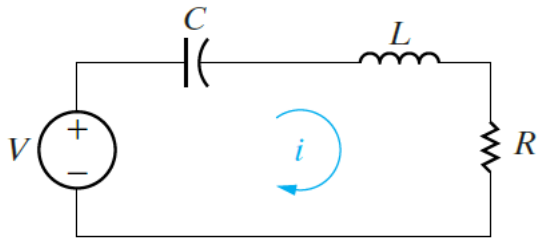
XMUT203 Analogue Circuits and Systems

Tutorial 6: Second-Order RLC Circuits (Solution)

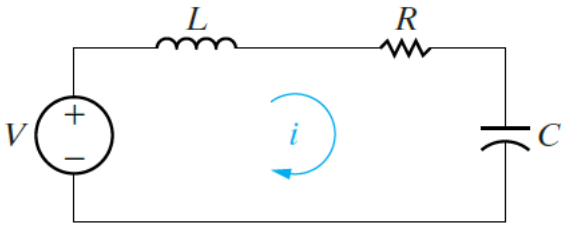
A. Second-Order RLC Circuits

1. Find the transfer functions $G(j\omega) = \tilde{V}_o(\omega)/\tilde{V}_i(\omega)$ for each of the circuits shown in the figures given below.

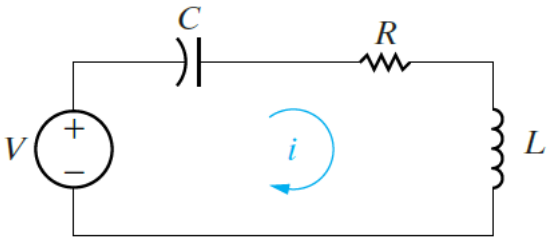
   a. Circuit 1 (e.g. V_o is taken across the resistor). [Answer: $G(\omega) = j\omega RC/(1 + j\omega RC - \omega^2 LC)$ or $G(\omega) = -[-j\omega(R/L)]/[\omega^2 - j\omega(R/L) - 1/LC]$] [10 marks]



   b. Circuit 2 (e.g. V_o is taken across the capacitor). [Answer: $G(\omega) = 1/[1 + j\omega RC - \omega^2 LC]$ or $G(\omega) = [-j\omega(1/LC)]/\omega^2 - j\omega(R/L) - 1/LC$] [10 marks]

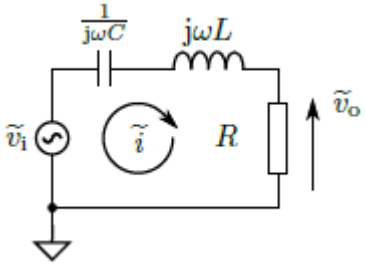


   c. Circuit 3 (e.g. V_o is taken across the inductor). [Answer: $G(\omega) = [-\omega^2 LC]/[1 + j\omega RC - \omega^2 LC]$ or $G(\omega) = [\omega^2]/[\omega^2 - j\omega(R/L) - 1/LC]$] [10 marks]



Solution

a. Applying Kirchhoff voltage law (KVL) across the input and output of the circuit.



Input voltage:

$$v_i(t) = v_C(t) + v_L(t) + v_R(t)$$

Taking Fourier transform and since $V(\omega) = I(\omega)Z(\omega)$, the voltages across the devices in the circuit are:

$$V_i(\omega) = I(\omega)\frac{1}{j\omega C} + I(\omega)j\omega L + I(\omega)R$$

Thus

$$V_i(\omega) = I(\omega)\left(\frac{1}{j\omega C} + j\omega L + R\right)$$

Output voltage:

$$V_o(\omega) = V_R(\omega) = I(\omega)R$$

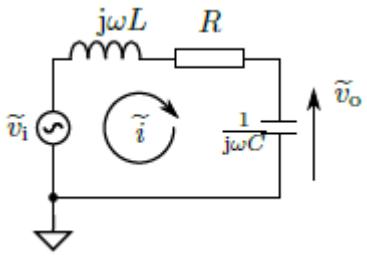
The transfer function of the circuit is:

$$G(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{I(\omega)R}{I(\omega)\left(\frac{1}{j\omega C} + j\omega L + R\right)}$$

The transfer function of the RLC circuit with R as output is given below:

$$G(\omega) = \frac{j\omega RC}{1 + j\omega RC - \omega^2 LC} \quad \text{or} \quad G(\omega) = -\frac{-j\omega\left(\frac{R}{L}\right)}{\omega^2 - j\omega\left(\frac{R}{L}\right) - \frac{1}{LC}}$$

b. Applying Kirchhoff voltage law (KVL) across the input and output of the circuit.



Input voltage:

$$v_i(t) = v_L(t) + v_R(t) + v_C(t)$$

Taking Fourier transform and since $V(\omega) = I(\omega)Z(\omega)$, the voltages across the devices in the circuit are:

$$V_i(\omega) = I(\omega)j\omega L + I(\omega)R + I(\omega)\frac{1}{j\omega C}$$

Thus

$$V_i(\omega) = I(\omega)\left(j\omega L + R + \frac{1}{j\omega C}\right)$$

Output voltage:

$$V_o(\omega) = V_C(\omega) = I(\omega)\frac{1}{j\omega C}$$

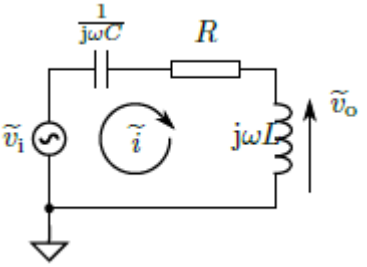
The transfer function of the circuit is:

$$G(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{I(\omega)\frac{1}{j\omega C}}{I(\omega)\left(\frac{1}{j\omega C} + j\omega L + R\right)}$$

The transfer function of the RLC circuit with C as output is given below:

$$G(\omega) = \frac{1}{1 + j\omega RC - \omega^2 LC} \quad \text{or} \quad G(\omega) = \frac{-j\omega\left(\frac{1}{LC}\right)}{\omega^2 - j\omega\left(\frac{R}{L}\right) - \frac{1}{LC}}$$

c. Applying Kirchhoff voltage law (KVL) across the input and output of the circuit.



Input voltage:

$$v_i(t) = v_C(t) + v_R(t) + v_L(t)$$

Taking Fourier transform and since $V(\omega) = I(\omega)Z(\omega)$, the voltages across the devices in the circuit are:

$$V_i(\omega) = I(\omega)\frac{1}{j\omega C} + I(\omega)R + I(\omega)j\omega L$$

Thus

$$V_i(\omega) = I(\omega)\left(\frac{1}{j\omega C} + R + j\omega L\right)$$

Output voltage:

$$V_o(\omega) = V_L(\omega) = I(\omega)j\omega L$$

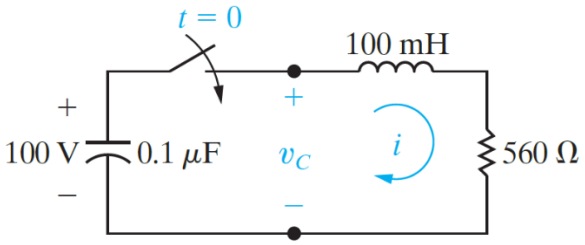
The transfer function of the circuit is:

$$G(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{I(\omega)j\omega L}{I(\omega)\left(\frac{1}{j\omega C} + j\omega L + R\right)}$$

The transfer function of the RLC circuit with L as output is given below:

$$G(\omega) = \frac{-\omega^2 LC}{1 + j\omega RC - \omega^2 LC} \quad \text{or} \quad G(\omega) = \frac{\omega^2}{\omega^2 - j\omega\left(\frac{R}{L}\right) - \frac{1}{LC}}$$

2. The 0.1 μF capacitor in the natural response series RLC circuit shown in figure below is charged to 100 V. At t = 0 the capacitor is discharged through a series combination of a 100 mH inductor and a 560 Ω resistor.



a. Find $i(t)$ for t > 0. [Answer: $i(t) = 0.1402e^{-2,800t}\sin 9,600t \quad t \geq 0$] [28 marks]

b. Find $v_c(t)$ for t > 0. [Answer: $v_c(t) = (100\cos 9,600t + 29.17\sin 9,600t)e^{-2,800t} \quad t \geq 0$] [8 marks]

Solution

a. The transfer function equation of the natural response in the series RLC given above is:

$$\frac{i(\omega)}{v_C(\omega)} = \frac{1}{1/(LC) + j\omega(R/L) - \omega^2}$$

The first step to finding $i(t)$ is to calculate the roots of the characteristic equation. For the given element values, the natural frequency of the circuit is:

$$\omega_o^2 = \frac{1}{LC} = \frac{1}{(100 \times 10^{-3})(0.1 \times 10^{-6})} = 10^8$$

Next, we compare ω_o^2 to α^2 and note that $\omega_o^2 > \alpha^2$, because:

$$\alpha^2 = \left(\frac{R}{2L}\right)^2 = \left[\frac{560}{2(100 \times 10^{-3})}\right]^2 = 7.84 \times 10^6$$

At this point, we know that the response is underdamped and that the solution for $i(t)$ is of the form

$$i(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

Where:

$\alpha = \sqrt{7.84 \times 10^6}$ = 2,800 rad/s.

$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{10^8 - (7.84 \times 10^6)}$ = 9,600 rad/s.

The numerical values of $B_1$and B_2 come from the initial conditions. The inductor current is zero before the switch has been closed, and hence it is zero immediately after. Therefore

$$i(0) = 0 = B_1$$

To find B_2, we evaluate $di(0^+)/dt$.

From the circuit, we note that, because $i(0) = 0$ immediately after the switch has been closed, there will be no voltage drop across the resistor.

Thus, the initial voltage on the capacitor appears across the terminals of the inductor, which leads to the expression,

$$L\left(\frac{di(0^+)}{dt}\right) = v_0(t)$$

Or

$$\frac{di(0^+)}{dt} = \frac{v_0(t)}{L} = \frac{100}{100 \times 10^{-3}} = 1{,}000 \text{ A}$$

Because B_1 = 0 from the calculation above, thus

$$i(t) = B_2 e^{-\alpha t} \sin \omega_d t$$

Taking the first derivative of the equation above (e.g. $y(t) = u(t)v(t)$, thus $dy/dt = (du/dt)v(t) + u(t)(dv/dt)$, it is:

$$\frac{di}{dt} = B_2(-\alpha)e^{-\alpha t} \sin \omega_d t + B_2 e^{-\alpha t} \omega_d \cos w_d t$$

From the previous calculations as above, $\alpha = \sqrt{7.84 \times 10^6}$ = 2,800 rad/s and $\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{10^8 - (7.84 \times 10^6)}$ = 9,600 rad/s. Thus

$$\frac{di}{dt} = B_2(-2800)e^{-2800t} \sin 9600\, t + B_2 e^{-2800t}(9600) \cos 9600t$$

As a result, the first derivative of the equation above is:

$$\frac{di}{dt} = 400B_2e^{-2,800t}(24\cos 9,600t - 7\sin 9,600t)$$

When $t = 0^+$, the equation above becomes:

$$\frac{di(0^+)}{dt} = 400B_2e^{2800(0)}(24\cos 9600(0) - 7\sin 9600(0))$$

$$= 9,600B_2$$

As a result

$$B_2 = \frac{1,000}{9,600} \approx 0.1402 \text{ A}$$

The solution for $i(t)$ is:

$$i(t) = 0.1402e^{-2,800t}\sin 9,600t \qquad t \geq 0$$

b. To find $v_c(t)$, we can use either of the following relationships:

$$v_c(t) = -\frac{1}{C}\int_0^t i\,d\tau + 100$$

Or

$$v_c(t) = i(t)R + L\left(\frac{di}{dt}\right)$$

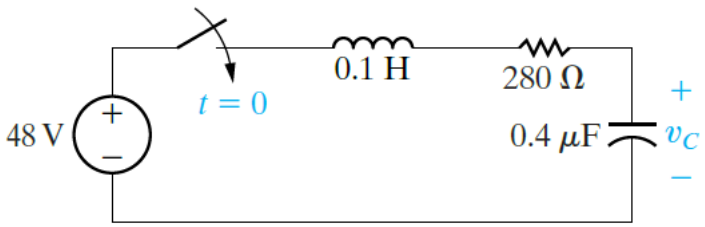
Since the second is recommended, the voltage across the capacitor is:

$$v_c(t) = 0.1402e^{-2,800t}\sin 9,600t\,(560) + (100 \times 10^{-3})(400(0.1402)e^{-2,800t}(24\cos 9,600t - 7\sin 9,600t))$$

Whichever expression is used, the result is:

$$v_c(t) = (100\cos 9,600t + 29.17\sin 9,600t)e^{-2,800t} \qquad t \geq 0$$

3. No energy is stored in the 100 mH inductor or the 0.4 μF capacitor of a step response series RLC circuit when the switch in the circuit shown in the figure below is closed. Find $v_c(t)$ for $t > 0$. [Answer: $v_c(t) = 48 - 48e^{-1,400t}\cos 4,800t - 14e^{-1,400t}\sin 4,800t \quad t \geq 0$] [24 marks]



Solution

The transfer function equation of the forced response in the series RLC given above is:

$$\frac{v_C(\omega)}{v_i(\omega)} = \frac{1}{1/(LC) + j\omega(R/L) - \omega^2}$$

The roots of the characteristic equation of the RLC circuit with voltage source given above are

$$\omega_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$= -\frac{280}{0.2} \pm \sqrt{\left(\frac{280}{0.2}\right)^2 - \frac{10^6}{(0.1)(0.4)}}$$

So, the roots of the equation above are

$$\omega_1 = -1{,}400 + j4{,}800 \text{ rad/s}$$

And

$$\omega_2 = -1{,}400 - j4{,}800 \text{ rad/s}$$

As shown in the results above, the roots are complex, so the voltage response is underdamped. Thus:

$$v(t) = 48 + B_1' e^{-1{,}400t} \cos 4{,}800t + B_2' e^{-1{,}400t} \sin 4{,}800t \qquad t \geq 0$$

No energy is stored in the circuit initially, so both $v_c(0)$ and $dv_c(0^+)/dt$ are zero. Then,

$$v_c(0) = 48 + B_1' e^{-1{,}400(0)} \cos 4{,}800(0) + B_2' e^{-1{,}400(0)} \sin 4{,}800(0) \qquad t \geq 0$$

Thus

$$v_c(0) = 48 + B_1' = 0$$

Taking the derivative of the equation given above, (e.g. $y(t) = u(t)v(t)$, thus $dy/dt = (du/dt)v(t)) + u(t)(dv/dt)$, so:

$$\frac{dv_c(t)}{dt} = B_1' e^{-1{,}400t}(-1{,}400)\cos 4{,}800t + B_1' e^{-1{,}400t} \sin 4{,}800t\,(4{,}800)$$
$$+ B_2' e^{-1{,}400t}(-1{,}400)\sin 4{,}800t + B_2' e^{-1{,}400t} \cos 4{,}800t\,(4{,}800) \qquad t \geq 0$$

At $t = 0^+$, the equation above becomes:

$$\frac{dv_c(0^+)}{dt} = B_1' e^{-1{,}400(0)}(-1{,}400)\cos 4{,}800(0) + B_1' e^{-1{,}400(0)} \sin 4{,}800(0)\,(4{,}800)$$
$$+ B_2' e^{-1{,}400(0)}(-1{,}400)\sin 4{,}800(0)$$
$$+ B_2' e^{-1{,}400(0)} \cos 4{,}800(0)\,(4{,}800) \qquad t \geq 0$$

Thus

$$\frac{dv_c(0^+)}{dt} = 4{,}800B_2' - 1{,}400B_1' = 0$$

Solving for B_1' and B_2' yields

$$B_1' = -48 \text{ V}$$

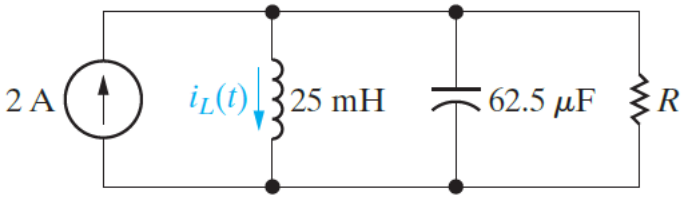
And

$$B_2' = -14 \text{ V}$$

Therefore, the solution for $v_c(t)$ is:

$$v_c(t) = 48 - 48e^{-1{,}400t}\cos 4{,}800t - 14e^{-1{,}400t}\sin 4{,}800t \qquad t \geq 0$$

4. Assume that at the instant the 2 A DC current source is applied to the circuit in the figure below, the initial current in the 25 mH inductor is 1 A, and the initial voltage on the capacitor is 50 V (i.e. positive at the upper terminal). Find the expression for $i_L(t)$ for $t \geq 0$ if R equals 12.5 Ω.
[Answer: $i_L(t) = 2 - e^{-640t}\cos 480t + 2.83e^{-640t}\sin 480t$ A, $t \geq 0$] [26 marks]



Solution

For the given step response parallel RLC circuit, its resonant frequency is:

$$\omega_o = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(25\times10^{-3})(62.5\times10^{-6})}} = 800 \text{ rad/s}$$

The neper frequency of the circuit is:

$$\alpha = \frac{1}{2RC} = \frac{1}{2(12.5)(62.5\times10^{-6})} = 640 \text{ rad/s}$$

As a result, the circuit is underdamped with $\omega_o > \alpha$ from calculations above. The equation for current that flows in the inductor in the circuit is:

$$i_L(t) = I_f + B_1'e^{-at}\cos\omega_d t + B_2'e^{-at}\sin\omega_d t$$

The damped frequency of the circuit is:

$$\omega_d = \sqrt{800^2 - 640^2} = 480$$

Since the current I_f is found to be 2 A, the equation for the inductor current $i_L(t)$ is:

$$i_L(t) = 2 + B_1'e^{-640t}\cos 480t + B_2'e^{-640t}\sin 480t$$

The next step is to determine the coefficients B_1' and B_2' from the initial conditions of the circuit. At t = 0, we can determine value of B_1' from:

$$i_L(0) = 2 + B_1' = 1$$

So, the value of B_1' is -1.

Considering $y(t) = uv$ and $dy(t)/dt = (du/dt)v + u(dv/dt)$, the first derivative of $i_L(t)$ is:

$$\frac{di_L(t)}{dt} = (-\alpha)B_1'e^{-at}\cos\omega_d t + B_1'e^{-at}\cos\omega_d t\,(\omega_d) + (-\alpha)B_2'e^{-at}\sin\omega_d t + B_2'e^{-at}\cos\omega_d t\,(\omega_d)$$

When $t = 0$, the equation above becomes:

$$\frac{di_L(0)}{dt} = (-\alpha)B_1'e^{-a(0)}\cos\omega_d(0) + B_1'e^{-a(0)}\cos\omega_d(0)\,(\omega_d) + (-\alpha)B_2'e^{-a(0)}\sin\omega_d(0) + B_2'e^{-a(0)}\cos\omega_d(0)\,(\omega_d)$$

Thus

$$\frac{di_L(0)}{dt} = -\alpha B_1' + \omega_d B_2'$$

At initial condition,

$$\frac{di_L(0)}{dt} = \frac{V_0}{L}$$

The value of B_2' is calculated from:

$$-\alpha B_1' + \omega_d B_2' = \frac{V_0}{L}$$

Thus

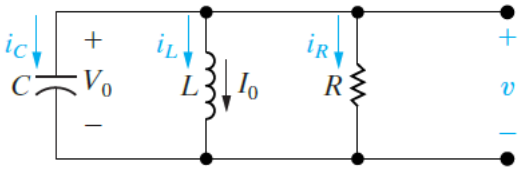
$$-640(-1) + 480B_2' = \frac{50}{25\times10^{-3}}$$

So, the value for B_2' is 2.83.

With all coefficients B_1' and B_2' are found, the expression for $i_L(t)$ is:

$$i_L(t) = 2 - e^{-640t}\cos 480t + 2.83e^{-640t}\sin 480t \text{ A}, \quad t \geq 0$$

5. The voltage $v(t)$ of a natural response parallel RLC circuit as shown in the figure below is $v(t) = 120e^{-400t}\cos 300t + 80e^{-400t}\sin 300t$ V.



When the capacitor is 250 F, find the values and expression for the following:

a. L. [Answer: 16 mH] [8 marks]

b. R. [Answer: 5 Ω] [2 marks]

c. V_0. [Answer: 120 V] [2 marks]

d. I_0. [Answer: -18 A] [12 marks]

e. $i_L(t)$. [Answer: $i_L(t) = e^{-400t}(-18\cos 300t + \sin 300t)$ A, $t \geq 0$] [10 marks]

Solution

a. From the equation for the given parallel RLC circuit, the parameters of the circuit are $\alpha = 400$ and ω_d = 300 rad/s.

Since the damped frequency is calculated from:

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2}$$

Thus

$$\omega_o^2 = \omega_d^2 + \alpha^2 = 9 \times 10^4 + 16 \times 10^4 = 25 \times 10^4$$

The value of the inductor is calculated from:

$$\omega_o^2 = \frac{1}{LC} = 25 \times 10^4$$

The value of inductor is:

$$L = \frac{1}{(25 \times 10^4)(250 \times 10^{-6})} = 16 \text{ mH}$$

b. For the parallel RLC circuit, the value of neper frequency is calculated from:

$$\alpha = \frac{1}{2RC}$$

As a result, the value of the resistor is:

$$R = \frac{1}{2\alpha C} = \frac{1}{2(400)(250 \times 10^{-6})} = 5\ \Omega$$

c. The voltage V_0 is calculated from:

$$V_o = v(0) = 120 \text{ V}$$

d. The input current I_0 is calculated from:

$$I_o = i_L(0) = -i_R(0) - i_C(0)$$

The current $i_R(0)$ is calculated from:

$$i_R(0) = \frac{120}{5} = 24 \text{ A}$$

Considering $y(t) = uv$ and $dy(t)/dt = (du/dt)v + u(dv/dt)$, the first derivative of the equation $v(t) = 120e^{-400t}\cos 300t + 80e^{-400t}\sin 300t$ V is:

$$\frac{dv(t)}{dt} = (120)(-400)e^{-400t}\cos 300t + 80e^{-400t}\cos 300t\,(300)$$

At $t = 0$, the equation above becomes:

$$\frac{dv(0)}{dt} = (120)(-400)e^{-400(0)}\cos 300(0) + 80e^{-400(0)}\cos 300(0)\,(300)$$

The current $i_C(0)$ is calculated from:

$$i_C(0) = C\frac{dv(0)}{dt} = (250 \times 10^{-6})[-400(120) + 300(80)] = -6\text{ A}$$

As a result, the input current I_0 is:

$$I_o = i_R(0) - i_C(0) = -24 + 6 = -18\text{ A}$$

e. The current $i_C(t)$ is calculated from:

$$i_C(t) = 250 \times 10^{-6}\frac{dv(t)}{dt} = e^{-400t}(-17\sin 300t - 6\cos 300t)\text{ A}$$

The current $i_R(t)$ is calculated from:

$$i_R(t) = \frac{v(t)}{5} = e^{-400t}(24\cos 300t + 16\sin 300t)\text{ A}$$

The current $i_L(t)$ is calculated from:

$$\begin{aligned} i_L &= -i_R(t) - i_C(t) \\ &= e^{-400t}(-18\cos 300t + \sin 300t)\text{ A}, \quad t \geq 0 \end{aligned}$$

Check the answer:

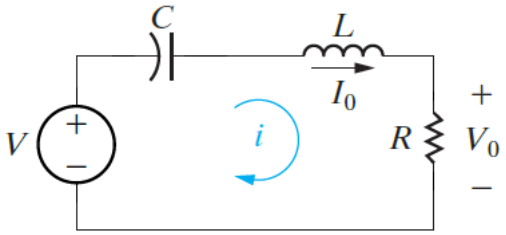
$$L\frac{di_L}{dt} = (16 \times 10^{-3})\{e^{-400t}[7500\cos 300t + 5000\sin 300t]\}$$

Thus, the voltage $v(t)$ is:

$$v(t) = e^{-400t}[120\cos 300t + 80\sin 300t]\text{ V}$$

B. Transient and Frequency Response of Second-Order RLC Circuits

6. For the series resonant RLC circuit given below.



a. Let's calculate the current flowing in the series resonant circuit as a function of the frequency of the input voltage. In this case, we have drawn the circuit directly in the phasor

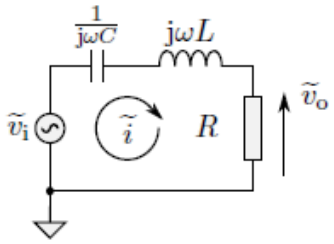
representation. [Answer: $\tilde{I}(\omega)/\tilde{V}_i(\omega) = 1/[R + j\omega L + 1/j\omega C]$] [6 marks]

b. Simulate the gain response of the system with $R = C = L = 1$. [8 marks]

c. Simulate the frequency response of the system in MATLAB. [8 marks]

Solution

a. For the given series RLC circuit, its equivalent circuit is as shown in the figure below.



The Ohm's law:

$$\tilde{I}(\omega) = \frac{\tilde{V}_i(\omega)}{Z_{tot}}$$

Thus

$$\tilde{I}(\omega) = \tilde{V}_i(\omega)\left(\frac{1}{R + Z_L + Z_C}\right) = \tilde{V}(\omega)\left(\frac{1}{R + j\omega L + 1/j\omega C}\right)$$

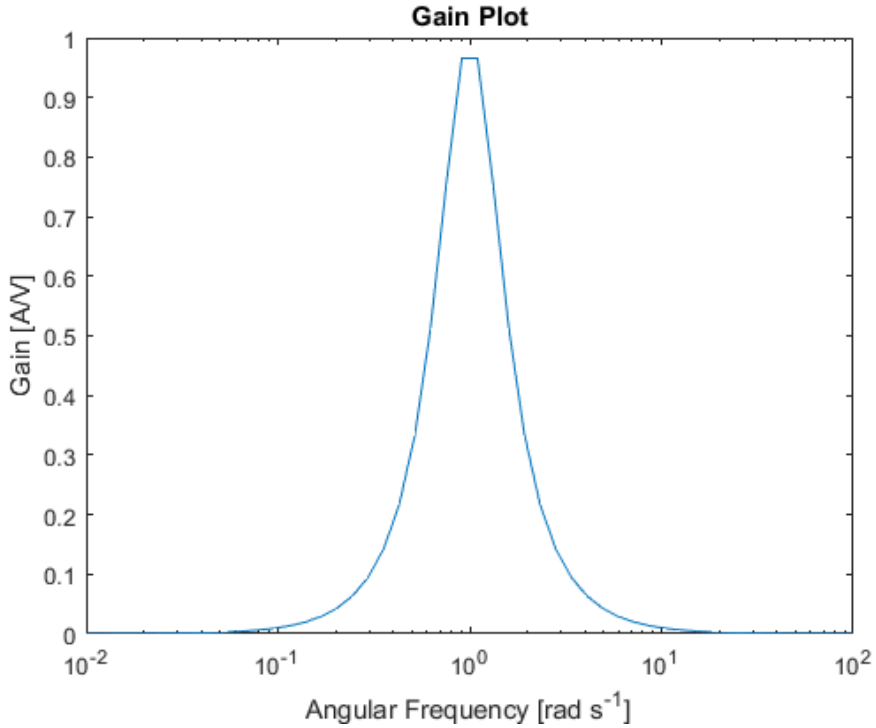
Then

$$\frac{\tilde{I}(\omega)}{\tilde{V}_i(\omega)} = \frac{1}{R + j\omega L + 1/j\omega C}$$

b. Let's see a plot of the gain response with $R = C = L = 1$.

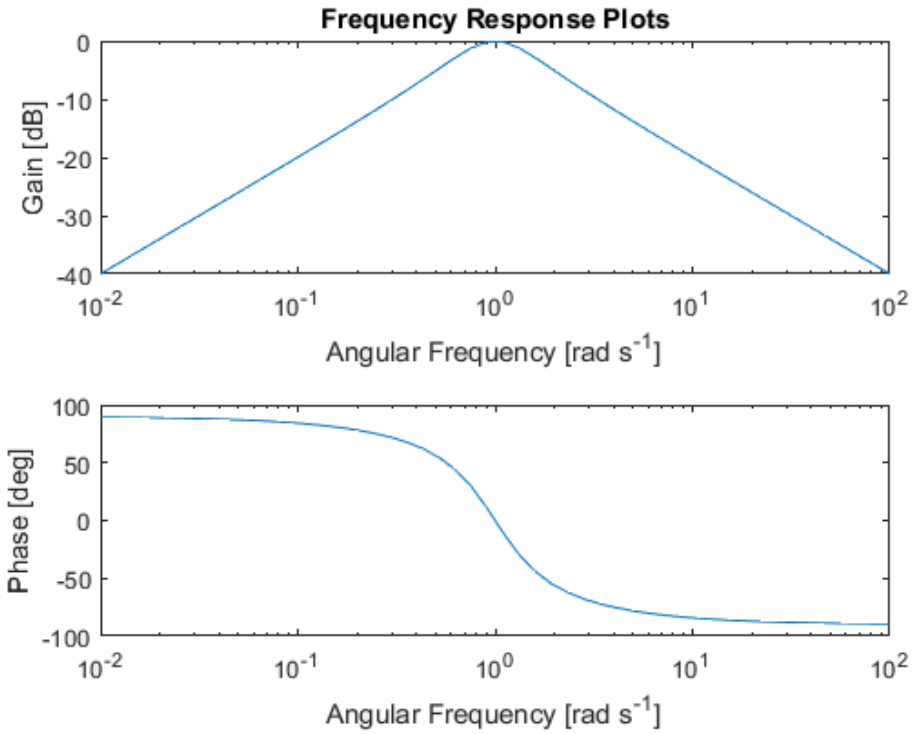
```
% Gain plot of resonant RLC circuit
w = logspace(-2,2);
G = 1./[1+j.*w+[1./(j.*w)]];
semilogx(w,G)
xlabel('Angular Frequency [rad s^{-1}]')
ylabel('Gain [A/V]')
title('Gain Plot')
```

The graph of magnitude or gain $|Y(\omega)| = \left|\tilde{I}(\omega)/\tilde{V}(\omega)\right|$ as a function of ω of the series resonant circuit is as shown in the figure below.

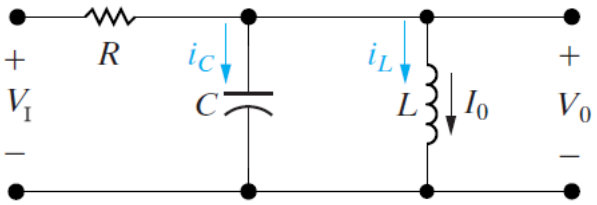


c. The magnitude $|Y(\omega)|$ and phase shift $\angle Y(\omega)$ of the frequency response of the series resonance circuit are plotted in the figures below.

```
% Series-resonance RLC circuit
w = logspace(-2,2);
G = 1./[1+j.*w+[1./(j.*w)]];
subplot(2,1,1)
semilogx(w,20*log10(abs(G)))
xlabel('Angular Frequency [rad s^{-1}]')
ylabel('Gain [dB]')
title('Frequency Response Plots')
subplot(2,1,2)
semilogx(w,180/pi*angle(G))
xlabel('Angular Frequency [rad s^{-1}]')
ylabel('Phase [deg]')
```



7. The following figure shows a series-parallel RLC circuit with the resistor R is connected in series with a parallel of inductor L and capacitor C. The input and output of the circuit are as indicated in the diagram.



The transfer function of the output voltage over the input voltage $G(\omega) = V_o(\omega)/V_i(\omega)$ of the circuit is:

$$G(\omega) = \frac{j\omega/(RC)}{(j\omega)^2 + j\omega/(RC) + 1/(LC)}$$

In this type of circuit, the product LC controls the resonant frequency while RC controls the bandwidth. To tune the resonant frequency of the circuit to the frequency of 1 rad/s, set $L = C = 1$ and vary the value of the R to tune the bandwidth of the frequency response. In the filter circuit applications, this type of RLC circuit is commonly termed as bandpass filter circuit.

a. Simulate the frequency response of the circuit for R = L = C = 1. Comment on the result of the simulation. Note: use `tf()` function in MATLAB to specify the circuit's transfer function and use bode function to plot the frequency response of the circuit. [8 marks]

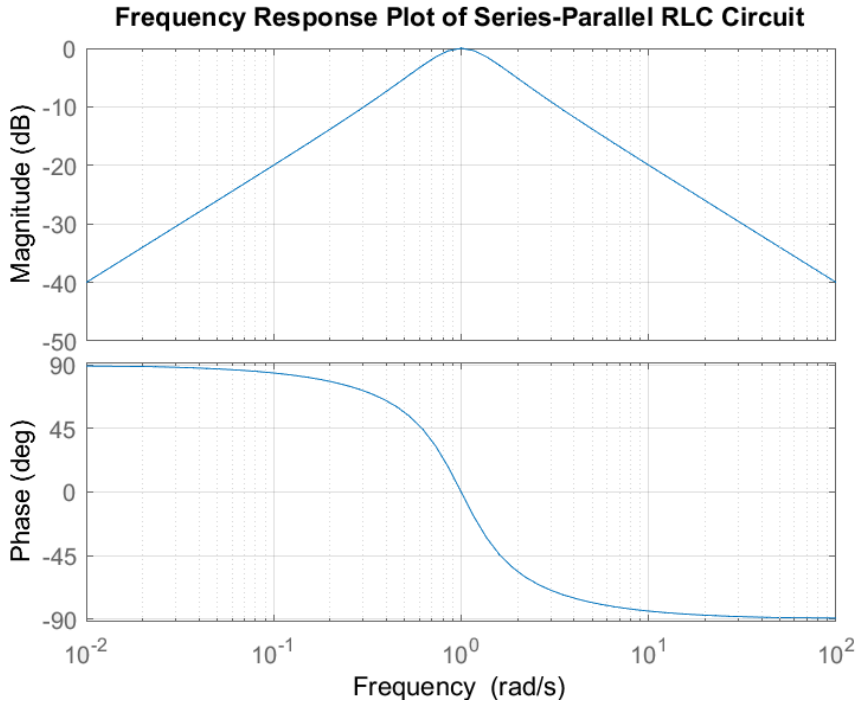
b. To get a narrower bandwidth, try increasing values of R. Simulate the circuit for additional values of R = 5 and 20 Ω. [8 marks]

Solution

a. The following MATLAB code is used to plot the frequency response plot of the circuit.

```
%|R=L=C=1|:
R = 1; L = 1; C = 1;
G = tf([1/(R*C) 0],[1 1/(R*C) 1/(L*C)])
bode(G), grid
title('Frequency Response Plot of Series-Parallel RLC Circuit')
```

The frequency response plots of the given RLC circuit in MATLAB are shown in the figure below.



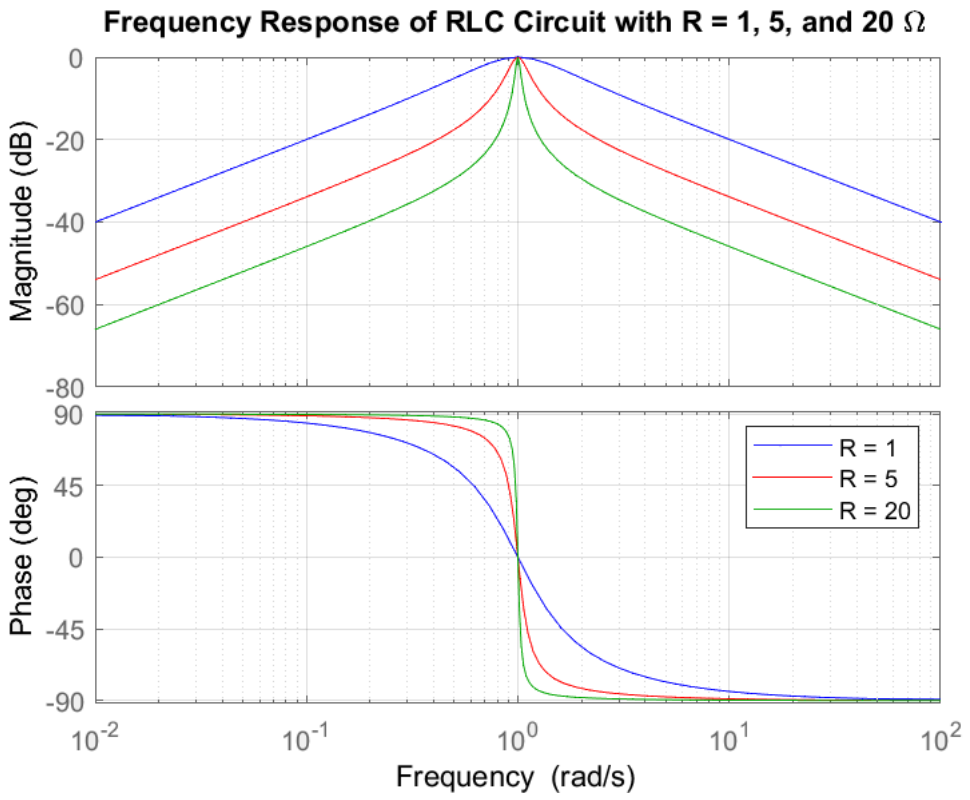
As expected, the RLC circuit has maximum gain at the resonant frequency of 1 rad/s. However, the attenuation is only -10dB half a decade away from this frequency.

b. The following MATLAB code is used to analyse the influence of varying the value of the resistor R towards the bandwidth of the frequency response of the circuit.

```
R = 1; L = 1; C = 1;
G = tf([1/(R*C) 0],[1 1/(R*C) 1/(L*C)])
R1 = 5;
G1 = tf([1/(R1*C) 0],[1 1/(R1*C) 1/(L*C)]);
```

```
R2 = 20;
G2 = tf([1/(R2*C) 0],[1 1/(R2*C) 1/(L*C)]);
bode(G,'b',G1,'r',G2,'g'), grid
legend('R = 1','R = 5','R = 20')
title('Frequency Response of Series-Parallel RLC Circuit with R = 1, 5, and 20 \Omega')
```

The frequency response plots of the given RLC circuit with various values of R = 1, 5, and 20 Ω are shown in the figure below.



8. For the series-parallel RC circuit that you have in the previous circuit, simulate its transient response. Confirm the attenuation properties of the circuit with R = 20 by simulating how this circuit transforms sine waves with frequency 0.9, 1, and 1.1 rad/s. [10 marks]

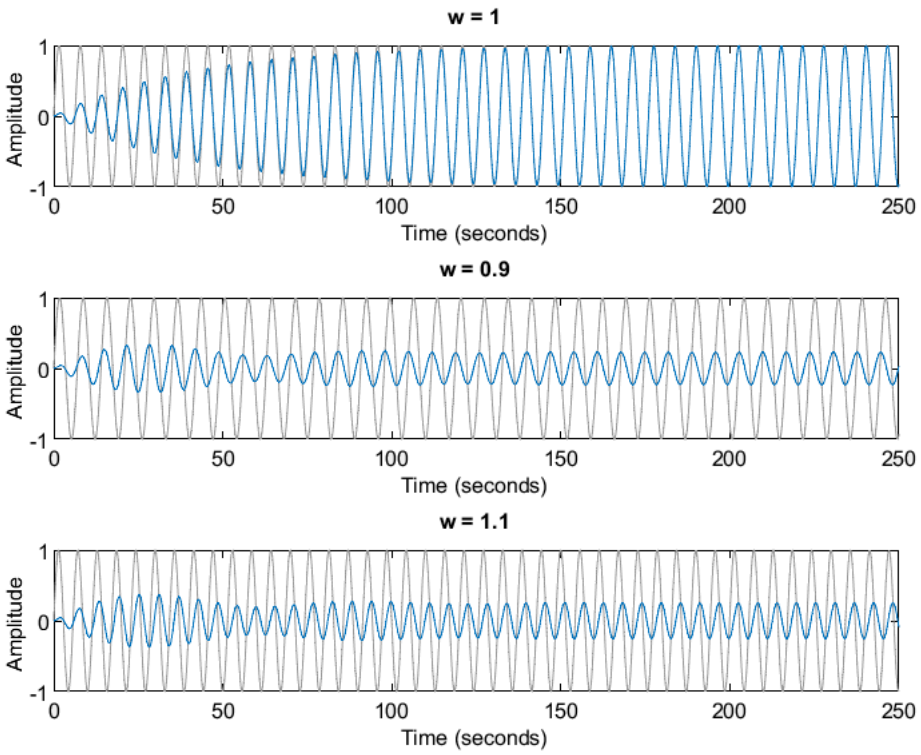
Solution

The following MATLAB code is used to simulate the transient response of the circuit when varying the frequency around the resonant frequency of the circuit.

```
R2 = 20; L = 1; C = 1;
G2 = tf([1/(R2*C) 0],[1 1/(R2*C) 1/(L*C)]);
```

```
t = 0:0.05:250;
opt = timeoptions;
opt.Title.FontWeight = 'Bold';
subplot(311), lsim(G2,sin(t),t,opt), title('w = 1')
subplot(312), lsim(G2,sin(0.9*t),t,opt), title('w = 0.9')
subplot(313), lsim(G2,sin(1.1*t),t,opt), title('w = 1.1')
```

The result of the transient response simulation of the circuit in MATLAB is shown in the figure below.



Notice that at resonant frequency the amplitude is bigger than amplitudes of the frequencies below and above it. Impedance for this series-parallel RLC circuit is minimum at its resonant frequency.

C. Design of Second-Order RLC Circuits

9. Design both series and parallel resonant RLC circuits that have a resonant frequency of 876 kHz, and a Q-factor of 10. Use a fixed capacitor value of 1 nF. What values of inductance and resistance are required? [Answer: L = 33 µH in both cases. R_{series} = 18.2 Ω, R_{parallel} = 1.82 kΩ]
[6 marks]

Solution

For a given resonant circuit, the natural frequency of the circuit (ω_n) is found from:

$$\omega_n = \frac{1}{\sqrt{LC}}$$

As a result, the inductance of the resonant circuit is:

$$L = \frac{1}{C\omega_n^2} = \frac{1}{(1\times10^{-9})(2\pi\times876\times10^3)} = 3.3\times10^{-6}\ \text{H}$$

The series resistance of the resonant circuit is:

$$R_{series} = \frac{1}{Q}\sqrt{\frac{L}{C}} = 18.2\ \Omega$$

The parallel resistance of the resonant circuit is:

$$R_{parallel} = Q\sqrt{\frac{L}{C}} = 1.82\ \Omega$$

10. A series-resonant RLC circuit has L = 1 mH and C = 10 µF. If the circuit is to have a bandwidth (B) of 15.9 Hz, what value of R will be needed, and what will be the resulting Q-factor of the circuit? [Answer: Q = 100, R = 0.1 Ω] [6 marks]

Solution

For a given series-resonant circuit, the cut-off frequency of the series-resonant circuit (ω_0) is found from:

$$\omega_0 = \frac{1}{\sqrt{LC}} = 10\times10^3$$

Now, the bandwidth is calculated from:

$$B = \frac{\omega_0}{Q}$$

Rearranging the equation given above, the Q-factor (Q) of the series-resonant circuit.

$$Q = \frac{10\times10^3}{2\pi(15.9)} = 100$$

To achieve this Q-factor, we need to choose R. The Q-factor of the series-resonant circuit is found from:

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Rearranging the equation given above, the resistor of the series-resonant circuit is:

$$R = \frac{1}{Q}\sqrt{\frac{L}{C}} = 0.1\ \Omega$$

11. Design a series-resonant RLC circuit that passes signals in the range of 1.75 MHz to 1.85 MHz (that is, the bandwidth must extend to these frequencies). Use a 10 pF capacitor. [Answer: L = 783 µH, R = 492 Ω] [6 marks]

Solution

Assume symmetric response around resonant frequency (ω_0), which requires $Q > 10$.

So, f_0 = 1.80 MHz ≡ 11.3 × 10^6 rads^{-1} and B = 0.1 MHz ≡ 0.63 × 10^6 rads^{-1}. So, the Q factor of the series-resonant circuit is

$$Q = \frac{f_0}{B} = \frac{1.8 \times 10^3}{0.1 \times 10^3} = 18$$

Now, the resonant frequency of the series-resonant circuit is:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Rearranging the equation given above, the inductance of the series-resonant circuit is:

$$L = \frac{1}{\omega_0^2 C} = 783\ \mu\text{H}$$

Also, the Q-factor of the series-resonant circuit is:

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Rearranging the equation given above, the resistance of the series-resonant circuit is:

$$R = \frac{1}{Q}\sqrt{\frac{L}{C}} = 492\ \Omega$$

12. You are working on a receiver that is a RLC circuit required to collect a signal that is being transmitted at 100 kHz. The circuit must use a fixed inductor of 10 µH.

    a. What value capacitor should be used to build a series-resonant circuit with a resonant frequency at 100 kHz? [Answer: C = 253 nF] [2 marks]

    b. You are now restricted to using a single capacitor with an E12 series value. What is the resonant frequency closest to the desired 100 kHz that you could achieve? [Answer: f_0 = 96.8 kHz] [4 marks]

    c. Choose a value of R that ensures that your modified resonant circuit contains the original

100 kHz within its bandwidth. [Answer: $R \geq 0.403\ \Omega$] [6 marks]

Solution

a. The resonant frequency of the series-resonant circuit is:

$$\omega_0 = \frac{1}{\sqrt{LC}} = 2\pi \times 100 \times 10^3$$

Rearranging the equation given above, the value of capacitor in the series-resonant circuit is:

$$C = \frac{1}{L\omega_0^2} = \frac{1}{(10 \times 10^{-6})(2\pi \times 100 \times 10^3)^2} = 253\ \text{nF}$$

b. Evaluate the value of the resonant frequency on two values of capacitor around 100 kHz

For C = 220 nF, this will give a resonant frequency of:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(10 \times 10^{-6})(220 \times 10^{-9})}} = 107.3\ \text{kHz}$$

For C = 270 nF, this will give a resonant frequency of:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(10 \times 10^{-6})(270 \times 10^{-9})}} = 96.8\ \text{kHz}$$

c. We need a bandwidth (B) that extends at least 100 kHz − 96.8 kHz = 3.2 kHz either side of the central peak (if we assume that Q is sufficiently high that we can presume that the resonance is symmetric about f_0).

$$B = 2 \times 3.2\ \text{kHz} = 6.4\ \text{kHz}$$

So, the Q-factor of the resonant circuit is:

$$Q = \frac{f_0}{B} = \frac{96.8\ \text{kHz}}{6.4\ \text{kHz}} = 15.1$$

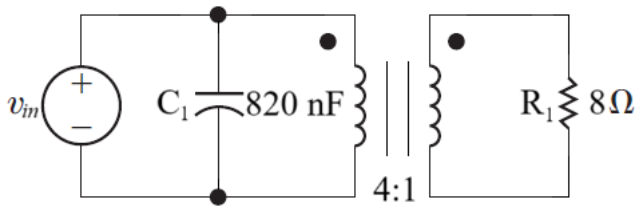
Which is high enough for our bandwidth approximation to hold. Now, Q-factor of the resonant circuit is:

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Rearranging the equation given above, the resistance in the resonant circuit is:

$$R \geq \frac{1}{Q}\sqrt{\frac{L}{C}} \geq \frac{1}{15.1}\sqrt{\frac{10 \times 10^{-6}}{270 \times 10^{-9}}} \geq 0.403\ \Omega$$

13. An 8 Ω speaker is connected as part of a resonant circuit with a transformer as shown in the figure below.



a. If the circuit is intended to sound a single tone at 6 kHz, what should the equivalent inductance of the transformer be as seen from the primary? [Answer: $L = 860 \times 10^{-6}$ H] [2 marks]

b. What will the Q-factor of the resulting circuit be? [Answer: $Q = 3.95$] [2 marks]

c. You find that the circuit has been incorrectly configured, with the capacitor placed in the secondary side of the transformer, rather than the primary. What would the resonant frequency and Q-factor of the circuit then be? [Answer: $f_0 = 24$ kHz, $Q \approx 1$] [8 marks]

Solution

a. The resonant frequency of the resonant circuit is:

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

Rearranging the equation given above, the resistance in the resonant circuit is:

$$L = \frac{1}{C\omega_o^2} = \frac{1}{(820 \times 10^{-9})(2\pi \times 6000)^2} = 860 \times 10^{-6}\ \text{H}$$

b. When taken to the primary side, the load resistance looks like $N^2 R_L = 128\ \Omega$.

The Q-factor of the given resonant circuit is found from:

$$Q = R\sqrt{\frac{C}{L}} = 128\sqrt{\frac{820 \times 10^{-9}}{860 \times 10^{-6}}} = 3.95$$

c. When brought to the primary side, the impedance of the capacitor becomes

$$Z_{c,primary} = N^2 Z_{c,secondary} = N^2\left(\frac{1}{j\omega C}\right) = \frac{1}{j\omega\left(\frac{C}{N^2}\right)}$$

That is, the capacitor in our misconfigured circuit looks like it is 16 times smaller than intended. So, the new resonant frequency is:

$$\omega_0 = \frac{1}{\sqrt{LC_{new}}} = \frac{1}{\sqrt{L\left(\frac{C_{old}}{16}\right)}} = \frac{4}{\sqrt{LC_{old}}}$$

Thus, the new resonant frequency is:

$$f_{new} = 4f_{old} = 4 \times 4\ \text{kHz} = 24\ \text{kHz}$$

Also, the Q-factor is:

$$Q = R\sqrt{\frac{C_{new}}{L}} = R\sqrt{\frac{\left(\frac{C_{old}}{4}\right)}{L}} = \frac{1}{4}\left(R\sqrt{\frac{C_{old}}{L}}\right) = \frac{1}{4}Q_{old} = \frac{3.95}{4} \approx 1$$