{RMS}\rvert = \sqrt{P^2 + Q^2}$
Real power	$P = \mathrm{Re}(S) = S\cos(\theta_v - \theta_i)$
Reactive power	$Q = \mathrm{Im}(S) = S\sin(\theta_v - \theta_i)$
Power factor:	$\frac{P}{S} = \cos(\theta_v - \theta_i)$

I. Three-Phase Systems

The following table lists some selected formulae for the three-phase systems.

Connection	**Phase Voltages/Currents**	**Line Voltages/Currents**
$Y - Y$	$V_{an} = V_m\angle 0°$ $V_{bn} = V_m\angle -120°$ $V_{cn} = V_m\angle +120°$	$V_{ab} = \sqrt{3}V_m\angle 30°$ $V_{bc} = V_{ab}\angle -120°$ $V_{ca} = V_{ab}\angle +120°$

	Same as line currents	$I_a = V_{an}/Z_Y$ $I_b = I_a\angle -120°$ $I_c = I_a\angle +120°$
$Y-\Delta$	$V_{an} = V_m\angle 0°$ $V_{bn} = V_m\angle -120°$ $V_{cn} = V_m\angle +120°$	$V_{ab} = V_{AB} = \sqrt{3}V_m\angle 30°$ $V_{bc} = V_{BC} = V_{ab}\angle -120°$ $V_{ca} = V_{CA} = V_{ab}\angle +120°$
	$I_{AB} = V_{AB}/Z_\Delta$ $I_{BC} = V_{BC}/Z_\Delta$ $I_{CA} = V_{CA}/Z_\Delta$	$I_a = I_{AB}\sqrt{3}\angle -30°$ $I_b = I_a\angle -120°$ $I_c = I_a\angle +120°$
$\Delta-\Delta$	$V_{ab} = V_m\angle 0°$ $V_{bc} = V_m\angle -120°$ $V_{ca} = V_m\angle +120°$	Same as phase voltages
	$I_{AB} = V_{ab}/Z_\Delta$ $I_{BC} = V_{bc}/Z_\Delta$ $I_{CA} = V_{ca}/Z_\Delta$	$I_a = I_{AB}\sqrt{3}\angle -30°$ $I_b = I_a\angle -120°$ $I_c = I_a\angle +120°$
$\Delta-Y$	$V_{ab} = V_m\angle 0°$ $V_{bc} = V_m\angle -120°$ $V_{ca} = V_m\angle +120°$	Same as phase voltages
	Same as line currents	$I_a = \dfrac{V_m\angle -30°}{\sqrt{3}Z_Y}$ $I_b = I_a\angle -120°$ $I_c = I_a\angle +120°$