XMUT203 Analogue Circuits and Systems

Tutorial 7a: Laplace Transforms and Its Circuit Applications (Solution)

A. Functional and Operational Transforms of Laplace Transform

1. Find the Laplace transform of each of the following functions:

   a. Function 1: [**Answer**: $1/[(s+a)^2]$] [2 marks]

   $$f_1(t) = te^{-at}$$

   b. Function 2: [**Answer**: $\omega/(s^2+\omega^2)$] [2 marks]

   $$f_2(t) = \sin \omega t$$

   c. Function 3: [**Answer**: $(\omega \cos\theta + s\sin\theta)/(s^2+\omega^2)$] [2 marks]

   $$f_3(t) = \sin(\omega t + \theta)$$

   d. Function 4: [**Answer**: $1/s^2$] [2 marks]

   $$f_4(t) = t$$

   e. Function 5: [**Answer**: $[\sinh\theta + s(\cosh\theta)]/(s^2-1)$] [6 marks]

   $$f_5(t) = \cosh(t+\theta)$$

Solution

a. Knowing that:

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

Therefore

$$\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$$

b. Knowing that:

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{j2}$$

Therefore

$$\mathcal{L}\{\sin \omega t\} = \frac{1}{2j}\left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right) = \frac{1}{2j}\left(\frac{2j\omega}{s^2 + \omega^2}\right) = \frac{\omega}{s^2 + \omega^2}$$

c. Knowing that:

$$\sin(\omega t + \theta) = \sin \omega t \cos \theta + \sin \theta$$

Therefore

$$\mathcal{L}\{\sin(\omega t + \theta)\} = \cos \theta \, \mathcal{L}\{\sin \omega t\} + \sin \theta \, \mathcal{L}\{\cos \omega t\} = \frac{\omega \cos \theta + s \sin \theta}{s^2 + \omega^2}$$

d. Knowing that $f(t) = t$, so:

$$\mathcal{L}\{t\} = \int_0^\infty t e^{-st} \, dt = \frac{e^{-st}}{s^2}(-st - 1)\Big|_0^\infty = 0 - \frac{1}{s^2}(0 - 1) = \frac{1}{s^2}$$

e. Knowing that:

$$f(t) = \cosh t \cosh \theta + \sinh t \sinh \theta$$

From AP 12.1(a)

$$\mathcal{L}\{\cosh t\} = \frac{s}{s^2 - 1}$$

From AP 12.1(b)

$$\mathcal{L}\{\sinh t\} = \frac{1}{s^2 - 1}$$

Therefore

$$\mathcal{L}\{\cosh(t + \theta)\} = \cosh \theta \left[\frac{s}{s^2 - 1}\right] + \sinh \theta \left[\frac{1}{s^2 - 1}\right] = \frac{\sinh \theta + s[\cosh \theta]}{s^2 - 1}$$

2. Find the Laplace transform for functions in parts (a) and (b).

a. Function A: [**Answer**: $F_A(s) = (-a^2 - sa - \omega^2)/[(s + a)^2 + \omega^2]$] [2 marks]

$$f_A(t) = \frac{d}{dt}(e^{-at} \cos \omega t)$$

b. Function B: [**Answer**: $F_B(s) = \omega/\{s[(s + a)^2 + \omega^2]\}$] [2 marks]

$$f_B(t) = \int_{0^-}^{t} e^{-ax} \sin \omega x \; dx$$

c. Verify the results obtained in (a) and (b) by first carrying out the indicated mathematical operation. [8 marks]

Solution

a. For $f_A(t) = e^{-at}\cos\omega t$, then

$$F_A(s) = \frac{s+a}{(s+a)^2+\omega^2}$$

Thus

$$F_A(s) = sF_A(s) - f_1(0^-) = \frac{s+a}{(s+a)^2+\omega^2} - 1 = \frac{-a^2-sa-\omega^2}{(s+a)^2+\omega^2}$$

b. For $f_B(t) = e^{-at}\sin\omega t$, then

$$F_B(s) = \frac{\omega}{(s+a)^2+\omega^2}$$

Thus

$$F_B(s) = \frac{F_1(s)}{s} = \frac{\omega}{s[(s+a)^2+\omega^2]}$$

c. Knowing that

$$\frac{d}{dt}[e^{-at}\cos\omega t] = -\omega e^{-at}\sin\omega t - ae^{-at}\cos\omega t$$

Therefore

$$F(s) = \frac{-\omega^2 - a(s+a)}{(s+a)^2+\omega^2} = \frac{-a^2-sa-\omega^2}{(s+a)^2+\omega^2}$$

Thus

$$\int_{0^-}^{t} e^{-ax}\sin\omega x\,dx = \frac{-ae^{-at}\sin\omega t - \omega e^{-at}\cos\omega t + \omega}{a^2+\omega^2}$$

Therefore

$$F(s) = \frac{1}{a^2+\omega^2}\left[-\frac{-a\omega}{(s+a)^2+\omega^2} - \frac{\omega(s+a)}{(s+a)^2+\omega^2} + \frac{\omega}{s}\right] = \frac{\omega}{s[(s+a)^2+\omega^2]}$$

3. Find the Laplace transform of each of the following functions:

a. Function 1: [**Answer**: $20e^{-10s}/(s+500)$] [2 marks]

$$f_1(t) = 20e^{-500(t-10)}u(t-10)$$

b. Function 2: [**Answer**: $5[e^{4s} - 2e^{2s} + 2e^{-2s} - e^{-4s}]/s^2$] [4 marks]

$$f_2(t) = (5t + 20)[u(t + 4) - u(t + 2)] - 5t[u(t + 2) - u(t - 2)] + (5t - 20)[u(t - 2) - u(t - 4)]$$

Solution

a. The Laplace transform of the function is:

$$\mathcal{L}\{20e^{-500(t-10)}u(t - 10)\} = \frac{20e^{-10s}}{(s + 500)}$$

b. First, rewrite $f_2(t)$ as:

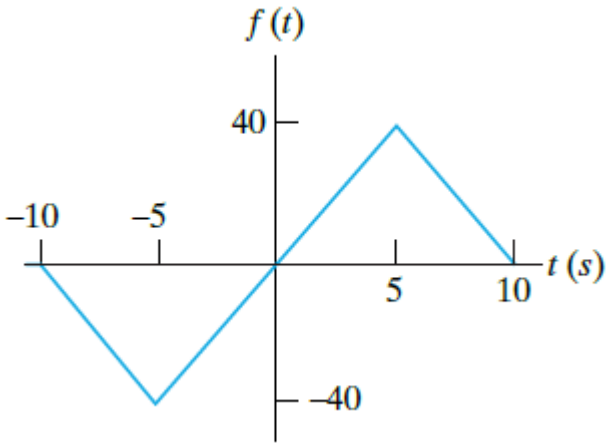
$$f_2(t) = (5t + 20)u(t + 4) - (10t + 20)u(t + 2) + (10t - 20)u(t - 2) - (5t - 20)u(t - 4)$$

$$= 5(t + 4)u(t + 4) - 10(t + 2)u(t + 2) + 10(t - 2)u(t - 2) - 5(t - 4)u(t - 4)$$

As a result,

$$F_2(s) = \frac{5[e^{4s} - 2e^{2s} + 2e^{-2s} - e^{-4s}]}{s^2}$$

4. With regard the function illustrated in the figure below, attempt the following tasks:



a. Find the Laplace transform of the function. [**Answer**: $F(s) = 8(-e^{10s} + 2e^{5s} - 2e^{-5s} + e^{-10s})/s^2$] [4 marks]

b. Find the Laplace transform of the first derivative of the function. [**Answer**: $\mathcal{L}\{f'(t)\} = 8(-e^{10s} + 2e^{5s} - 2e^{-5s} + e^{-10s})/s$] [6 marks]

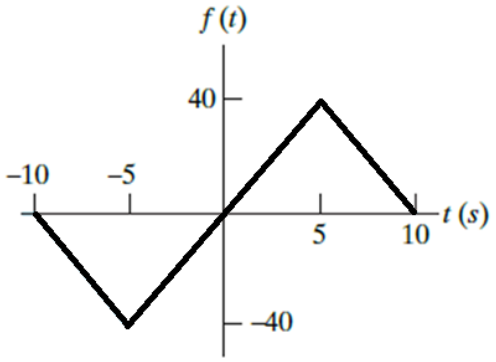
c. Find the Laplace transform of the second derivative of the function. [**Answer**: $\mathcal{L}\{f''(t)\} = 8[-e^{10s} + 2e^{5s} - 2e^{-5s} + e^{-10s}]$] [6 marks]

Solution

a. The function $f(t)$ as shown in the figure below is:

$$f(t) = (-8t - 80)[u(t+10) - u(t+5)] + 8t[u(t+5) - u(t-5)] + (-8t + 80)[u(t-5) - u(t-10)]$$

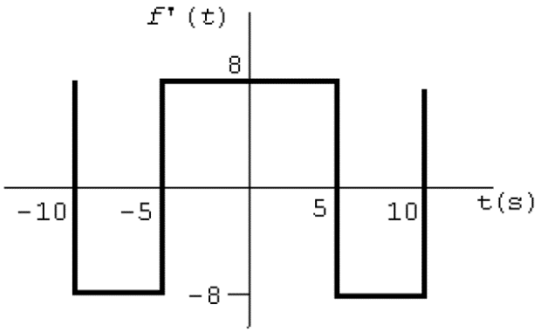
$$= 8(t+10)u(t+10) + 16(t+5)u(t+5) - 16(t-5)u(t-5) + 8(t-10)u(t-10)$$



Thus, the Laplace transform of the function is:

$$F(s) = \frac{8[-e^{10s} + 2e^{5s} - 2e^{-5s} + e^{-10s}]}{s^2}$$

b. The graph for the given function $f(t)$ is as shown in the figure below



The first derivative of function $f(t)$ is:

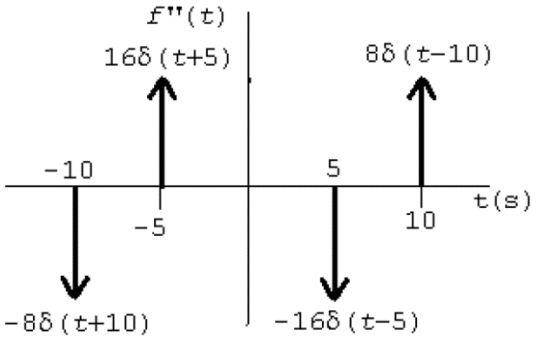
$$f'(t) = -8[u(t+10) - u(t+5)] + 8[u(t+5) - u(t-5)] + (-8)[u(t-5) - u(t-10)]$$

$$= -8u(t+10) + 16u(t+5) - 16u(t-5) + 8u(t-10)$$

As a result, the Laplace transform of the first derivative of $f(t)$ is:

$$\mathcal{L}\{f'(t)\} = \frac{8[-e^{10s} + 2e^{5s} - 2e^{-5s} + e^{-10s}]}{s}$$

c. The graph for the second derivative of function $f(t)$ is as shown in the figure below



The second derivative of function $f(t)$ is:

$$f''(t) = -8\delta(t+10) + 16\delta(t+5) - 16\delta(t-5) + 8\delta(t-10)$$

As a result, the Laplace transform of the second derivative of $f(t)$ is:

$$\mathcal{L}\{f''(t)\} = 8[-e^{10s} + 2e^{5s} - 2e^{-5s} + e^{-10s}]$$

B. Inverse Transforms and Partial Fractions Expansions in Laplace Transform

1. By applying partial fraction expansion and inverse Laplace transform, find $f(t)$ for each of the following s-domain functions:

a. Function 1: [**Answer**: $f(t) = [10e^{-5t} - 4e^{-8t}]u(t)$] [8 marks]

$$F_1(s) = \frac{6(s+10)}{(s+5)(s+8)}$$

b. Function 2: [**Answer**: $f(t) = [15 - 6e^{-3t} + 11e^{-7t}]u(t)$] [10 marks]

$$F_2(s) = \frac{20s^2 + 141s + 315}{s(s^2 + 10s + 21)}$$

c. Function 3: [**Answer**: $f(t) = [8e^{-2t} - 5e^{-4t} + 12e^{-6t}]u(t)$] [10 marks]

$$F_2(s) = \frac{15s^2 + 112s + 228}{(s+2)(s+4)(s+6)}$$

d. Function 4: [**Answer**: $f(t) = [9 + 4e^{-t} - 8e^{-2t} - 3e^{-3t}]u(t)$] [12 marks]

$$F_3(s) = \frac{2s^3 + 33s^2 + 93s + 54}{s(s+1)(s^2 + 5s + 6)}$$

Solution

a. Applying partial fraction expansion, the equation becomes:

$$F_1(s) = \frac{K_1}{s+5} + \frac{K_2}{s+8}$$

Constants K_1 and K_2 in the equation above are:

$$K_1 = \left.\frac{6(s+10)}{(s+8)}\right|_{s=-5} = 10$$

$$K_2 = \left.\frac{6(s+10)}{(s+5)}\right|_{s=-8} = -4$$

Thus

$$F_1(s) = \frac{10}{s+5} - \frac{4}{s+8}$$

Taking inverse Laplace transform of the equation above, the equation is now:

$$f_1(t) = [10e^{-5t} - 4e^{-8t}]u(t)$$

b. Applying partial fraction expansion, the equation becomes:

$$F_2(s) = \frac{K_1}{s} + \frac{K_2}{s+3} + \frac{K_3}{s+7}$$

Constants K_1, K_2 and K_3 in the equation above are:

$$K_1 = \left.\frac{20s^2 + 141s + 315}{(s+3)(s+7)}\right|_{s=0} = 15$$

$$K_2 = \left.\frac{20s^2 + 141s + 315}{s(s+7)}\right|_{s=-3} = -6$$

$$K_3 = \left.\frac{20s^2 + 141s + 315}{s(s+3)}\right|_{s=-7} = 11$$

Thus

$$F_2(s) = \frac{15}{s} - \frac{6}{s+3} + \frac{11}{s+7}$$

Taking inverse Laplace transform of the equation above, the equation is now:

$$f_2(t) = [15 - 6e^{-3t} + 11e^{-7t}]u(t)$$

c. Applying partial fraction expansion, the equation becomes:

$$F_3(s) = \frac{K_1}{s+2} + \frac{K_2}{s+4} + \frac{K_3}{s+6}$$

Constants K_1, K_2 and K_3 in the equation above are:

$$K_1 = \left.\frac{15s^2 + 112s + 228}{(s+4)(s+6)}\right|_{s=-2} = 8$$

$$K_2 = \left.\frac{15s^2 + 112s + 228}{(s+2)(s+6)}\right|_{s=-4} = -5$$

$$K_3 = \left.\frac{15s^2 + 112s + 228}{(s+2)(s+4)}\right|_{s=-6} = 12$$

Thus

$$F_3(s) = \frac{8}{s+2} - \frac{5}{s+4} + \frac{12}{s+6}$$

Taking inverse Laplace transform of the equation above, the equation is now:

$$f_3(t) = [8e^{-2t} - 5e^{-4t} + 12e^{-6t}]u(t)$$

d. Applying partial fraction expansion, the equation becomes:

$$F_4(s) = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2} + \frac{K_4}{s+3}$$

Constants K_1, K_2, K_3 and K_4 in the equation above are:

$$K_1 = \left.\frac{2s^3 + 33s^2 + 93s + 54}{(s+1)(s+2)(s+3)}\right|_{s=0} = 9$$

$$K_2 = \left.\frac{2s^3 + 33s^2 + 93s + 54}{s(s+2)(s+3)}\right|_{s=-1} = 4$$

$$K_3 = \left.\frac{2s^3 + 33s^2 + 93s + 54}{s(s+1)(s+3)}\right|_{s=-2} = -8$$

$$K_4 = \left.\frac{2s^3 + 33s^2 + 93s + 54}{s(s+1)(s+2)}\right|_{s=-3} = -3$$

Thus

$$F_4(s) = \frac{9}{s} + \frac{4}{s+1} - \frac{8}{s+2} - \frac{3}{s+3}$$

Taking inverse Laplace transform of the equation above, the equation is now:

$$f_4(t) = [9 + 4e^{-t} - 8e^{-2t} - 3e^{-3t}]u(t)$$

2. By applying partial fraction expansion and inverse Laplace transform, find $f(t)$ for each of the following s-domain functions:

a. Function 1: [**Answer**: $f_1(t) = [40t - 5 + 5e^{-8t}]u(t)$] [10 marks]

$$F_1(s) = \frac{320}{s^2(s+8)}$$

b. Function 2: [**Answer**: $f_2(t) = [60 - 40te^{-2t} - 60e^{-2t}]u(t)$] [10 marks]

$$F_2(s) = \frac{80(s+3)}{s(s+2)^2}$$

c. Function 3: [**Answer**: $f_3(t) = [12te^{-t} + 0.6e^{-t} + 3.35e^{-3t}\cos(4t + 100.305°)]u(t)$]

[12 marks]

$$F_3(s) = \frac{60(s+5)}{(s+1)^2(s^2+6s+25)}$$

d. Function 4: [**Answer**: $f_4(t) = [16t + 1.6 + te^{-5t} - 1.6e^{-5t}]u(t)$] [12 marks]

$$F_4(s) = \frac{25(s+4)^2}{s^2(s+5)^2}$$

Solution

a. Applying partial fraction expansion, the equation becomes:

$$F_1(s) = \frac{K_1}{s^2} + \frac{K_2}{s} + \frac{K_2}{s+8}$$

Constants K_1, K_2 and K_3 in the equation above are:

$$K_1 = \frac{320}{(s+8)}\Big|_{s=0} = 40$$

$$K_2 = \frac{d}{ds}\left|\frac{320}{(s+8)}\right|_{s=0} = \left|\frac{-320}{(s+8)^2}\right|_{s=0} = -5$$

$$K_3 = \frac{320}{s^2}\Big|_{s=-8} = 5$$

Thus

$$F_1(s) = \frac{40}{s^2} - \frac{5}{s} + \frac{5}{s+8}$$

Taking inverse Laplace transform of the equation above, the equation is now:

$$f_1(t) = [40t - 5 + 5e^{-8t}]u(t)$$

b. Applying partial fraction expansion, the equation becomes:

$$F_2(s) = \frac{K_1}{s} + \frac{K_2}{(s+2)^2} + \frac{K_3}{s+2}$$

Constants K_1, K_2 and K_3 in the equation above are:

$$K_1 = \frac{80(s+3)}{(s+2)^2}\Big|_{s=0} = 60$$

$$K_2 = \frac{80(s+3)}{s}\Big|_{s=-2} = -40$$

$$K_3 = \frac{d}{ds}\left|\frac{80(s+3)}{s}\right|_{s=-2} = \left|\frac{80s - 80(s+3)}{s^2}\right|_{s=-2} = -60$$

Thus

$$F_2(s) = \frac{60}{s} - \frac{40}{(s+2)^2} - \frac{60}{s+2}$$

Taking inverse Laplace transform of the equation above, the equation is now:

$$f_2(t) = [60 - 40te^{-2t} - 60e^{-2t}]u(t)$$

c. Applying partial fraction expansion, the equation becomes:

$$F_3(s) = \frac{K_1}{(s+1)^2} + \frac{K_2}{s+1} + \frac{K_3}{s+3-j4} + \frac{K_3^*}{s+3+j4}$$

Constants K_1, K_2, and K_3 and K_3^* in the equation above are:

$$K_1 = \frac{60(s+5)}{(s^2+6s+25)}\bigg|_{s=-1} = 12$$

$$K_2 = \frac{d}{ds}\left|\frac{60(s+5)}{(s^2+6s+25)}\right|_{s=-1}$$

$$= \left|\frac{60(s^2+6s+25) - 60(s+5)(2s+6)}{(s^2+6s+25)^2}\right|_{s=-1} = 0.6$$

$$K_3 = \frac{60(s+5)}{(s+1)^2(s+3+j4)}\bigg|_{s=-3+j4} = 1.68\angle 100.305°$$

Thus

$$F_3(s) = \frac{12}{(s+2)^2} + \frac{0.6}{s+1} + \frac{1.68\angle 100.305°}{s+3-j4} + \frac{1.68\angle -100.305°}{s+3+j4}$$

Taking inverse Laplace transform of the equation above,

$$i_o(t) = 12te^{-t} + 0.6e^{-t} + 1.68e^{-j100.305°}e^{-(3-j4)t} + 1.68e^{j100.305°}e^{-(3+j4)t}$$

$$= 12te^{-t} + 0.6e^{-t} + 1.68e^{-3t}\left[e^{j(4-100.305°)} + e^{-j(4-100.305°)}\right]$$

Finally, the equation is now:

$$f_3(t) = [12te^{-t} + 0.6e^{-t} + 3.35e^{-3t}\cos(4t + 100.305°)]u(t)$$

d. Applying partial fraction expansion, the equation becomes:

$$F_4(s) = \frac{K_1}{s^2} + \frac{K_2}{s} + \frac{K_3}{(s+5)^2} + \frac{K_4}{s+5}$$

Constants K_1, K_2, K_3 and K_4 in the equation above are:

$$K_1 = \frac{25(s+4)^2}{(s+5)^2}\bigg|_{s=0} = 16$$

$$K_2 = \frac{d}{ds}\left|\frac{25(s+4)^2}{(s+5)^2}\right|_{s=0}$$

$$= \left| \frac{25(2)(s+4)(s+5) - 25(2)(s+4)^2}{(s+5)^3} \right| s = 0 = 1.6$$

$$K_3 = \frac{25(s+4)^2}{s^2} \Big| s = -5 = 1$$

$$K_4 = \frac{d}{ds} \left| \frac{25(s+4)^2}{s^2} \right| s = -5$$

$$= \left| \frac{25(2)(s+4)s - 25(2)(s+4)^2}{s^3} \right| s = -5 = -1.6$$

Thus

$$F_4(s) = \frac{16}{s^2} + \frac{1.6}{s} + \frac{1}{(s+5)^2} - \frac{1.6}{s+5}$$

Taking inverse Laplace transform of the equation above, the equation is now:

$$f_4(t) = [16t + 1.6 + te^{-5t} - 1.6e^{-5t}]u(t)$$

C. Initial and Final Value Theorems in Laplace Transform

1. Apply the initial- and final-value theorems to each transform pair given below and determine the initial and final conditions of the functions.

   a. Function 1: [**Answer**: $f_1(\infty) = 0$ and $f_1(0^+) = 6$] [6 marks]

   $$F_1(s) = \frac{6(s+10)}{(s+5)(s+8)}$$

   b. Function 2: [**Answer**: $f_2(\infty) = 15$ and $f_2(0^+) = 20$] [6 marks]

   $$F_2(s) = \frac{20s^2 + 141s + 315}{s(s^2 + 10s + 21)}$$

   c. Function 3: [**Answer**: $f_3(\infty) = 0$ and $f_3(0^+) = 15$] [6 marks]

   $$F_3(s) = \frac{15s^2 + 112s + 228}{(s+2)(s+4)(s+6)}$$

   d. Function 4: [**Answer**: $f_4(\infty) = 9$ and $f_4(0^+) = 2$] [6 marks]

   $$F_4(s) = \frac{2s^3 + 33s^2 + 93s + 54}{s(s+1)(s^2 + 5s + 6)}$$

   Solution

   a. The s-domain equation for the circuit is:

   $$sF_1(s) = \frac{6s^2 + 60s}{(s+5)(s+8)}$$

Applying initial-value and final-value theorems to the equation above:

$$\lim_{s\to 0} sF_1(s) = \frac{0}{(5)(8)} = 0, \quad f_1(\infty) = 0$$

$$\lim_{s\to \infty} sF_1(s) = \frac{6s^2}{s^2} = 6, \quad f_1(0^+) = 6$$

b. The s-domain equation for the circuit is:

$$sF_2(s) = \frac{20s^2 + 141s + 315}{(s^2 + 10s + 21)}$$

Applying initial-value and final-value theorems to the equation above:

$$\lim_{s\to 0} sF_2(s) = \frac{315}{21} = 15, \quad f_2(\infty) = 15$$

$$\lim_{s\to \infty} sF_2(s) = \frac{20s^2}{s^2} = 20, \quad f_2(0^+) = 20$$

c. The s-domain equation for the circuit is:

$$sF_3(s) = \frac{15s^3 + 112s^2 + 228s}{(s+2)(s+4)(s+6)}$$

Applying initial-value and final-value theorems to the equation above:

$$\lim_{s\to 0} sF_3(s) = \frac{0}{(2)(4)(6)} = 0, \quad f_3(\infty) = 0$$

$$\lim_{s\to \infty} sF_3(s) = \frac{15s^3}{s^3} = 15, \quad f_3(0^+) = 15$$

d. The s-domain equation for the circuit is:

$$sF_4(s) = \frac{2s^3 + 33s^2 + 93s + 54}{(s+1)(s^2 + 5s + 6)}$$

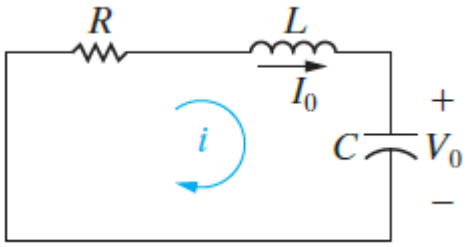
Applying initial-value and final-value theorems to the equation above:

$$\lim_{s\to 0} sF_4(s) = \frac{54}{6} = 9, \quad f_4(\infty) = 9$$

$$\lim_{s\to \infty} sF_4(s) = \frac{2s^3}{s^3} = 2, \quad f_4(0^+) = 2$$

D. Poles and Zeros in Laplace Transform

1. A 400 Ω resistor, a 12.5 mH inductor, and a 0.2 μF capacitor are in series.



   a. Express the s-domain impedance of this series combination as a rational function. [**Answer**: $Z(s) = [s^2 + 32{,}000s + 4 \times 10^8]/80s$] [4 marks]

   b. Give the numerical value of the poles and zeros of the impedance. [**Answer**: z_1 = -16,000 + j12,000 rad/s, z_2 = -16,000 – j12,000 rad/s, and $p_1 = 0$] [6 marks]

Solution

a. The total impedance of the circuit is calculated from:

$$Z(s) = R + sL + \frac{1}{sC}$$

$$= \frac{L\left[s^2 + \left(\frac{R}{L}\right)s + \left(\frac{1}{LC}\right)\right]}{s}$$

Thus, entering values of the components into the equation above, the Laplace transform of the impedance of the circuit is:

$$Z(s) = \frac{s^2 + 32{,}000s + 4 \times 10^8}{80s}$$

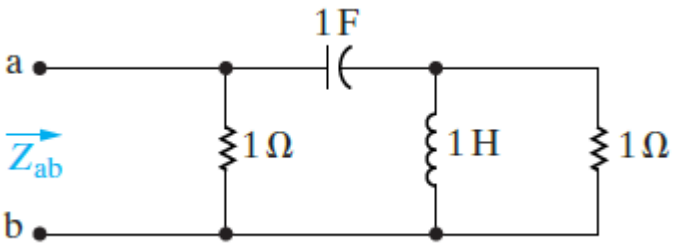
b. The roots of the second order equation on the numerator in part (a) are determined from:

$$s_{1,2} = -16{,}000 \pm j12{,}000 \text{ rad/s}$$

Thus, the zeros and poles of the circuit are:

- Zeros at z_1 = -16,000 + j12,000 rad/s and z_2 = -16,000 – j12,000 rad/s.
- Poles at $p_1 = 0$.

2. Find the poles and zeros of the impedance seen looking into the terminals a, b of the circuit shown in the figure below. [**Answer**: $z_1 = -0.5 + j0.866$ rad/s, $z_2 = -0.5 - j0.866$ rad/s, $p_1 = -0.5 + j0.5$ rad/s, and $p_2 = -0.5 - j0.5$ rad/s] [8 marks]



Solution

The impedance of the circuit is determined from:

$$Z_{ab}(s) = \left(1 \parallel s + \frac{1}{s}\right) \parallel 1 = \left(\frac{(s)(1)}{s+1} + \frac{1}{s}\right) \parallel 1 = \left[\frac{s^2+s+1}{s(s+1)}\right] \parallel 1$$

Thus, the impedance of the circuit across terminals a, b is:

$$Z_{ab}(s) = \frac{\left[\frac{s^2+s+1}{s(s+1)}\right](1)}{\frac{s^2+s+1}{s(s+1)} + 1}$$

$$= \frac{s^2+s+1}{2s^2+2s+1}$$

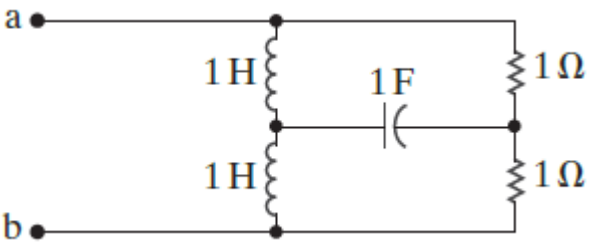
$$= \frac{0.5(s^2+s+1)}{s^2+s+0.5}$$

The poles and zeros of the circuit are:

- Zeros at $z_1 = -0.5 + j0.866$ rad/s and $z_2 = -0.5 - j0.866$ rad/s.
- Poles at $p_1 = -0.5 + j0.5$ rad/s and $p_2 = -0.5 - j0.5$ rad/s.

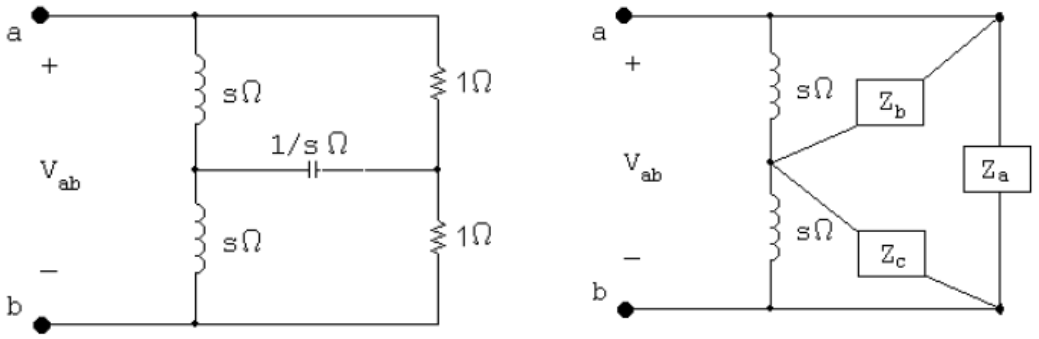
3. Find the poles and zeros of the impedance seen looking into the terminals a, b of the circuit shown in the figure below. [**Answer**: z_1 = 0, z_2 = -2 rad/s, p_1 = -1 rad/s, and p_2 = -2 rad/s]

[24 marks]

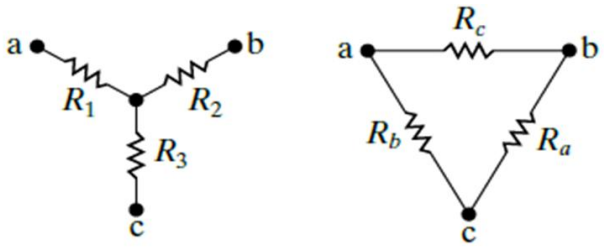


Solution

Transform the star (wye) connection of two 1-Ω resistors and 1 F capacitor into its equivalent delta connection:



Transformation from star-to-delta circuit requires the standard equations as given below.



Where:

$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1}$$

$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}$$

$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}$$

Referring to the circuits given in the above figure, the results of the impedance transformation are listed below.

$$Z_a = \frac{\left(\frac{1}{s}\right)(1) + (1)\left(\frac{1}{s}\right) + (1)(1)}{\left(\frac{1}{s}\right)} = s + 2$$

$$Z_b = \frac{\left(\frac{1}{s}\right)(1) + (1)\left(\frac{1}{s}\right) + (1)(1)}{1} = \frac{s+2}{s}$$

$$Z_c = \frac{\left(\frac{1}{s}\right)(1) + (1)\left(\frac{1}{s}\right) + (1)(1)}{1} = \frac{s+2}{s}$$

Referring to the transformation equations given above, notice that impedances $Z_b = Z_c$.

Then, the impedance between terminals a and b in the circuit is now a parallel between Z_a and the sum of 1 H inductor with Z_b and 1H inductor with Z_c.

$$Z_{ab} = Z_a \parallel [(s \parallel Z_c) + (s \parallel Z_b)] = Z_a \parallel 2(s \parallel Z_b)$$

Where:

$$s \parallel Z_b = \frac{s\left(\frac{s+2}{s}\right)}{s+\left(\frac{s+2}{s}\right)} = \frac{s(s+2)}{s^2+s+2}$$

Thus, the impedance of the circuit is:

$$Z_{ab} = (s+2) \parallel \frac{2s(s+2)}{s^2+s+2}$$

$$= \frac{(s+2)\left[\frac{2s(s+2)}{s^2+s+2}\right]}{(s+2)+\frac{2s(s+2)}{s^2+s+2}}$$

$$= \frac{2s(s+2)^2}{(s+2)(s^2+s+2)+2s(s+2)}$$

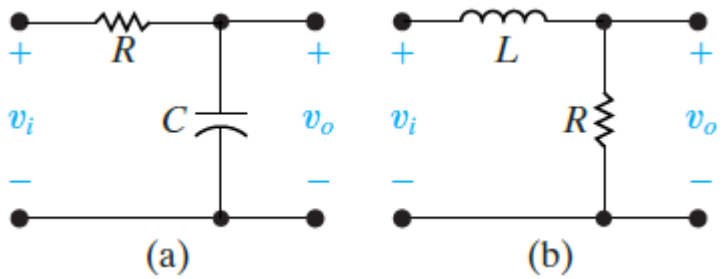
$$= \frac{2s(s+2)}{s^2+3s+2}$$

The poles and zeros of the circuit are:

- Zeros at z_1 = 0 and z_2 = -2 rad/s.
- Poles at p_1 = -1 rad/s and p_2 = -2 rad/s.

E. Transfer Functions with Laplace Transform

1. For the circuit given in the figure below, attempt the following tasks.



a. Find the transfer function $H(s) = V_o(s)/V_i(s)$ for the circuit shown in Fig. (a). [**Answer**: $H(s) = (1/RC)/[s + (1/RC)]$] [4 marks]

b. Find the transfer function $H(s) = V_o(s)/V_i(s)$ for the circuit shown in Fig. (b). [**Answer**: $H(s) = (R/L)/[s + (R/L)]$] [2 marks]

c. Create two different circuits that have the transfer function:

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1000}{s+1000}$$

Use components selected from E24 component standards and Figs. (a) and (b). [6 marks]

Solution

a. The transfer function of the circuit in figure (a) is determined from:

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\left(\frac{1}{sC}\right)}{R + \left(\frac{1}{sC}\right)}$$

$$= \frac{1}{RsC + 1} = \frac{\left(\frac{1}{RC}\right)}{s + \left(\frac{1}{RC}\right)}$$

Thus, there are no zeros and a single pole at -1/RC rad/s.

b. The transfer function of the circuit in figure (b) is determined from:

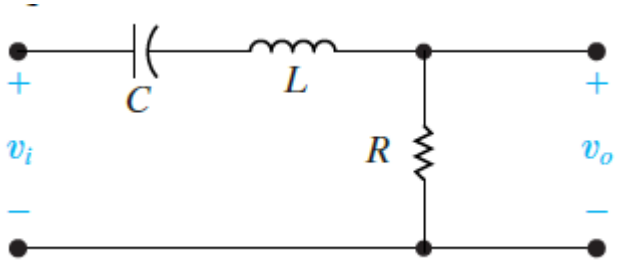
$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{R + sL}$$

$$= \frac{\left(\frac{R}{L}\right)}{s + \left(\frac{R}{L}\right)}$$

c. There are several possible solutions. One of them is:

$$R = 10\Omega; \qquad L = 10\ \text{mH}; \qquad C = 100\ \mu\text{F}$$

2. For the circuit as shown in the figure below, perform the following tasks.



a. Find the transfer function $H(s) = V_o(s)/V_i(s)$ for the circuit shown above. Identify the poles and zeros for this transfer function. [**Answer**: $H(s) = [(R/L)s]/[s^2 + (R/L)s + (1/LC)]$ and $p_{1,2} = -(R/2L) \pm \sqrt{(R/2L)^2 - (1/LC)}$] [6 marks]

b. Find three components from E24 component standards which when used in the circuit that will result in a transfer function with the following characteristics. Calculate the values of the poles.

i. Two poles that are distinct real numbers. [**Answer**: $R = 250\ \Omega; L = 10\ \text{mH}; C = 1\ \mu\text{F}$] [4 marks]

ii. Two poles, both with the same value. [**Answer**: $R = 200\ \Omega; L = 10\ \text{mH}; C = 1\ \mu\text{F}$] [4 marks]

iii. Two poles that are complex conjugate complex numbers. [**Answer**: $R = 120\ \Omega;\ L = 10\ \text{mH}; C = 1\ \mu\text{F}$] [4 marks]

Solution

a. The transfer function of the circuit given above is determined from:

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{\left(\frac{1}{sC}\right) + sL + R} = \frac{\left(\frac{R}{L}\right)s}{s^2 + \left(\frac{R}{L}\right)s + \left(\frac{1}{LC}\right)}$$

There is a single zero at 0 rad/s and two poles at:

$$p_1 = -\left(\frac{R}{2L}\right) + \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)}$$

$$p_2 = -\left(\frac{R}{2L}\right) - \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)}$$

b. For the three components for the circuit with stated characteristics:

$$\left(\frac{R}{2L}\right)^2 = \frac{1}{LC} \qquad \text{thus} \qquad R = 2\sqrt{\frac{L}{C}}$$

As a result,

- For repeated roots, make the value of $R = 2\sqrt{L/C}$.
- For real roots, make the value of resistor > R.
- For complex roots, make the value of resistor < R.

i. There are several possibilities. One of the solutions is:

$$R = 250\ \Omega; \qquad L = 10\ \text{mH}; \qquad C = 1\ \mu\text{F}$$

These component values in the circuit yield the following poles:

p_1 = -5000 rad/s and p_2 = -20,000 rad/s

ii. There are several possibilities. One of the solutions is:

$$R = 200\ \Omega; \qquad L = 10\ \text{mH}; \qquad C = 1\ \mu\text{F}$$

These component values in the circuit yield the following poles:

p_1 = -10,000 rad/s and p_2 = -10,000 rad/s

iii. There are several possibilities. One of the solutions is:

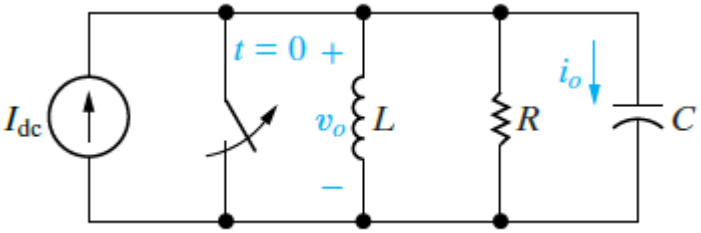
$$R = 120\ \Omega; \quad L = 10\ \text{mH}; \quad C = 1\ \mu\text{F}$$

These component values in the circuit yield the following poles:

p_1 = -6000 + j8000 rad/s and p_2 = -6000 – j8000 rad/s

F. Circuit Applications of Laplace Transform

1. There is no energy stored in the circuit shown in the figure below at the time the switch is opened.



a. Derive the integro-differential equation that governs the behaviour of the voltage v_o.
[**Answer**: $I_{dc} = 1/L \int_0^t v_o\, dx + v_o/R + C dv_o/dt$] [4 marks]

b. Show that: [4 marks]

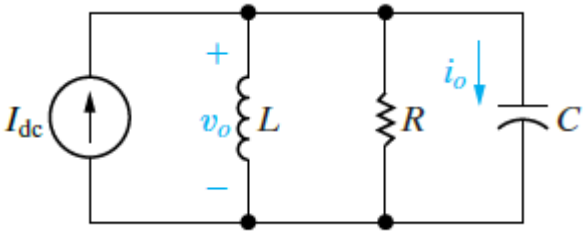
$$V_o(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}$$

c. Show that: [4 marks]

$$I_o(s) = \frac{sI_{dc}}{s^2 + (1/RC)s + (1/LC)}$$

Solution

a. The equivalent circuit of the circuit given above when the switch is closed is as shown in the figure below.



When the switch is open at $t = 0$, by applying KCL in the v_o node in the top of the circuit,

the current I_{dc} is calculated from:

$$I_{dc} = \frac{1}{L}\int_0^t v_o\,dx + \frac{v_o}{R} + C\frac{dv_o}{dt}$$

b. Applying Laplace transform, the equation for the circuit becomes:

$$\frac{I_{dc}(s)}{s} = \frac{V_o(s)}{sL} + \frac{V_o(s)}{R} + sCV_o(s)$$

Rearrange the equation

$$V_o(s)\left[\frac{1}{sL} + \frac{1}{R} + sC\right] = \frac{I_{dc}(s)}{s}$$

Thus

$$V_o(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}$$

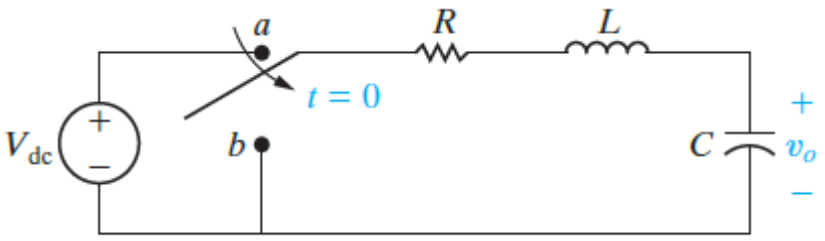
c. The output current is determined from the following equation

$$i_o(t) = C\frac{dV_o}{dt}$$

As a result, the Laplace transform of the output current is:

$$I_o(s) = sCV_o(s) = \frac{sI_{dc}(s)}{s^2 + (1/RC)s + (1/LC)}$$

2. The switch in the circuit in the figure below has been in position a for a long time. At $t = 0$, the switch moves instantaneously to position b.



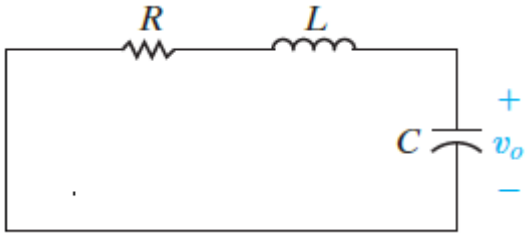
a. Derive the integro-differential equation that governs the behaviour of the voltage v_o for a period $t \geq 0^+$. [**Answer**: $d^2v_o/dt^2 + (R/L)dv_o/dt + (1/LC)v_o = 0$] [4 marks]

b. Show that: [6 marks]

$$V_o(s) = \frac{V_{dc}\left[s + \left(\frac{R}{L}\right)\right]}{s^2 + \left(\frac{R}{L}\right)s + \left(\frac{1}{LC}\right)}$$

Solution

a. At $t \geq 0^+$, the equivalent circuit for the circuit is shown in the figure below:



For $t \geq 0^+$, applying KVL for the closed loop in the circuit, the equation for the circuit is:

$$Ri_o + L\frac{di_o}{dt} + v_o = 0$$

Where:

$$i_o(t) = \frac{Cdv_o}{dt}$$

Thus, the first derivative of output current is:

$$\frac{di_o}{dt} = C\frac{d^2v_o}{dt^2}$$

As a result, the equation for the circuit becomes:

$$RC\frac{dv_o}{dt} + LC\frac{d^2v_o}{dt^2} + v_o = 0$$

Or

$$\frac{d^2v_o}{dt^2} + \left(\frac{R}{L}\right)\frac{dv_o}{dt} + \left(\frac{1}{LC}\right)v_o = 0$$

b. Applying Laplace transform to the equation in part (a), it becomes:

$$s^2V_o(s) - sV_{dc}(s) - 0 + \frac{R}{L}[sV_o(s) - V_{dc}(s)] + \frac{1}{LC}V_o(s) = 0$$

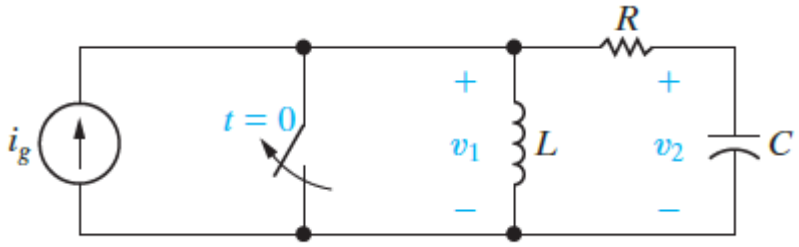
Gather all variables of to the left-hand side and to the right-hand side, it is now:

$$V_o(s)\left[s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC}\right] = V_{dc}\left(s + \frac{R}{L}\right)$$

Rearrange the equation given above, the s-domain equation for the circuit is:

$$V_o(s) = \frac{V_{dc}\left(s + \frac{R}{L}\right)}{\left[s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC}\right]}$$

3. There is no energy stored in the circuit shown in the figure below at the time the switch is opened.



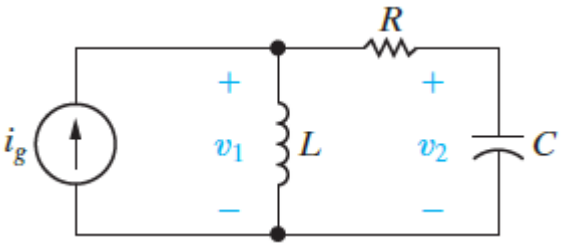
a. Derive the integro-differential equations that govern the behaviour of the node voltages v_1 and v_2. [**Answer**: $C(dv_2/dt) + (v_2 - v_{1)}/R = 0$] [4 marks]

b. Show that: [6 marks]

$$V_2(s) = \frac{sI_g(s)}{C[s^2 + (R/L)s + (1/LC)]}$$

Solution

a. The equivalent circuit when $t \geq 0^+$ is shown in the figure below.



By applying KCL on the node v_1 of the circuit, the equations for the circuit are:

$$\frac{1}{L}\int_0^t v_1\, d\tau + \frac{v_1 - v_2}{R} = i_g u(t)$$

And

$$C\left(\frac{dv_2}{dt}\right) + \frac{v_2 - v_1}{R} = 0$$

b. Taking Laplace transform of the equation in part (a), it becomes:

$$\frac{V_1(s)}{sL} + \frac{V_1(s) - V_2(s)}{R} = I_g(s)$$

And

$$sCV_2 + \frac{V_2 - V_1}{R} = 0$$

Or

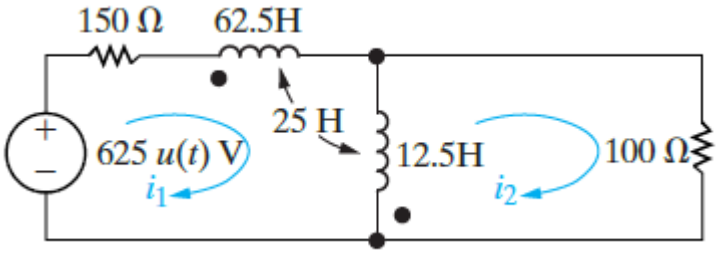
$$(R + sL)V_1(s) - sLV_2(s) = RLsI_g(s)$$

$$-V_1(s) + (RCs + 1)V_2(s) = 0$$

Solving the equation by substitution, then:

$$V_2(s) = \frac{sI_g(s)}{C[s^2 + (R/L)s + (1/LC)]}$$

4. For the circuit shown in the figure below, perform the following tasks.



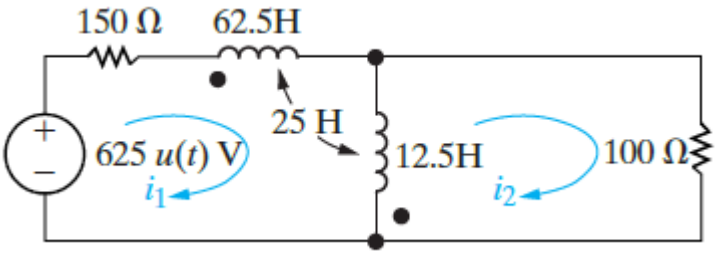
a. Write the two simultaneous differential equations that describe the circuit in terms of the mesh currents i_1 and i_2. [**Answer**: $150i_1 + 25di_1/dt + 12.5di_2/dt = 625$ and $100i_2 + 12.5di_1/dt + 12.5di_2/dt = 0$] [6 marks]

b. Laplace-transform the equations derived in (a). Assume that the initial energy stored in the circuit is zero. [**Answer**: $(25s + 150)I_1(s) + 12.5sI_2(s) = 625/s$ and $12.5sI_1(s) + (12.5s + 100)I_2(s) = 0$] [4 marks]

c. Solve the equations in (b) for $I_1(s)$ and $I_2(s)$. [**Answer**: $I_1(s) = [50(s + 8)]/[s(s + 4)(s + 24)]$ and $I_2(s) = -50/[(s + 4)(s + 24)]$] [4 marks]

Solution

a. The equivalent circuit of the circuit given above is:



By applying KVL across the closed loop meshes i_1 and i_2, the equation for the circuit are:

$$150i_1 + 62.5\frac{di_1}{dt} + 25\frac{d}{dt}(i_2 - i_1) + 12.5\frac{d}{dt}(i_1 - i_2) - 25\frac{di_1}{dt} = 625$$

$$12.5\frac{d}{dt}(i_2 - i_1) + 25\frac{di_1}{dt} + 100i_2 = 0$$

Simplifying the above equations gives:

$$150i_1 + 25\frac{di_1}{dt} + 12.5\frac{di_2}{dt} = 625$$

$$100i_2 + 12.5\frac{di_1}{dt} + 12.5\frac{di_2}{dt} = 0$$

b. Taking Laplace transform of the equation in part (a)

$$(25s + 150)I_1(s) + 12.5sI_2(s) = \frac{625}{s}$$

$$12.5sI_1(s) + (12.5s + 100)I_2(s) = 0$$

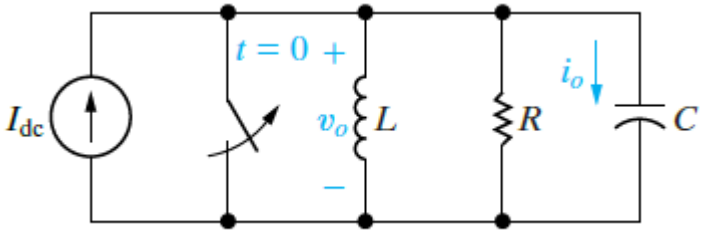
c. Solve the equations in part (b) by substitution, the equations for the currents in the circuit are

$$I_1(s) = \frac{50(s+8)}{s(s+4)(s+24)}$$

And

$$I_2(s) = \frac{-50}{(s+4)(s+24)}$$

5. The circuit parameters in the circuit seen in the figure below have the following values: R = 20 Ω, L = 50 mH, C = 20 μF, and $i_{dc} = 75$ mA.



a. Find $v_o(t)$ for $t \geq 0$. [**Answer**: $v_o(t) = 2.5[e^{-500t} - e^{-2000t}]u(t)$] [14 marks]

b. Find $i_o(t)$ for $t \geq 0$. [**Answer**: $i_o(t) = [-0.025e^{-500t} + 0.1e^{-2000t}]u(t)$ A] [14 marks]

c. Does your solution for make sense when $t = 0$? Explain. [4 marks]

Solution

a. From the results obtained in the previous question, the s-domain equation for the output voltage in the circuit is:

$$V_o(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}$$

Where:

$$\frac{1}{RC} = \frac{1}{(20)(20 \times 10^{-6})} = 2500$$

$$\frac{1}{LC} = \frac{1}{(0.05)(20 \times 10^{-6})} = 10^6$$

$$\frac{1}{C} = \frac{1}{20 \times 10^{-6}} = 50{,}000$$

Knowing that $i_{dc} = 75$ mA, the equation for the output voltage in the circuit becomes:

$$V_o(s) = \frac{50{,}000 I_{dc}}{s^2 + 2500s + 10^6} = \frac{3750}{(s + 500)(s + 2000)}$$

Applying partial fraction expansion, the equation above becomes

$$V_o(s) = \frac{K_1}{s + 500} + \frac{K_2}{s + 2000}$$

Constants K_1 and K_2 in the equation above are:

$$K_1 = \left.\frac{3750}{s + 2000}\right|_{s = -500} = 2.5$$

$$K_2 = \left.\frac{3750}{s + 500}\right|_{s = -2000} = -2.5$$

Thus

$$V_o(s) = \frac{2.5}{s + 500} - \frac{2.5}{s + 2000}$$

Taking inverse Laplace transform, the equation for the output voltage is:

$$v_o(t) = 2.5[e^{-500t} - e^{-2000t}]u(t)$$

b. From the results obtained in the previous question, the s-domain equation for the output voltage in the circuit is:

$$I_o(s) = \frac{sI_{dc}}{s^2 + (1/RC)s + (1/LC)}$$

Where:

$$\frac{1}{RC} = \frac{1}{(20)(20 \times 10^{-6})} = 2500$$

$$\frac{1}{LC} = \frac{1}{(0.05)(20 \times 10^{-6})} = 10^6$$

Knowing that $i_{dc} = 75$ mA, the equation for the output current in the circuit becomes:

$$I_o(s) = \frac{sI_{dc}}{s^2 + 2500s + 10^6} \frac{0.075s}{(s + 500)(s + 2000)}$$

Applying partial fraction expansion, the equation above becomes

$$I_o(s) = \frac{K_1}{s+500} + \frac{K_2}{s+2000}$$

Constants K_1 and K_2 in the equation above are:

$$K_1 = \left.\frac{0.075s}{s+2000}\right|_{s=-500} = -0.025$$

$$K_2 = \left.\frac{0.075s}{s+500}\right|_{s=-2000} = -0.1$$

Thus

$$I_o(s) = \frac{-0.025}{s+500} + \frac{0.1}{s+2000}$$

Taking inverse Laplace transform, the equation for the output voltage is:

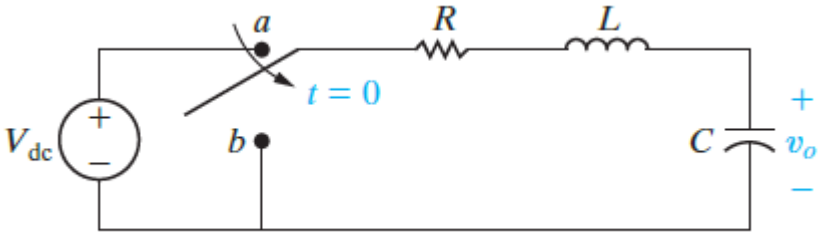
$$i_o(t) = [-0.025e^{-500t} + 0.1e^{-2000t}]u(t) \text{ A}$$

c. At $t = 0$, the output current is determined from:

$$i_o(0) = 0.1 - 0.025 = 0.075 \text{ A}$$

Yes. The initial inductor current is zero by hypothesis. The initial resistor current is zero because the initial capacitor voltage is zero by hypothesis. Thus, at $t = 0$, the source current appears in the capacitor.

6. The circuit parameters in the circuit in the figure below are R = 2 kΩ, L = 400 mH, and C = 156.25 nF. If V_{dc}= 120 V, find $v_o(t)$ for $t \geq 0$. [**Answer**: $v_o(t) = [160e^{-2000t} - 40e^{-8000t}]u(t)$ V] [14 marks]



Solution

From previous question, the s-domain equation for the circuit above is:

$$V_o(s) = \frac{V_{dc}[s + (R/L)]}{s^2 + (R/L)s + (1/LC)}$$

Where:

$$\frac{R}{L} = 100{,}000$$

$$\frac{1}{LC} = 1600 \times 10^6$$

Knowing V_{dc}= 120 V, the equation for the output voltage in the circuit becomes:

$$V_o(s) = \frac{120(s + 10{,}000)}{s^2 + 10{,}000s + 16 \times 10^6} = \frac{120(s + 10{,}000)}{(s + 2000)(s + 8000)}$$

Applying partial fraction expansion, the equation above becomes

$$V_o(s) = \frac{K_1}{s + 2000} + \frac{K_2}{s + 8000}$$

Constants K_1 and K_2 in the equation above are:

$$K_1 = \left.\frac{120(s + 10{,}000)}{s + 8000}\right|_{s = -2000} = 160$$

$$K_2 = \left.\frac{120(s + 10{,}000)}{s + 2000}\right|_{s = -8000} = -40$$

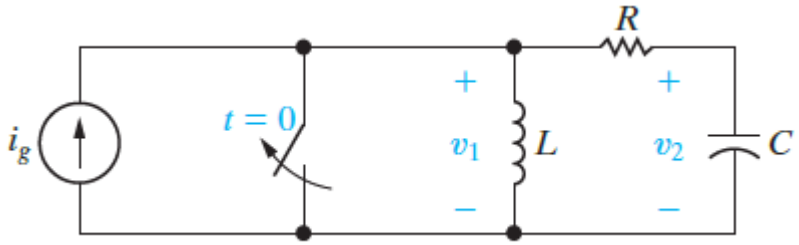
Thus

$$V_o(s) = \frac{160}{s + 2000} - \frac{40}{s + 8000}$$

Taking inverse Laplace transform, the equation for the output voltage in the circuit is:

$$v_o(t) = [160e^{-2000t} - 40e^{-8000t}]u(t) \text{ V}$$

7. The circuit parameters in the circuit in the figure below are R = 4000 Ω, L = 40 mH and C = 15.625 nF. If $i_g(t)$ = 150 mA, find $v_2(t)$. [**Answer**: $v_2(t) = [160e^{-20{,}000t} - 160e^{-80{,}000t}]u(t)$ V] [14 marks]



Solution

From the results obtained in the previous question, the s-domain equation for the output voltage in the circuit is:

$$V_2(s) = \frac{sI_g(s)}{C[s^2 + (R/L)s + (1/LC)]}$$

Where:

$$\frac{1}{C} = 64 \times 10^6$$

$$\frac{1}{LC} = 1600 \times 10^6$$

$$\frac{R}{L} = 100{,}000$$

$$I_g(s) = \frac{0.15}{s}$$

The equation for the output current in the circuit becomes:

$$V_2(s) = 64 \times 10^6 \frac{s(0.15/s)}{s^2 + 100{,}000s + 1600 \times 10^6} = \frac{96 \times 10^6}{s^2 + 10^5 s + 1600 \times 10^6}$$

Applying partial fraction expansion, the equation above becomes

$$V_2(s) = \frac{K_1}{s + 20{,}000} + \frac{K_2}{s + 80{,}000}$$

Constants K_1 and K_2 in the equation above are:

$$K_1 = \left.\frac{96 \times 10^6}{s + 80{,}000}\right|_{s = -20{,}000} = 160$$

$$K_2 = \left.\frac{96 \times 10^6}{s + 20{,}000}\right|_{s = -80{,}000} = -160$$

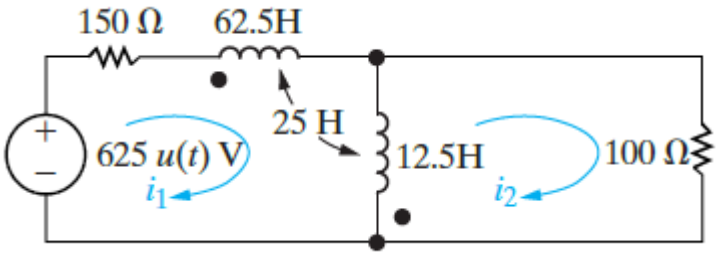
Thus

$$V_2(s) = \frac{160}{s + 20{,}000} - \frac{160}{s + 80{,}000}$$

Taking inverse Laplace transform, the equation for the voltage v_2 in the circuit is:

$$v_2(t) = [160e^{-20{,}000t} - 160e^{-80{,}000t}]u(t) \text{ V}$$

8. Use the results from previous question and the circuit shown in the figure below to do the following tasks:



a. Find $i_1(t)$ and $i_2(t)$. [**Answer**: $i_1(t) = [4.167 - 2.5e^{-4t} - 1.667e^{-24t}]u(t)$ A and $i_2(t) = 2.5[e^{-24t} - e^{-4t}]u(t)$ A] [16 marks]

b. Find $i_1(\infty)$ and $i_2(\infty)$. [**Answer**: $i_1(\infty) = 4.167$ A and $i_1(\infty) = 0$ A] [4 marks]

c. Do the solutions for i_1 and i_2 make sense? Explain. [4 marks]

Solution

a. The currents in the circuit are:

$$I_1(s) = \frac{50(s+8)}{s(s+4)(s+24)}$$

Applying partial fraction expansion, the equation above becomes

$$I_1(s) = \frac{K_1}{s} + \frac{K_2}{s+4} + \frac{K_3}{s+24}$$

Constants K_1, K_2 and K_3 in the equation above are:

$$K_1 = \frac{50(s+8)}{(s+4)(s+24)}\Big|_{s=0} = 4.167$$

$$K_2 = \frac{50(s+8)}{s(s+24)}\Big|_{s=-4} = -2.5$$

$$K_3 = \frac{50(s+8)}{s(s+4)}\Big|_{s=-24} = -1.667$$

Thus

$$I_1(s) = \frac{4.167}{s} - \frac{2.5}{s+4} - \frac{-1.667}{s+24}$$

Taking inverse Laplace transform, the equation for the output voltage is:

$$i_1(t) = [4.167 - 2.5e^{-4t} - 1.667e^{-24t}]u(t)\ \text{A}$$

And

$$I_2(s) = \frac{-50}{(s+4)(s+24)}$$

Applying partial fraction expansion, the equation above becomes

$$I_2(s) = \frac{K_1}{s+4} + \frac{K_2}{s+24}$$

Constants in the equation above are:

$$K_1 = \frac{-50}{s+24}\Big|_{s=-4} = -2.5$$

$$K_2 = \frac{-50}{s+4}\Big|_{s=-24} = 2.5$$

Thus

$$I_2(s) = \frac{-2.5}{s+4} + \frac{2.5}{s+24}$$

Taking inverse Laplace transform, the equation for the output voltage is:

$$i_2(t) = 2.5[e^{-24t} - e^{-4t}]u(t)\ \text{A}$$

b. At steady-state, $t = \infty$, the currents i_1 and i_2 in the circuit are calculated as follow.

At steady-state condition, the current i_1 is:

$$i_1(\infty) = \left[4.167 - 2.5e^{-4(\infty)} - 1.667e^{-24(\infty)}\right] = 4.167 \text{ A}$$

At steady-state condition, the current i_2 is:

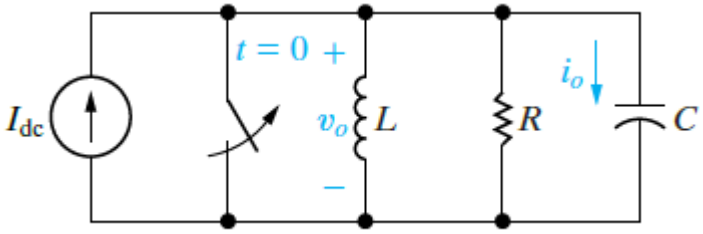
$$i_2(\infty) = \left[-0.025e^{-500(\infty)} + 0.1e^{-2000(\infty)}\right] = 0 \text{ A}$$

c. Yes. At, $t = \infty$, the current is:

$$i_1 = \frac{625}{150} = 4.167 \text{ A}$$

Since i_1 is a dc current at $t = \infty$, there is no voltage induced in the 12.5 H inductor. Hence, $i_2 = 0$. Also, note that $i_1(0) = 0$ and $i_2(0) = 0$. Thus, the solutions satisfy the condition of no initial energy stored in the circuit.

9. Use the initial- and final-value theorems to check the initial and final values of the current and voltage in the circuit given below. [**Answer**: $i_o(\infty) = 0$ and $i_o(0^+) = I_{dc}$] [12 marks]



For the circuit given above, the output voltage and output current are:

$$V_o(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}$$

And

$$I_o(s) = \frac{sI_{dc}}{s^2 + (1/RC)s + (1/LC)}$$

Solution

The s-domain equation for the output voltage v_o in the circuit is:

$$sV_o(s) = \frac{sI_{dc}/C}{s^2 + (1/RC)s + (1/LC)}$$

Applying initial-value and final-value theorems to the equation above:

$$\lim_{s\to 0} sV_o(s) = \frac{0}{(1/LC)}0, \quad v_o(\infty) = 0$$

$$\lim_{s\to \infty} sV_o(s) = \frac{1}{s} = 0, \quad v_o(0^+) = 0$$

The s-domain equation for the output current i_o in the circuit is:

$$sI_o(s) = \frac{s^2 I_{dc}}{s^2 + (1/RC)s + (1/LC)}$$

Applying initial-value and final-value theorems to the equation above:

$$\lim_{s\to 0} sI_o(s) = \frac{0}{(1/LC)} = 0, \quad i_o(\infty) = 0$$

$$\lim_{s\to \infty} sI_o(s) = \frac{s^2 I_{dc}}{s^2} = I_{dc}, \quad i_o(0^+) = I_{dc}$$