XMUT203 Analogue Circuits and Systems

Tutorial 7a: Laplace Transforms and Its Circuit Applications

A. Functional and Operational Transforms of Laplace Transform

1. Find the Laplace transform of each of the following functions:

   a. Function 1: [**Answer**: $1/[(s+a)^2]$] [2 marks]

   $$f_1(t) = te^{-at}$$

   b. Function 2: [**Answer**: $\omega/(s^2+\omega^2)$] [2 marks]

   $$f_2(t) = \sin \omega t$$

   c. Function 3: [**Answer**: $(\omega \cos\theta + s \sin\theta)/(s^2+\omega^2)$] [2 marks]

   $$f_3(t) = \sin(\omega t + \theta)$$

   d. Function 4: [**Answer**: $1/s^2$] [2 marks]

   $$f_4(t) = t$$

   e. Function 5: [**Answer**: $[\sinh\theta + s(\cosh\theta)]/(s^2-1)$] [6 marks]

   $$f_5(t) = \cosh(t+\theta)$$

2. Find the Laplace transform for functions in parts (a) and (b).

   a. Function A: [**Answer**: $F_A(s) = (-a^2 - sa - \omega^2)/[(s+a)^2+\omega^2]$] [2 marks]

   $$f_A(t) = \frac{d}{dt}(e^{-at}\cos\omega t)$$

   b. Function B: [**Answer**: $F_B(s) = \omega/\{s[(s+a)^2+\omega^2]\}$] [2 marks]

   $$f_B(t) = \int_{0^-}^{t} e^{-ax}\sin\omega x \; dx$$

   c. Verify the results obtained in (a) and (b) by first carrying out the indicated mathematical operation. [8 marks]

3. Find the Laplace transform of each of the following functions:

a. Function 1: [**Answer**: $20e^{-10s}/(s+500)$] [2 marks]

$$f_1(t) = 20e^{-500(t-10)}u(t-10)$$

b. Function 2: [**Answer**: $5[e^{4s} - 2e^{2s} + 2e^{-2s} - e^{-4s}]/s^2$] [4 marks]

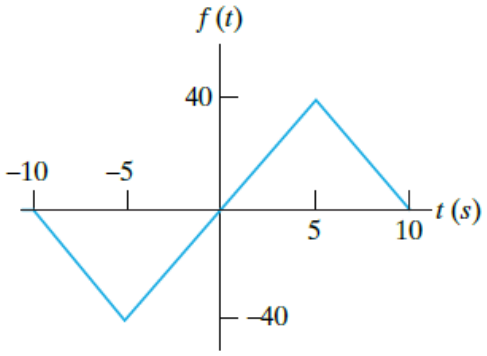
$$f_2(t) = (5t+20)[u(t+4) - u(t+2)] - 5t[u(t+2) - u(t-2)] + (5t - 20)[u(t-2) - u(t-4)]$$

4. With regard the function illustrated in the figure below, attempt the following tasks:

a. Find the Laplace transform of the function. [**Answer**: $F(s) = 8(-e^{10s} + 2e^{5s} - 2e^{-5s} + e^{-10s})/s^2$] [4 marks]

b. Find the Laplace transform of the first derivative of the function. [**Answer**: $\mathcal{L}\{f'(t)\} = 8(-e^{10s} + 2e^{5s} - 2e^{-5s} + e^{-10s})/s$] [6 marks]

c. Find the Laplace transform of the second derivative of the function. [**Answer**: $\mathcal{L}\{f''(t)\} = 8(-e^{10s} + 2e^{5s} - 2e^{-5s} + e^{-10s})$] [6 marks]



B. Inverse Laplace Transforms and Partial Fractions Expansions

1. By applying partial fraction expansion and inverse Laplace transform, find $f(t)$ for each of the following s-domain functions:

a. Function 1: [**Answer**: $f(t) = [10e^{-5t} - 4e^{-8t}]u(t)$] [8 marks]

$$F_1(s) = \frac{6(s+10)}{(s+5)(s+8)}$$

b. Function 2: [**Answer**: $f(t) = [15 - 6e^{-3t} + 11e^{-7t}]u(t)$] [10 marks]

$$F_2(s) = \frac{20s^2 + 141s + 315}{s(s^2 + 10s + 21)}$$

c. Function 3: [**Answer**: $f(t) = [8e^{-2t} - 5e^{-4t} + 12e^{-6t}]u(t)$] [10 marks]

$$F_3(s) = \frac{15s^2 + 112s + 228}{(s+2)(s+4)(s+6)}$$

d. Function 4: [**Answer**: $f(t) = [9 + 4e^{-t} - 8e^{-2t} - 3e^{-3t}]u(t)$] [12 marks]

$$F_4(s) = \frac{2s^3 + 33s^2 + 93s + 54}{s(s+1)(s^2 + 5s + 6)}$$

2. By applying partial fraction expansion and inverse Laplace transform, find $f(t)$ for each of the following s-domain functions:

a. Function 1: [**Answer**: $f(t) = [40t - 5 + 5e^{-8t}]u(t)$] [10 marks]

$$F_1(s) = \frac{320}{s^2(s+8)}$$

b. Function 2: [**Answer**: $f(t) = [60 - 40te^{-2t} - 60e^{-2t}]u(t)$] [10 marks]

$$F_2(s) = \frac{80(s+3)}{s(s+2)^2}$$

c. Function 3: [**Answer**: $f(t) = [12te^{-t} + 0.6e^{-t} + 3.35e^{-3t}\cos(4t + 100.305°)]u(t)$] [12 marks]

$$F_3(s) = \frac{60(s+5)}{(s+1)^2(s^2 + 6s + 25)}$$

d. Function 4: [**Answer**: $f(t) = [16t + 1.6 + te^{-5t} - 1.6e^{-5t}]u(t)$] [12 marks]

$$F_4(s) = \frac{25(s+4)^2}{s^2(s+5)^2}$$

C. Initial and Final Value Theorems in Laplace Transform

1. Apply the initial- and final-value theorems to each transform pair given below and determine the initial and final conditions of the functions.

a. Function 1: [**Answer**: $f(\infty) = 0$ and $f(0^+) = 6$] [6 marks]

$$F_1(s) = \frac{6(s+10)}{(s+5)(s+8)}$$

b. Function 2: [**Answer**: $f(\infty) = 15$ and $f(0^+) = 20$] [6 marks]

$$F_2(s) = \frac{20s^2 + 141s + 315}{s(s^2 + 10s + 21)}$$

c. Function 3: [**Answer**: $f(\infty) = 0$ and $f(0^+) = 15$] [6 marks]

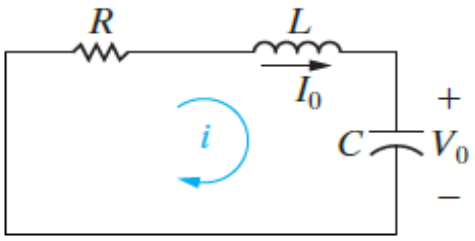
$$F_3(s) = \frac{15s^2 + 112s + 228}{(s+2)(s+4)(s+6)}$$

d. Function 4: [**Answer**: $f(\infty) = 9$ and $f(0^+) = 2$] [6 marks]

$$F_4(s) = \frac{2s^3 + 33s^2 + 93s + 54}{s(s+1)(s^2 + 5s + 6)}$$

D. Poles and Zeros in Laplace Transform

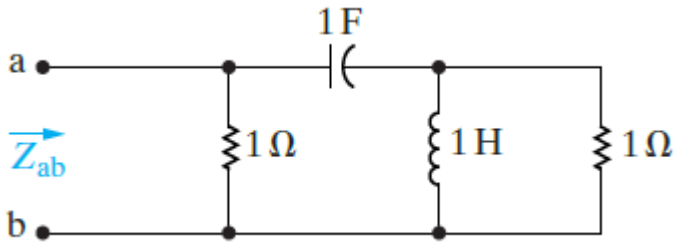
1. A 400 Ω resistor, a 12.5 mH inductor, and a 0.2 μF capacitor are in series.



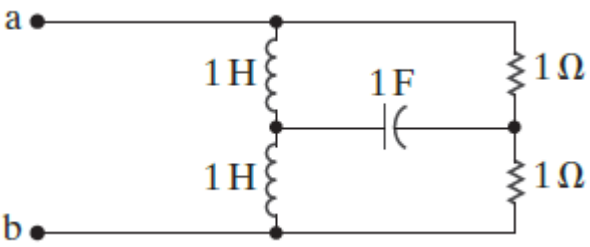
   a. Express the s-domain impedance of this series combination as a rational function. [**Answer**: $Z(s) = [s^2 + 32{,}000s + 4 \times 10^8]/80s$] [4 marks]

   b. Give the numerical value of the poles and zeros of the impedance. [**Answer**: z_1 = -16,000 + j12,000 rad/s, z_2 = -16,000 – j12,000 rad/s, and $p_1 = 0$] [6 marks]

2. Find the poles and zeros of the impedance seen looking into the terminals a, b of the circuit shown in the figure below. [**Answer**: $z_1 = -0.5 + j0.866$ rad/s, $z_2 = -0.5 - j0.866$ rad/s, $p_1 = -0.5 + j0.5$ rad/s, and $p_2 = -0.5 - j0.5$ rad/s] [8 marks]

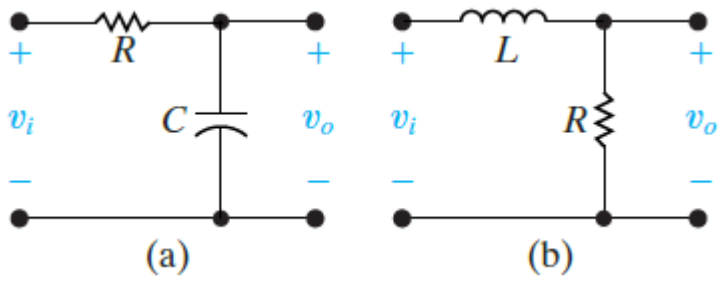


3. Find the poles and zeros of the impedance seen looking into the terminals a, b of the circuit shown in the figure below. [**Answer**: z_1 = 0, z_2 = -2 rad/s, p_1 = -1 rad/s, and p_2 = -2 rad/s] [24 marks]



E. Transfer Functions with Laplace Transform

1. For the circuit given in the figure below, attempt the following tasks.



   a. Find the transfer function $H(s) = V_o(s)/V_i(s)$ for the circuit shown in Fig. (a). [**Answer**: $H(s) = (1/RC)/[s + (1/RC)]$] [4 marks]

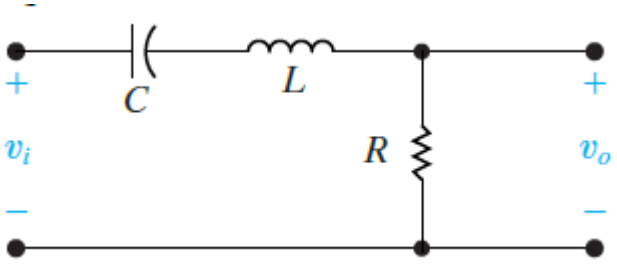
   b. Find the transfer function $H(s) = V_o(s)/V_i(s)$ for the circuit shown in Fig. (b). [**Answer**: $H(s) = (R/L)/[s + (R/L)]$] [2 marks]

   c. Create two different circuits that have the transfer function:

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1000}{s + 1000}$$

   Use components selected from E24 component standards and Figs. (a) and (b). [6 marks]

2. For the circuit as shown in the figure below, perform the following tasks.



   a. Find the transfer function $H(s) = V_o(s)/V_i(s)$ for the circuit shown above. Identify the poles and zeros for this transfer function. [**Answer**: $H(s) = [(R/L)s]/[s^2 + (R/L)s + (1/LC)]$ and $p_{1,2} = -(R/2L) \pm \sqrt{(R/2L)^2 - (1/LC)}$] [6 marks]

   b. Find three components from E24 component standards which when used in the circuit that will result in a transfer function with the following characteristics. Calculate the values of the poles.

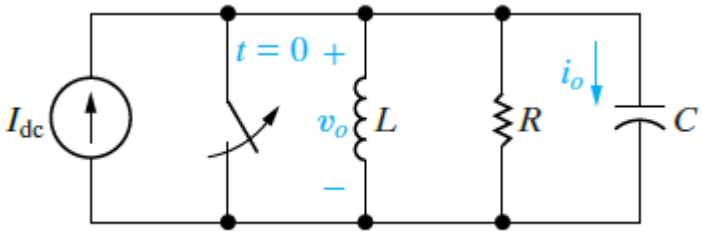
      i. Two poles that are distinct real numbers. [**Answer**: $R = 250\ \Omega; L = 10\ \text{mH}; C = 1\ \mu\text{F}$] [4 marks]

ii. Two poles, both with the same value. [**Answer**: $R = 200\ \Omega; L = 10\ \text{mH}; C = 1\ \mu\text{F}$] [4 marks]

iii. Two poles that are complex conjugate complex numbers. [**Answer**: $R = 120\ \Omega;\ L = 10\ \text{mH}; C = 1\ \mu\text{F}$] [4 marks]

F. Circuit Applications of Laplace Transform

1. There is no energy stored in the circuit shown in the figure below at the time the switch is opened.



a. Derive the integro-differential equation that governs the behaviour of the voltage v_o. [**Answer**: $I_{dc} = 1/L \int_0^t v_o\, dx + v_o/R + C dv_o/dt$] [2 marks]

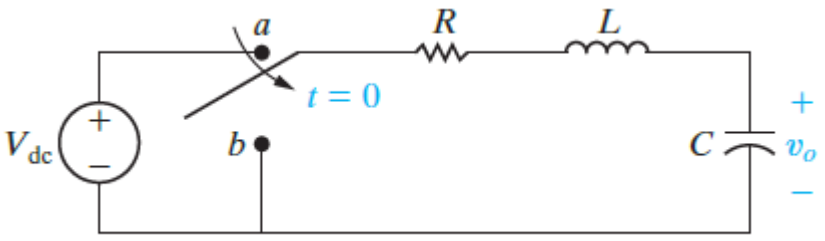
b. Show that: [4 marks]

$$V_o(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}$$

c. Show that: [4 marks]

$$I_o(s) = \frac{sI_{dc}}{s^2 + (1/RC)s + (1/LC)}$$

2. The switch in the circuit in the figure below has been in position a for a long time. At $t = 0$, the switch moves instantaneously to position b.

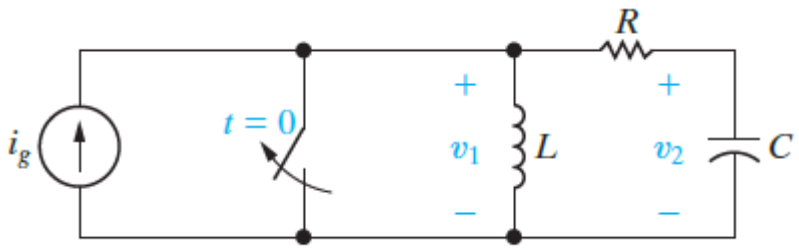


a. Derive the integro-differential equation that governs the behaviour of the voltage v_o for a period $t \geq 0^+$. [**Answer**: $d^2v_o/dt^2 + (R/L)dv_o/dt + (1/LC)v_o = 0$] [4 marks]

b. Show that: [6 marks]

$$V_o(s) = \frac{V_{dc}[s + (R/L)]}{s^2 + (R/L)s + (1/LC)}$$

3. There is no energy stored in the circuit shown in the figure below at the time the switch is opened.

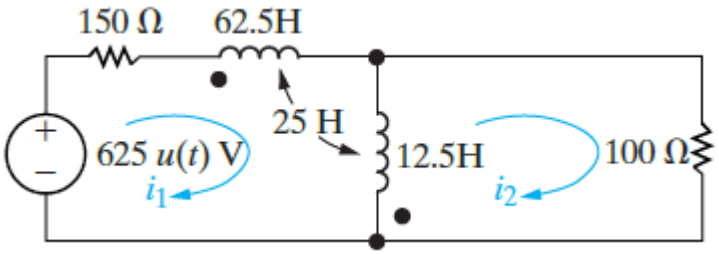


   a. Derive the integro-differential equations that govern the behaviour of the node voltages v_1 and v_2. [**Answer**: $C(dv_2/dt) + (v_2 - v_{1)}/R = 0$] [4 marks]

   b. Show that: [6 marks]

$$V_2(s) = \frac{sI_g(s)}{C[s^2 + (R/L)s + (1/LC)]}$$

4. For the circuit shown in the figure below, perform the following tasks.

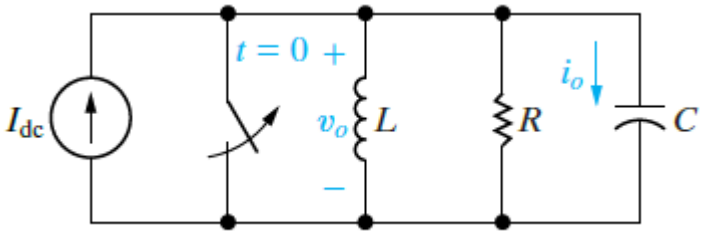


   a. Write the two simultaneous differential equations that describe the circuit in terms of the mesh currents i_1 and i_2. [**Answer**: $150i_1 + 25di_1/dt + 12.5di_2/dt = 625$ and $100i_2 + 12.5di_1/dt + 12.5di_2/dt = 0$] [6 marks]

   b. Laplace-transform the equations derived in (a). Assume that the initial energy stored in the circuit is zero. [**Answer**: $(25s + 150)I_1(s) + 12.5sI_2(s) = 625/s$ and $12.5sI_1(s) + (12.5s + 100)I_2(s) = 0$] [4 marks]

   c. Solve the equations in (b) for $I_1(s)$ and $I_2(s)$. [**Answer**: $I_1(s) = [50(s + 8)]/[s(s + 4)(s + 24)]$ and $I_2(s) = -50/[(s + 4)(s + 24)]$] [4 marks]

5. The circuit parameters in the circuit seen in the figure below have the following values: R = 20 Ω, L = 50 mH, C = 20 μF, and $i_{dc} = 75$ mA.

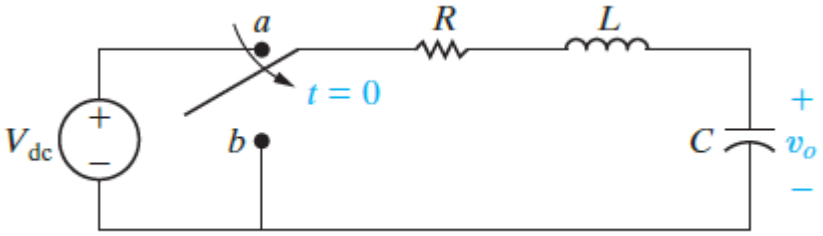


a. Find $v_o(t)$ for $t \geq 0$. [**Answer**: $v_o(t) = 2.5[e^{-500t} - e^{-2000t}]u(t)$] [14 marks]

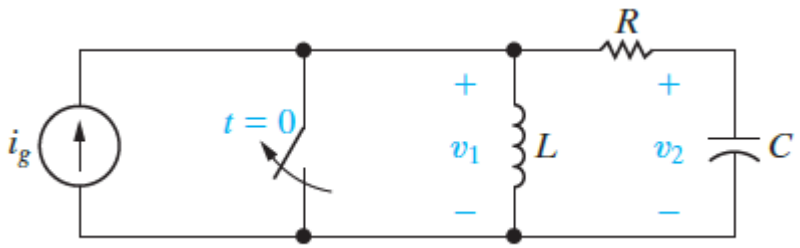
b. Find $i_o(t)$ for $t \geq 0$. [**Answer**: $i_o(t) = [-0.025e^{-500t} + 0.1e^{-2000t}]u(t)$ A] [14 marks]

c. Does your solution for make sense when $t = 0$? Explain. [4 marks]

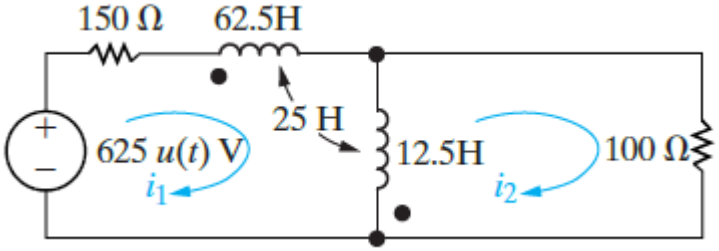
6. The circuit parameters in the circuit in the figure below are R = 2 kΩ, L = 400 mH, and C = 156.25 nF. If V_{dc}= 120 V, find $v_o(t)$ for $t \geq 0$. [**Answer**: $v_o(t) = [160e^{-2000t} - 40e^{-8000t}]u(t)$ V] [14 marks]



7. The circuit parameters in the circuit in the figure below are R = 4000 Ω, L = 40 mH and C = 15.625 nF. If $i_g(t)$ = 150 mA, find $v_2(t)$. [**Answer**: $v_2(t) = [160e^{-20,000t} - 160e^{-80,000t}]u(t)$ V] [14 marks]



8. Use the results from previous question and the circuit shown in the figure below to do the following tasks:

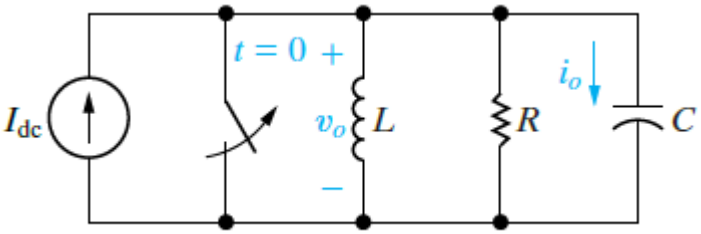


a. Find $i_1(t)$ and $i_2(t)$. [**Answer**: $i_1(t) = [4.167 - 2.5e^{-4t} - 1.667e^{-24t}]u(t)$ A and $i_2(t) = 2.5[e^{-24t} - e^{-4t}]u(t)$ A] [16 marks]

b. Find $i_1(\infty)$ and $i_2(\infty)$. [**Answer**: $i_1(\infty) = 4.167$ A and $i_1(\infty) = 0$ A] [4 marks]

c. Do the solutions for i_1 and i_2 make sense? Explain. [4 marks]

9. Use the initial- and final-value theorems to check the initial and final values of the current and voltage in the circuit given below. [**Answer**: $i_o(\infty) = 0$ and $i_o(0^+) = I_{dc}$] [12 marks]



For the circuit given above, the output voltage and output current are:

$$V_o(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}$$

And

$$I_o(s) = \frac{sI_{dc}}{s^2 + (1/RC)s + (1/LC)}$$

Appendix

Table 1: Laplace Transforms Pairs

Type of Function	$f(t)(t > 0^-)$	$F(s)$
Impulse	$\delta(t)$	1
Step	$u(t)$	$\frac{1}{s}$
Ramp	t	$\frac{1}{s^2}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Sine	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
Damped ramp	te^{-at}	$\frac{1}{(s+a)^2}$
Damped sine	$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
Damped cosine	$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Table 2: Selected Laplace Transform Properties

$f(t)$	$F(s)$	**Operation**
$Kf(t)$	$KF(s)$	Multiplication by a constant
$f_1(t)+f_2(t)+f_3(t)$	$F_1(s)+F_2(s)+F_3(s)$	Addition/subtraction
$\frac{df(t)}{dt}$	$sF(s)-f(0^-)$	First derivative (time)
$\frac{d^2f(t)}{dt^2}$	$s^2F(s)-sf(0^-)-\frac{df(0^-)}{dt}$	Second derivative (time)
$\frac{d^nf(t)}{dt^n}$	$s^nF(s)-s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - s^{n-3}\frac{d^2f(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$	nth derivative (time)

$\int_0^t f(x)\,dx$	$\frac{F(s)}{s}$	Time integral
$f(t-a)u(t-a), \quad a>0$	$e^{-as}F(s)$	Translation in time
$f(at), \quad a>0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	Scale changing
$tf(t)$	$-\frac{dF(s)}{ds}$	First derivative (s)
$t^n f(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$	nth derivative (s)
$\frac{f(t)}{t}$	$\int_s^\infty F(u)\,du$	s integral

Table 3: Four useful transform pairs

#Pair	$\boldsymbol{F(s)}$	$\boldsymbol{f(t)}$	**Description**
1	$\frac{K}{s+a}$	$Ke^{-at}u(t)$	Distinct real
2	$\frac{K}{(s+a)^2}$	$Kte^{-at}u(t)$	Repeated real
3	$\frac{K}{s+\alpha-j\beta}+\frac{K^*}{s+\alpha+j\beta}$	$2\lvert K\rvert e^{-at}\cos(\beta t+\theta)\,u(t)$	Distinct complex
4	$\frac{K}{(s+\alpha-j\beta)^2}+\frac{K^*}{(s+\alpha+j\beta)^2}$	$2t\lvert K\rvert e^{-at}\cos(\beta t+\theta)\,u(t)$	Repeated complex

Note: in pairs 1 and 2, K is a real quantity, whereas in pairs 3 and 4, K is the complex quantity $|K|\angle\theta$.