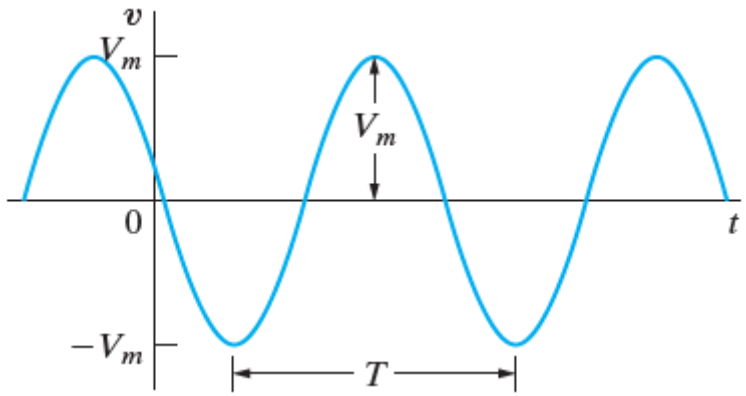
XMUT203 Analogue Circuits and Systems

Tutorial 8a: Electrical Power and Three-Phase Systems (Solution)

A. AC Analysis

1. Consider the sinusoidal voltage as given below:

$$v(t) = 25\cos(400\pi t + 60°)\ \text{V}$$



a. What is the maximum amplitude of the voltage? (Answer: 25 V) [2 marks]

b. What is the frequency in Hertz? (Answer: 200 Hz) [2 marks]

c. What is the frequency in radians per second? (Answer: 1256.64 rad/s) [2 marks]

d. What is the phase angle in radians? (Answer: 1.0472 rad/s) [2 marks]

e. What is the phase angle in degrees? (Answer: 60°) [2 marks]

f. What is the period in milliseconds? (Answer: 5 ms) [2 marks]

g. What is the first time after t = 0 that v = 0 V? (Answer: 416.67 μs) [4 marks]

h. The sinusoidal function is shifted 5/6 ms to the right along the time axis. What is the expression for $v(t)$? (Answer: $v = 25\cos[400\pi t]$) [2 marks]

i. What is the minimum number of milliseconds that the function must be shifted to the left if the expression for $v(t)$ is $25\sin 400\pi t$ V? (Answer: 2.92 ms) [4 marks]

Solution

a. As the equation for the sinusoidal function is:

$$v(t) = V_m \cos(2\pi f t + \theta)\ \text{V}$$

Thus, the maximum amplitude of the voltage is 25 V.

b. The frequency in Hertz is:

$$\omega = 2\pi f = 400\pi$$

Thus, the frequency f is 200 Hz.

c. The frequency in radians per second is:

$$\omega = 400\pi = 1256.64\ \text{rad/s}$$

d. The phase angle in radians is:

$$\theta(\text{rad}) = \left(\frac{60°}{180°}\right)\pi\ = \frac{\pi}{3} = 1.0472\ \text{rad/s}$$

e. The phase angle in degrees is θ = 60°.

f. The period in milliseconds is:

$$T = \frac{1}{f} = \frac{1}{200} = 5\ \text{ms}$$

g. The first time after t = 0 that v = 0 V is calculated when:

$$400\pi t + \frac{\pi}{3} = \frac{\pi}{2}$$

Thus

$$400\pi t = \frac{\pi}{6}$$

As a result, the time after t = 0 when v = 0 V is:

$$t = \frac{1}{2400} = 416.67\ \mu\text{s}$$

h. The expression for $v(t)$ when the sinusoidal function is shifted 5/6 ms to the right along the time axis is derived as follow.

$$v = 25\cos\left[400\pi\left(t - \frac{0.005}{6}\right) + \frac{\pi}{3}\right]$$
$$= 25\cos[400\pi t - \left(\frac{\pi}{3}\right) + \left(\frac{\pi}{3}\right)] = 25\cos[400\pi t]$$

i. The minimum number of milliseconds that the function must be shifted to the left if the

expression for $v(t)$ is $25\sin 400\pi t$ V is:

$$400\pi(t+t_0)+\left(\frac{\pi}{3}\right)=400\pi t+\left(\frac{3\pi}{2}\right)$$

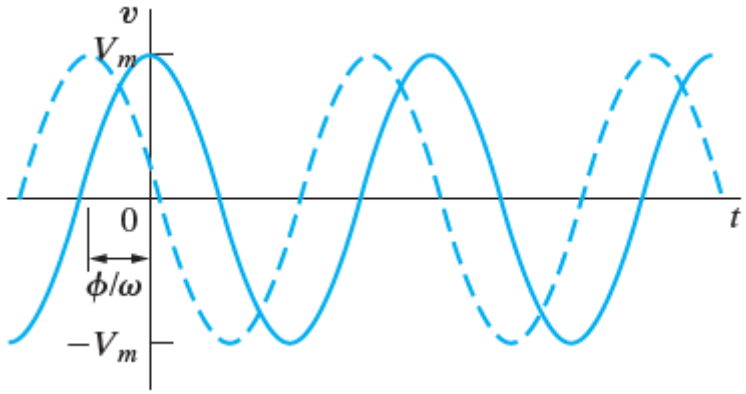
Rearrange the equation above, thus

$$400\pi t_0=\frac{7\pi}{6}$$

As a result, the required minimum time is:

$$t_0=\frac{7}{2400}=2.92\text{ ms}$$

2. For a given AC circuit, the phase shift determines the displacement of the waveform.



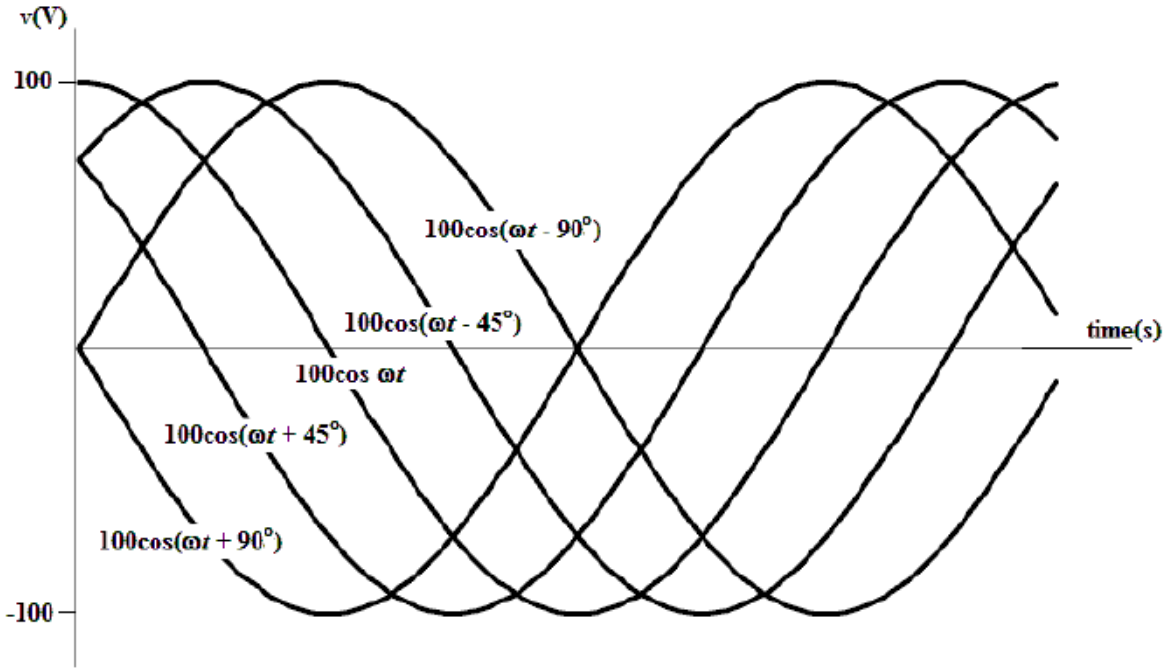
a. In a single graph, sketch $v=100\cos(\omega t+\phi)$ versus ωt for ϕ = 90°, 45°, 0°, -45° and -90°. [4 marks]

b. State whether the voltage function is shifting to the right or left as ϕ becomes more negative. (Answer: Right) [2 marks]

c. What is the direction of shift if ϕ changes from 0° to 45°? (Answer: Left) [2 marks]

Solution

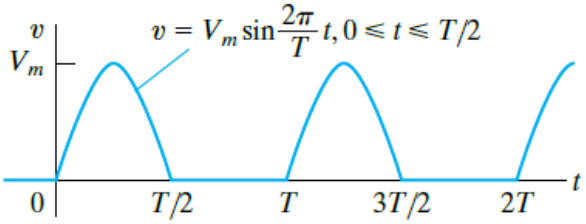
a. Sketch of $v=100\cos(\omega t+\phi)$ versus ωt for ϕ = 90°, 45°, 0°, -45° and -90° is as shown in the figure below.



b. The voltage function is shifting to the right as ϕ becomes more negative.

c. The direction of shift if ϕ changes from 0° to 45° is left.

3. Find the rms value of the half-wave rectified sinusoidal voltage shown in the figure below. (Answer: $V_m/2$) [6 marks]



Solution

The rms voltage waveform is calculated from

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^{T/2} V_m^2 \sin^2\left(\frac{2\pi}{T}t\right) dt}$$

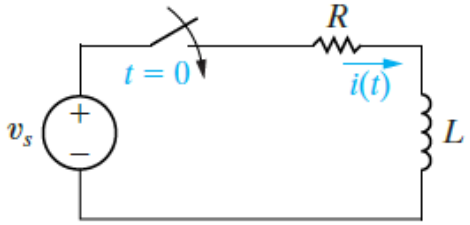
Taking the integral part of the equation given above, the equation is now:

$$\int_0^{T/2} V_m^2 \sin^2\left(\frac{2\pi}{T}t\right) dt = \frac{V_m^2}{2}\int_0^{T/2}\left(1 - \cos\frac{4\pi}{T}t\right) dt = \frac{V_m^2 T}{4}$$

Therefore, the rms value of the half-wave rectified sinusoidal voltage is:

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\frac{V_m^2 T}{4}} = \frac{V_m}{2}$$

4. The voltage applied to the circuit shown in the figure below at t = 0 is $75\cos(4000t - 60°)$ V. The circuit resistance is 400 Ω and the initial current in the 75 mH inductor is zero.



a. Find $i(t)$ for ≥ 0. (Answer: $i(t) = -17.94e^{-5333.33t} + 150\cos(4000t - 96.87°)$ mA) [6 marks]

b. Write the expressions for the transient and steady-state components of $i(t)$. (Answer: Transient component = $-17.94e^{-5333.33t}$ and steady-state component = $150\cos(4000t - 96.87°)$) [4 marks]

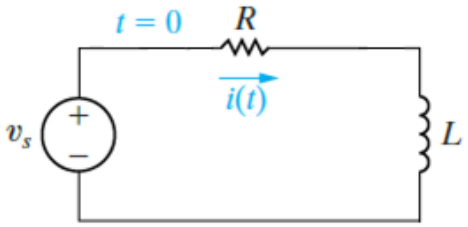
c. Find the numerical value of i after the switch has been closed for 750 μs. (Answer: 38.44 mA) [2 marks]

d. What are the maximum amplitude, frequency (in radians per second), and phase angle of the steady-state current? (Answer: 150 mA, 4000 rad/s, and -96.87°) [2 marks]

e. By how many degrees are the voltage and the steady-state current out of phase? (Answer: 36.87°) [2 marks]

Solution

a. Based on the equivalent circuit shown below, the current $i(t)$ for $t \geq 0$ is calculated as follow.



Applying KVL to the circuit, the differential equation of the circuit is:

$$L\frac{di}{dt} + Ri = V_m\cos(\omega t + \phi) \qquad (Eq.1)$$

Thus

$$i(t) = \frac{-V_m}{\sqrt{R^2+\omega^2L^2}}\cos(\phi-\theta)\,e^{-(R/L)t} + \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t+\phi-\theta) \qquad (Eq.2)$$

The numerical values of the terms in the equation above are:

$V_m = 75$ V, R/L = 5333.33, $\omega L = 300$, $\sqrt{R^2+\omega^2L^2} = 500$, ϕ = -60°, $\theta = \tan^{-1}(300/400)$, and θ = 36.87°.

Substitute these values into equation (2) above, this equation becomes:

$$i(t) = -17.94e^{-5333.33t} + 150\cos(4000t - 96.87°)\ \text{mA}$$

b. The expressions for the transient and steady-state components of $i(t)$ are:

$$\text{Transient component} = -17.94e^{-5333.33t}$$

And

$$\text{Steady-state component} = 150\cos(4000t - 96.87°)$$

c. The numerical value of i after the switch has been closed for 750 μs is by direct substitution.

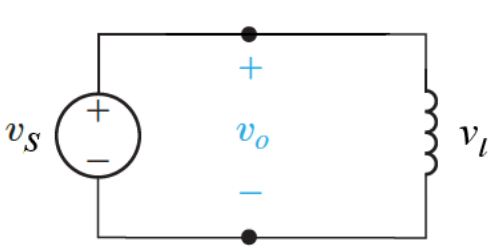
$$i(750\ \mu s) = 38.44\ \text{mA}$$

d. The maximum amplitude, frequency (in radians per second), and phase angle of the steady-state current are 150 mA, 4000 rad/s, and -96.87° respectively.

e. The current lags the voltage by 36.87°.

B. Impedances

1. A 400 Hz sinusoidal voltage with a maximum amplitude of 100 V at t = 0 is applied across the terminals of an inductor. The maximum amplitude of the steady-state current in the inductor is 20 A.



a. What is the frequency of the inductor current? (Answer: 400 Hz) [2 marks]

b. If the phase angle of the voltage is zero, what is the phase angle of the current? (Answer: -

90°) [4 marks]

c. What is the inductive reactance of the inductor? (Answer: 5 Ω) [2 marks]

d. What is the inductance of the inductor in milli Henrys? (Answer: 1.99 mH) [2 marks]

e. What is the impedance of the inductor? (Answer: $j5\ \Omega$) [2 marks]

Solution

a. The frequency of the inductor current is 400 Hz.

b. The phase angle of the current when the phase angle of the voltage is zero is calculated as follow.

Knowing that the phase shift of the voltage is:

$$\theta_v = 0°$$

Thus

$$\mathbf{I} = \frac{100\angle 0°}{j\omega L} = \frac{100}{\omega L}\angle -90°$$

As a result, the phase shift of the circuit is:

$$\theta_i = -90°$$

c. The inductive reactance of the inductor is:

$$\frac{100}{\omega L} = 20$$

Thus

$$\omega L = 5\ \Omega$$

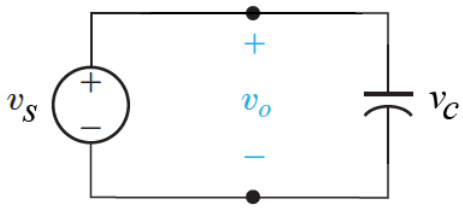
d. The inductance of the inductor in milli Henrys is:

$$L = \frac{5}{800\pi} = 1.99\ \text{mH}$$

e. The impedance of the inductor is:

$$Z_L = j\omega L = j5\ \Omega$$

2. An 80 kHz sinusoidal voltage has zero phase angle and a maximum amplitude of 25 mV. When this voltage is applied across the terminals of a capacitor, the resulting steady-state current has a maximum amplitude of 628.32 μA.



a. What is the frequency of the current in radians per second? (Answer: 502.65 krad/s) [2 marks]

b. What is the phase angle of the current? (Answer: 90°) [2 marks]

c. What is the capacitive reactance of the capacitor? (Answer: -39.79 Ω) [2 marks]

d. What is the capacitance of the capacitor in microfarads? (Answer: 0.05 μF) [2 marks]

e. What is the impedance of the capacitor? (Answer: $-j39.79\ \Omega$) [2 marks]

Solution

a. The frequency of the current in radians per second is:

$$\omega = 2\pi f = 160\pi \times 10^3 = 502.65 \text{ krad/s}$$

b. The phase angle of the current is:

$$I = \frac{25 \times 10^{-3}\angle 0°}{1/j\omega C} = j\omega C(25 \times 10^{-3})\angle 0° = 25 \times 10^{-3}\omega C\angle 90°$$

As a result, θ_i = 90°.

c. The capacitive reactance of the capacitor is calculated from:

$$628.32 \times 10^{-6} = 25 \times 10^{-3}\omega C$$

Thus

$$\frac{1}{\omega C} = \frac{25 \times 10^{-3}}{628.32 \times 10^{-6}} = 39.79\ \Omega$$

As a result, reactance of the capacitor (X_c) is -39.79 Ω.

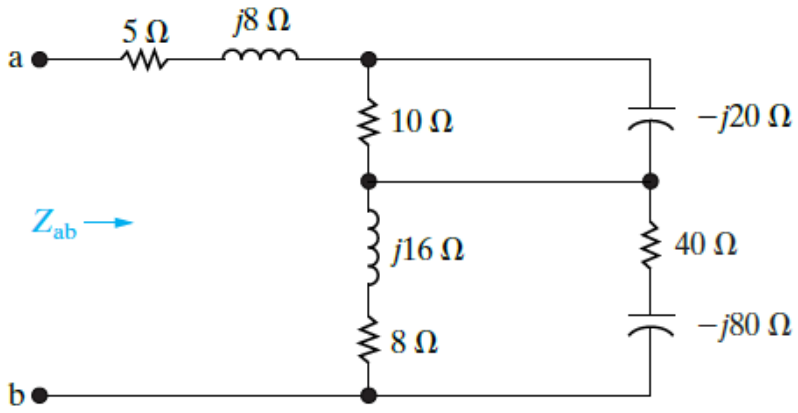
d. The capacitance of the capacitor in microfarads is:

$$C = \frac{1}{39.79(\omega)} = \frac{1}{(39.79)(160\pi \times 10^3)} = 0.05 \times 10^{-6}\ \text{F} = 0.05\ \mu\text{F}$$

e. The impedance of the capacitor is:

$$Z_c = j\left(\frac{-1}{\omega C}\right) = -j39.79\ \Omega$$

3. Find the impedance in the circuit seen in the figure below. Express in both polar and rectangular form. (Answer: $32.02\angle 38.66°$) [4 marks]



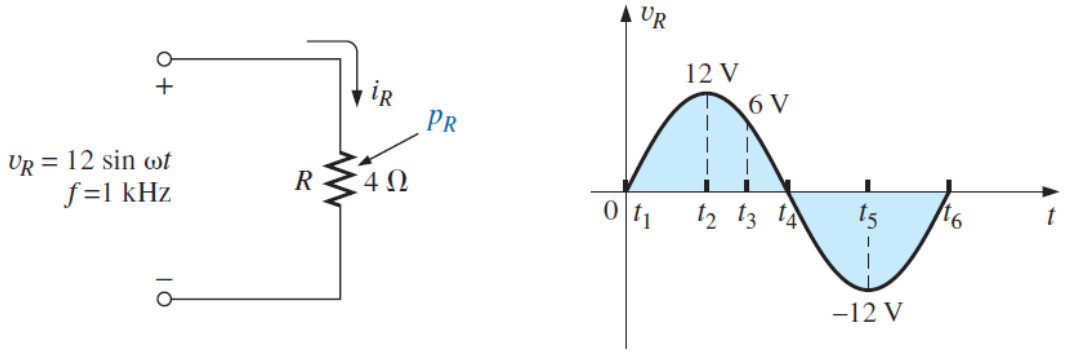
Solution

The impedance across a and b is calculated as follow.

$$\begin{aligned} Z_{ab} &= 5 + j8 + 10 \parallel -j20(8 + j16) \parallel (40 - j80) \\ &= 5 + j8 + 8 - j4 + 12 + j16 \\ &= 25 + j20\ \Omega \\ &= 32.02\angle 38.66° \end{aligned}$$

C. Electrical Power

1. For the resistive circuit given below, it is supplied with a voltage source $v_R(t) = 12 \sin \omega t$ with its frequency of oscillation is 1 kHz. The graph of the voltage, v_R against time, t of the circuit is also given in the figure below.



a. Find the instantaneous power delivered to the resistor at times t_1 through t_6. [12 marks]

b. Plot the results of part (a) for one full period of the applied voltage. [4 marks]

c. Find the average value of the curve of part (b) and compare the level to that determined by the following equation: [2 marks]

$$P = VI = \frac{V_m I_m}{2} = I^2 R = \frac{V_2}{R}$$

d. Find the energy dissipated by the resistor over one full period of the applied voltage. (Answer: 18 mJ) [2 marks]

Solution

a. For the given durations of time, the instantaneous real powers delivered to the resistor are calculated as follows.

t_1: v_R = 0 V and $P_R = v_R i_R = 0$ W.

t_2: v_R =12 V and $i_R = 12\ \text{V}/4\ \Omega = 3$ A. Thus,

$$P_R = v_R i_R = (12\ \text{V})(3\ \text{A}) = 36\ \text{W}$$

t_3: v_R = 6 V and $i_R = 6\ \text{V}/4\ \Omega = 1.5$ A. Thus,

$$P_R = v_R i_R = (6\ \text{V})(1.5\ \text{A}) = 9\ \text{W}$$

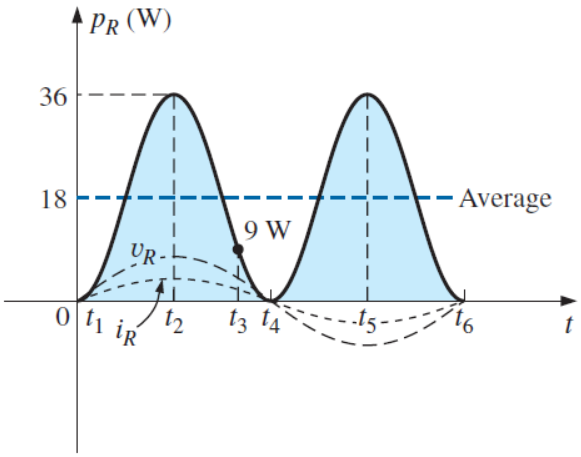
t_4: v_R = 0 V and $P_R = v_R i_R = 0$ W.

t_5: v_R = 12 V and $i_R = 12\ \text{V}/4\ \Omega = 3$ A. Thus,

$$P_R = v_R i_R = (-12\ \text{V})(-3\ \text{A}) = 36\ \text{W}$$

t_6: v_R = 0 V and $P_R = v_R i_R = 0$ W.

b. The resulting plot of v_R, i_R, and P_R appears in the figure below.



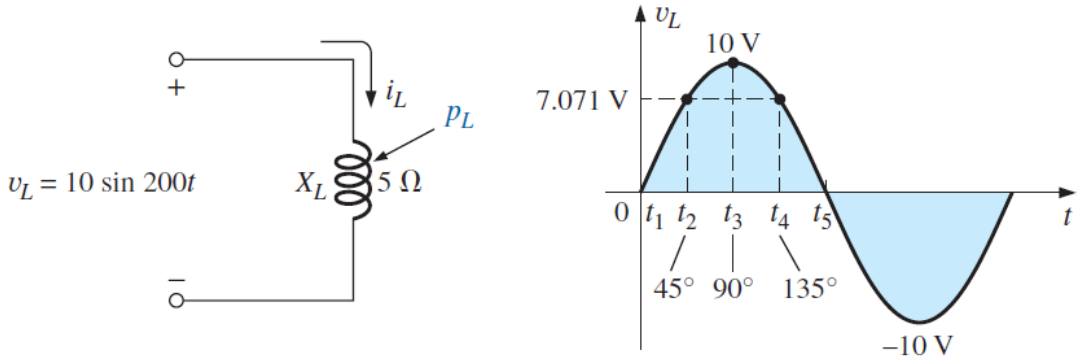
c. The average value of real power in the curve is 18 W as shown in the figure in part (b). This result is an exact match with that obtained using the following equation, since $T = 1/f$:

$$P = \frac{V_m I_m}{2} = \frac{(12\text{ V})(3\text{ A})}{2} = 18\text{ W}$$

d. The area under the curve is determined by the following equation:

$$W_R = \frac{VI}{f} = \frac{V_m I_m}{2f} = \frac{(12\text{ V})(3\text{ A})}{2(10^3)} = 18\text{ mJ}$$

2. For the inductive circuit in the figure below, it is supplied with a voltage source $v_L(t) = 10\sin 200t$. The graph of the voltage, v_L against time, t of the circuit is also given in the figure below.



a. Find the instantaneous power level for the inductor at times t_1 through t_5. [10 marks]

b. Plot the results of part (a) for one full period of the applied voltage. [4 marks]

c. Find the average value of the curve of part (b) over one full cycle of the applied voltage and compare the peak value of each pulse with the value determined by the following equation:

$$Q_L = VI \quad (\text{i. e. unit in VAr})$$ [2 marks]

d. Find the energy stored or released for any one pulse of the power curve. (Answer: 50 mJ)

[2 marks]

Solution

a. The currents at any given period of time are calculated as follows.

t_1: $v_L = 0$ V and $P_L = v_L i_L = 0$ W.

t_2: $v_L = 7.071$ V and the current that flows in the inductor at this time is:

$$i_L = \left(\frac{V_m}{X_L}\right)\sin(\alpha - 90°) = \left(\frac{10\text{ V}}{5\ \Omega}\right)\sin(\alpha - 90°) = 2\sin(\alpha - 90°)$$

At $\alpha = 45°$

$$i_L = 2\sin(45° - 90°) = 2\sin(-45°) = -1.414\text{ A}$$

And

$$P_L = v_L i_L = (7.071\text{ V})(-1.414\text{ A}) = -10\text{ W}$$

t_3: $i_L = 0$ A and $P_L = v_L i_L = 0$ W.

t_4: $v_L = 7.071$ V, and the current that flows in the inductor at this time is:

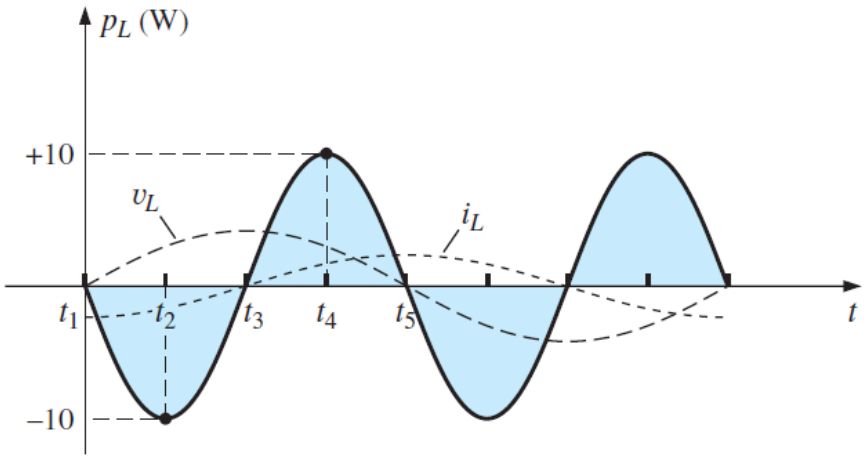
$$i_L = 2\sin(\alpha - 90°) = 2\sin(135° - 90°) = 2\sin 45° = 1.414\text{ A}$$

And

$$P_L = v_L i_L = (7.071\text{ V})(1.414\text{ A}) = 10\text{ W}$$

t_5: $v_L = 0$ V and $P_L = v_L i_L = 0$ W.

b. The resulting plot of v_L, i_L, and P_L of the inductor appears in the figure below.



c. The average value for the curve in the figure in part (b) is 0 W over one full cycle of the applied voltage. The peak value of the curve is 10 W which compares directly with that obtained from the product

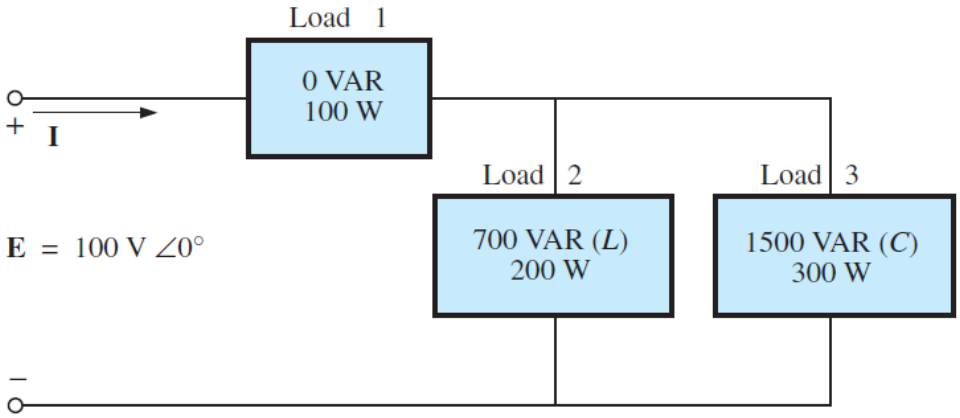
$$P = VI = \frac{V_m I_m}{2} = \frac{(10\text{ V})(2\text{ A})}{2} = 10\text{ W}$$

d. The energy stored or released during each pulse of the power curve is:

$$W_L = \frac{VI}{\omega} = \frac{V_m I_m}{2\omega} = \frac{(10\text{ V})(2\text{ A})}{2(200\text{ rad/s})} = 50\text{ mJ}$$

3. Find the total number of watts, volt-amperes reactive, and volt-amperes, and the power factor pf of the network circuit given in the figure below. Draw the power triangle and find the current

in phasor form. (Answer: 600 W, 800 VAr, 1000 VA, pf=0.6) [12 marks]



Solution

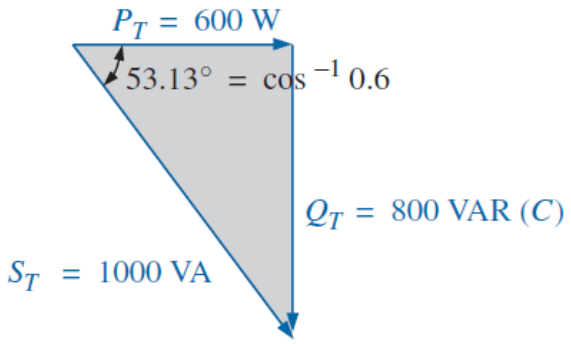
Construct a load power calculation table for the circuit given above such as shown in the table below.

Load	P (W)	Q (VAr)	S (VA)
1	100	0	100
2	200	700 (inductive)	$\sqrt{(200)^2 + (700)^2} = 728$
3	300	1500 (capactive)	$\sqrt{(300)^2 + (1500)^2} = 1529.71$
Total	600	800	$\sqrt{(600)^2 + (800)^2} = 1000$

Thus, the power factor of the network circuit is:

$$pf = \frac{P_T}{S_T} = \frac{600\ \text{W}}{1000\ \text{VA}} = 0.6\ (\text{lead})$$

The power triangle is shown in the figure below.



Since $S_T = VI$ = 1000 VA, I = 1000 VA/100 V = 10 A, and since θ of pf = cos θ is the angle between the input voltage and current:

$$I = 10\ \text{A} \angle 53.13°$$

The plus sign is associated with the phase angle since the circuit is predominantly capacitive.

4. A small industrial plant has a 10 kW heating load and a 20 kVA inductive load due to a bank of induction motors. The heating elements are considered purely resistive (pf = 1), and the induction motors have a lagging power factor of 0.7.

   a. If the supply is 1000 V at 60 Hz, determine the capacitive element required to raise the power factor to 0.95. (Answer: 16.93 µF) [24 marks]

   b. Compare the levels of current drawn from the supply. (Answer: 9.5% reduction) [6 marks]

Solution

a. For the induction motors, its apparent power is:

$$S = VI = 20 \text{ kVA}$$

Thus, its real power is:

$$P = S\cos\theta = (20 \times 10^3 \text{ VA})(0.7) = 14 \text{ kW}$$

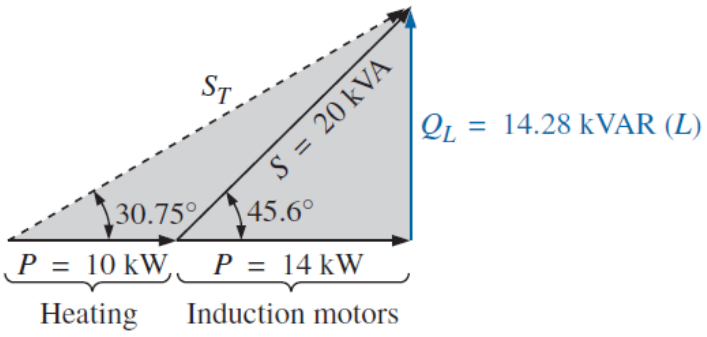
The angle for the power factor is:

$$\theta = \cos^{-1} 0.7 = 45.6°$$

And

$$Q_L = S\sin\theta = (20 \text{ kVA})(\sin 45.6) = 14.28 \text{ kVAr}$$

The power triangle for the plant electricity system appears in the figure below.



Note the addition of real powers and the resulting S_T:

$$S_T = \sqrt{P^2 + Q_L^2} = \sqrt{(24 \text{ kW})^2 + (14.28 \text{ kVAr})^2} = 27.93 \text{ kVA}$$

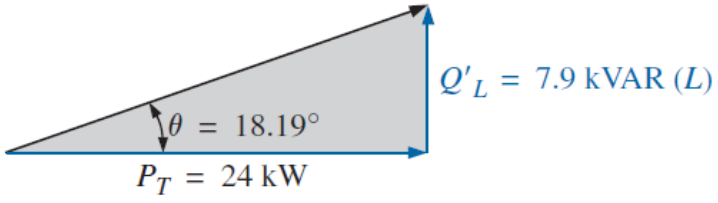
With

$$I_T = \frac{S_T}{E} = \frac{27.93 \text{ kVA}}{1000 \text{ V}} = 27.93 \text{ A}$$

The desired power factor of 0.95 results in an angle between S and P of:

$$\theta = \cos^{-1} 0.95 = 18.19°$$

This is changing the power triangle of the plant electricity system to that in the figure below.



As shown above, the angle for the power factor is calculated from:

$$\tan\theta = \frac{Q'_L}{P_T}$$

Thus

$$Q'_L = P_T \tan\theta = (24\text{ kW})(\tan 18.19°) = (24\text{ kW})(0.329) = 7.9\text{ kVAr}$$

The inductive reactive power must therefore be reduced by:

$$Q_L - Q'_L = 14.28\text{ kVAr} - 7.9\text{ kVAr} = 6.38\text{ kVAr}$$

Therefore, Q_C = 6.38 kVAr, and using the following equation:

$$Q_C = \frac{E^2}{X_C}$$

We obtain the reactance of the capacitor:

$$X_C = \frac{E^2}{Q_C} = \frac{(10^3\text{ V})^2}{6.38 \times 10^3\text{ VAr}} = 156.74\ \Omega$$

Thus, the capacitance of the compensating capacitor is:

$$C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi(60\text{ Hz})(156.74\ \Omega)} = 16.93\ \mu\text{F}$$

b. The total apparent power of the circuit is:

$$S_T = \sqrt{P^2 + Q_L^2} = \sqrt{(24\text{ kW})^2 + (7.9\text{ kVAr})^2} = 25.27\text{ kVA}$$

Thus, the total current is:

$$I'_T = \frac{S_T}{E} = \frac{25.27 \times 10^3\text{ VA}}{10^3\text{ V}} = 25.27\text{ A}$$

The new I_T is:

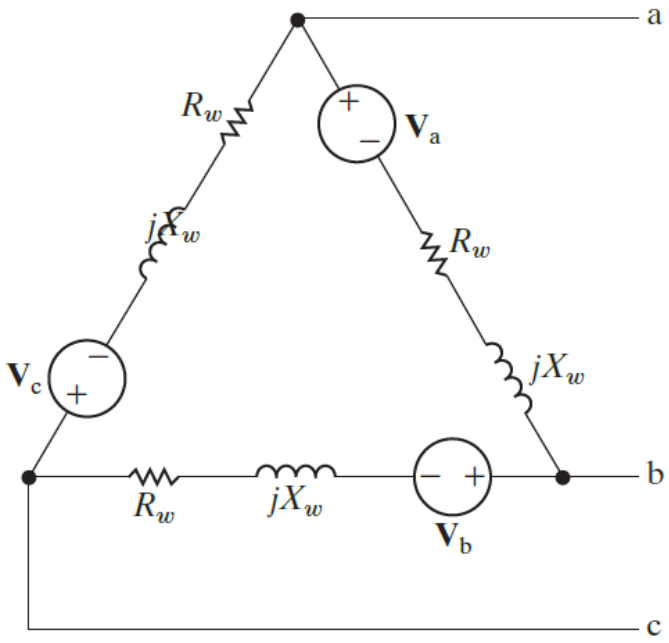
$$I'_T = 25.27\angle 27.93°\text{ A} \qquad \text{(original)}$$

Thus, the percentage reduction of current drawn from the supply is:

$$\Delta I = \left(I_T - \frac{I_T'}{I_T}\right) \times 100\% = \left(\frac{27.93\ \text{A} - 25.27\ \text{A}}{27.93\ \text{A}}\right) \times 100\% = 9.5\%$$

D. Three-Phase Systems

1. Refer to the circuit in the figure below. Assume that there are no external connections to the terminals a, b, and c.



Assume further that the three windings are from a three-phase generator whose voltages are those described below.

$$v_a = 188\cos(250t + 60°)\ \text{V}$$
$$v_b = -188\cos 250t\ \text{V}$$
$$v_c = 188\cos(250t - 60°)\ \text{V}$$

Determine the current circulating in the Δ-connected generator. (Answer: 0) [2 marks]

Solution

The current circulating in the generator circuit is:

$$I = \frac{V_a + V_b + V_c}{Z_a + Z_b + Z_c} = \frac{188\angle 60° + 188\angle 180° + 188\angle -60°}{3(R_\omega + jX_\omega)} = 0$$

2. The time-domain expressions for three line-to-neutral voltages at the terminals of a Y-connected load are:

$$v_{AN} = 288\cos(\omega t - 45°)\ \text{V}$$
$$v_{BN} = 288\cos(\omega t - 165°)\ \text{V}$$
$$v_{CN} = 288\cos(\omega t + 75°)\ \text{V}$$

What are the time-domain expressions for the three line-to-line voltages v_{AB}, v_{BC}, and v_{CA}? (Answer: $498.83\cos(\omega t - 15°)\ \text{V}$, $498.83\cos(\omega t - 135°)\ \text{V}$, and $498.83\cos(\omega t + 105°)\ \text{V}$)

[9 marks]

Solution

The phase voltages to neutral are:

$$V_{AN} = 285\angle -45°\ \text{V}$$
$$V_{BN} = 285\angle -165°\ \text{V}$$
$$V_{CN} = 285\angle 75°\ \text{V}$$

The line-to-line voltages in polar forms are:

$$V_{AB} = V_{AN} - V_{BN} = (285\angle -45°) - (285\angle -165°) = 498.83\angle -15°\ \text{V}$$
$$V_{BC} = V_{BN} - V_{CN} = (285\angle -165°) - (285\angle 75°) = 498.83\angle -135°\ \text{V}$$
$$V_{CA} = V_{CN} - V_{AN} = (285\angle 75°) - (285\angle -45°) = 498.83\angle 105°\ \text{V}$$

The line-to-line voltages in sinusoidal forms are:

$$v_{AB} = 498.83\cos(\omega t - 15°)\ \text{V}$$
$$v_{BC} = 498.83\cos(\omega t - 135°)\ \text{V}$$
$$v_{CA} = 498.83\cos(\omega t + 105°)\ \text{V}$$

3. A balanced three-phase circuit has the following characteristics:
   - Y-Y connected.
   - The line voltage at the source, V_{ab}, is $110\sqrt{3}\angle$-60° V.
   - The phase sequence is positive.
   - The line impedance is $3 + j2\ \Omega/\phi$.
   - The load impedance is $37 + j28\ \Omega/\phi$.

   a. Draw the single-phase equivalent circuit for the a-phase. [6 marks]

   b. Calculate the line current in the a-phase. (Answer: $2.2\angle -126.87°\ \text{A (rms)}$) [2 marks]

   c. Calculate the line voltage at the load in the a-phase. (Answer: $176.81\angle -59.75\ \text{A (rms)}$)

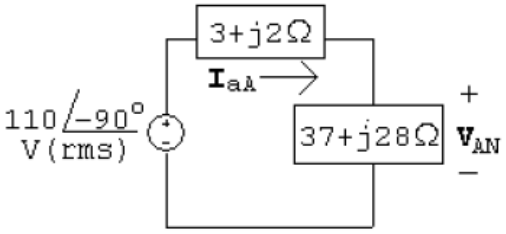
   [4 marks]

Solution

a. The phase voltage at the source in the circuit is:

$$V_{an} = \left[\left(\frac{1}{\sqrt{3}}\right)\angle -30°\right] V_{ab} = \left[\left(\frac{1}{\sqrt{3}}\right)\angle -30°\right]\left(110\sqrt{3}\angle -60°\right) = 110\angle -90° \text{ V (rms)}$$

The single-phase equivalent circuit for the a-phase is as shown in the figure below.



b. The a-phase line current is:

$$I_{aA} = \frac{V_a}{R_a + R_A} = \frac{110\angle -90°}{(3 + j2) + (37 + j28)} = \frac{110\angle -90°}{40 + j30} = 2.2\angle -126.87° \text{ A (rms)}$$

c. The phase voltage at the load is:

$$V_{AN} = (37 + j28)I_{aA} = (37 + j28)(2.2\angle -126.87°) = 102.08\angle -89.75° \text{ V (rms)}$$

Thus, the line-to-line voltage at the source is:

$$V_{AB} = \sqrt{3}\angle 30° V_{AN} = \left(\sqrt{3}\angle 30°\right)(102.08\angle -89.75°) = 176.81\angle -59.75 \text{ A (rms)}$$

4. A balanced Y-connected load having an impedance of 60 – *j*45 Ω/ϕ is connected in parallel with a balanced Δ-connected load having an impedance of $90\sqrt{2}$ ∠45°. The paralleled loads are fed from a line having an impedance of 2 + *j*2 Ω/ϕ. The magnitude of the line-to-line voltage of the Δ-load is $300\sqrt{3}$ V. Calculate the magnitudes of:

   a. Phase current in the Y-connected load. (Answer: 8.6 A (rms)) [6 marks]

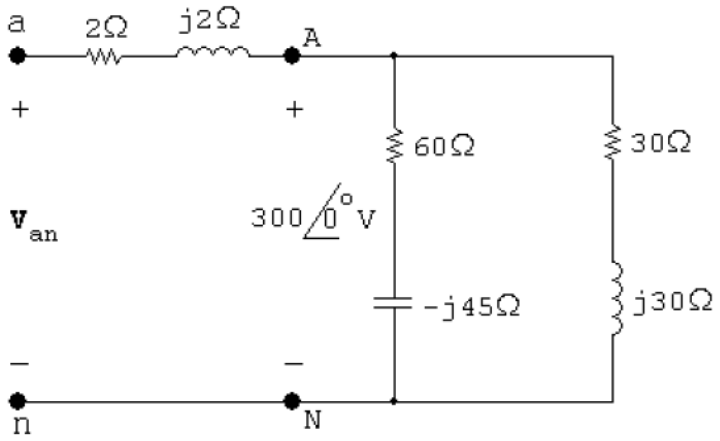
   b. Phase current in the Δ-connected load. (Answer: 4.08 A (rms)) [2 marks]

   c. Current in the line feeding the loads. (Answer: 4 A (rms)) [2 marks]

   d. Line voltage at the sending end of the line. (Answer: 557.37 V (rms)) [4 marks]

Solution

a. The a-phase equivalent circuit is as shown in the figure below.



The phase current in the Y-connected load is:

$$I_{aA} = \frac{V_{AN}}{Z_Y} + \frac{V_{AN}}{(Z_\Delta/3)} = \frac{300\angle 0°}{60 - j45} + \frac{300\angle 0°}{30 + j30} = 8.6\angle -17.59° \text{ A (rms)}$$

So, its magnitude $|I_{aA}| = 8.6 \text{ A (rms)}$.

b. The phase current in the Δ-connected load is:

$$I_{AB} = \frac{V_{AB}}{Z_\Delta} = \frac{300\sqrt{3}\angle 30°}{90 + j90} = 4.08\angle -15° \text{ A (rms)}$$

So, its magnitude $|I_{AB}| = 4.08 \text{ A (rms)}$.

c. The current in the line feeding the loads is:

$$I_{AN} = \frac{V_{AN}}{Z_Y} = \frac{300\angle 0°}{60 - j45} = 4\angle 36.87° \text{ A (rms)}$$

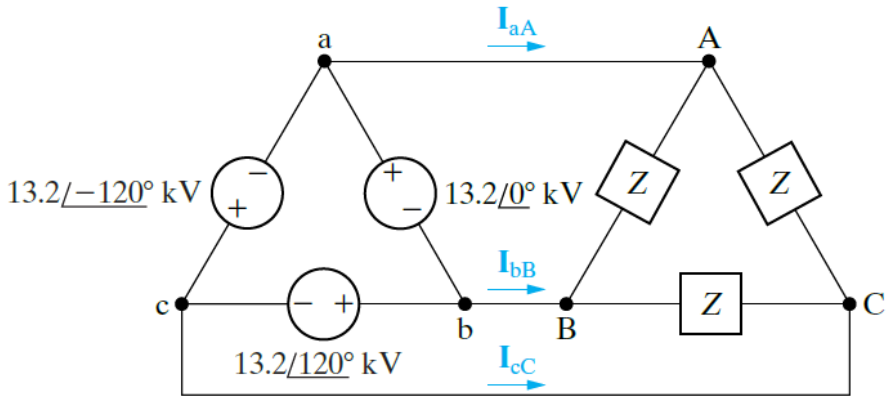
So, its magnitude $|I_{AN}| = 4 \text{ A (rms)}$.

d. The line voltage at the sending end of the line is:

$$V_{an} = I_{aA}Z_{line} + V_{AN} = (8.6\angle -17.59°)(2 + j2) + 300\angle 0° = 321.79\angle 1.99° \text{ V (rms)}$$

So, its magnitude $|V_{ab}| = \sqrt{3}V_{an} = \sqrt{3}(321.79) = 557.37 \text{ V (rms)}$.

5. The impedance Z in the balanced three-phase circuit in the figure below is 100 – j75 Ω. Find:



a. I_{AB}, I_{BC}, and I_{CA}, (Answer: $105.6\angle 36.87°$ A (rms), $105.6\angle 156.87°$ A (rms), and $105.6\angle -83.13°$ A (rms)) [6 marks]

b. I_{aA}, I_{bB}, and I_{cC}, and (Answer: $182.9\angle 66.87°$ A (rms), $182.9\angle -173.13°$ A (rms), and $182.9\angle -53.13°$ A (rms)) [6 marks]

c. I_{ba}, I_{cb}, and I_{ac}. (Answer: $105.6\angle 36.87°$ A (rms), $105.6\angle 156.87°$ A (rms), and $105.6\angle -83.13°$ A (rms)) [6 marks]

Solution

a. The phase currents I_{AB}, I_{BC}, and I_{CA} at the load are:

$$I_{AB} = \frac{V_{AB}}{Z} = \frac{13{,}200\angle 0°}{100 - j75} = 105.6\angle 36.87° \text{ A (rms)}$$

Thus

$$I_{BC} = 105.6\angle 156.87° \text{ A (rms)}$$

$$I_{CA} = 105.6\angle -83.13° \text{ A (rms)}$$

b. The line currents I_{aA}, I_{bB}, and I_{cC} at the source are:

$$I_{aA} = \left(\sqrt{3}\angle -30°\right) I_{AB} = 182.9\angle 66.87° \text{ A (rms)}$$

Thus

$$I_{bB} = 182.9\angle -173.13° \text{ A (rms)}$$

$$I_{cC} = 182.9\angle -53.13° \text{ A (rms)}$$

c. The phase currents I_{ba}, I_{cb}, and I_{ac} at the source are:

$$I_{ba} = I_{AB} = 105.6\angle 36.87° \text{ A (rms)}$$

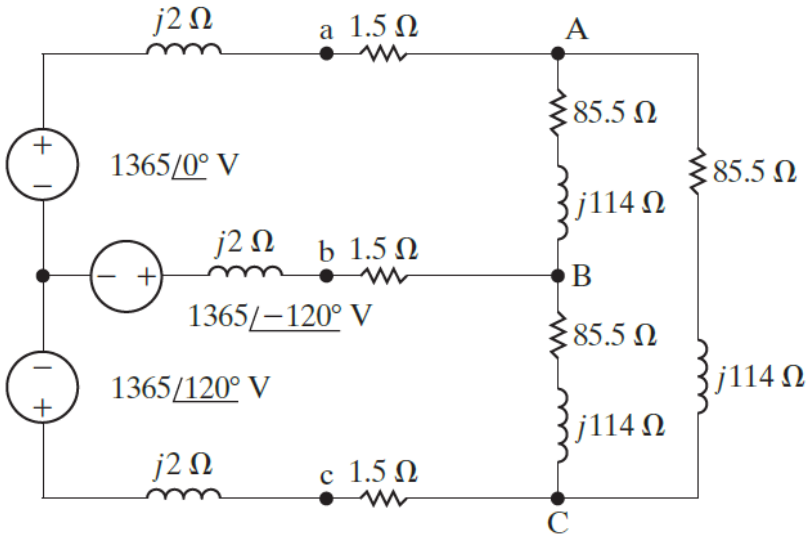
Thus

$$I_{cb} = I_{BC} = 105.6\angle 156.87° \text{ A (rms)}$$

$$I_{ac} = I_{CA} = 105.6\angle -83.13° \text{ A (rms)}$$

E. Power in Three-Phase Systems

1. Given a balanced three phase system circuit as shown in the figure below.

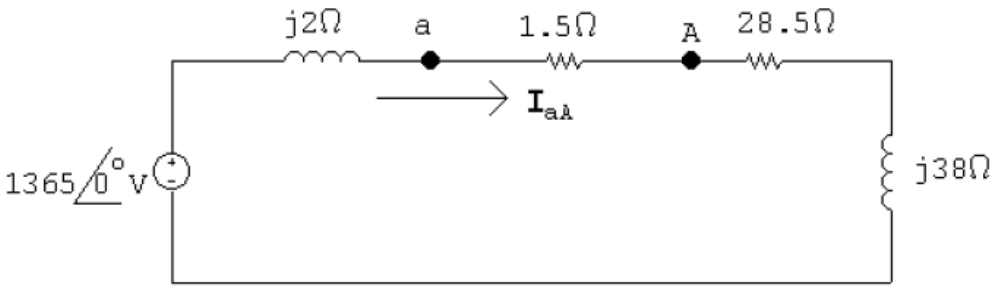


a. Find the rms magnitude and the phase angle of I_{CA} in the circuit. (Answer: $15.76\angle 96.87°$ A (rms)) [8 marks]

b. What percent of the average power delivered by the three-phase source is dissipated in the three-phase load? (Answer: 95%) [8 marks]

Solution

a. The a-phase equivalent circuit is as shown in the figure below.



The line current is:

$$I_{aA} = \frac{V_{an}}{Z_{an} + Z_{aA} + Z_{AN}} = \frac{1365}{(j2) + (1.5) + (28.5 + j38)} = 27.3\angle -53.13° \text{ A (rms)}$$

The line current at the load is:

$$I_{CA} = \left(\frac{1}{\sqrt{3}}\angle 150°\right) I_{aA} = 15.76\angle 96.87° \text{ A (rms)}$$

b. The total complex power at the source per phase is:

$$S_{g/\phi} = V_g I_{aA}^* = -1365(27.3\angle 53.13°) = -22{,}358.75 - j29{,}811.56 \text{ VA}$$

Since $S_{g/\phi} = P_{g/\phi} + Q_{g/\phi}$, the developed power per phase is:

$$P_{dev/\phi} = 22.359 \text{ kW}$$

The dissipated power per phase is:

$$P_{dis/\phi} = |I_{aA}|^2 Z_{AN} = (27.3)^2(28.5) = 21.241 \text{ kW}$$

The percentage delivered power is:

$$\eta = \left(\frac{P_{dis/\phi}}{P_{dev/\phi}}\right) \times 100\% = \left(\frac{21.241}{22.359}\right) \times 100\% = 95\%$$

2. A balanced three-phase source is supplying 60 kVA at 0.6 lagging to two balanced Y-connected parallel loads. The distribution line connecting the source to the load has negligible impedance. Load 1 is purely resistive and absorbs 30 kW. Find the per-phase impedance of Load 2 if the line voltage is $120\sqrt{3}$ V and the impedance components are in series. (Answer: $0.11 + j0.89\ \Omega$)

[8 marks]

Solution

The complex power of the source per phase is:

$$S_{g/\phi} = \frac{(S_g/3)}{\cos^{-1}\theta} = \frac{(60{,}000/3)}{\cos^{-1} 0.6} = 20{,}000\angle 53.13° = 12{,}000 + j16{,}000 \text{ VA}$$

Thus, knowing the dissipated power in load 1 is $S_2 = (30{,}000/3) = 10{,}000$ W, the complex power per phase must equal the sum of the impedances per phase of the two loads:

$$S_{g/\phi} = S_1 + S_2$$

So

$$12{,}000 + j16{,}000 = 10{,}000 + S_2$$

As a result,

$$S_2 = 2000 + j16{,}000 \text{ VA}$$

Also

$$S_2 = \frac{|V_{rms}|^2}{Z_2^*}$$

The magnitude of the RMS voltage is:

$$|V_{rms}| = \frac{|V_{load}|}{\sqrt{3}} = \frac{120\sqrt{3}}{\sqrt{3}} = 120 \text{ V (rms)}$$

Thus

$$Z_2^* = \frac{|V_{rms}|^2}{S_2} = \frac{(120)^2}{2000 + j16{,}000} = 0.11 - j0.89\ \Omega$$

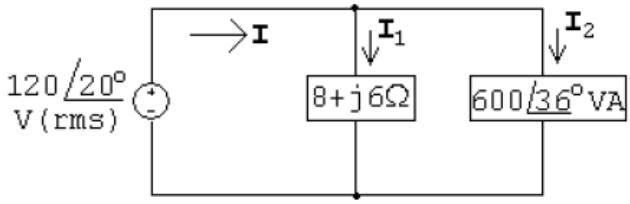
The impedance is:

$$Z_2 = 0.11 + j0.89\ \Omega$$

3. In a balanced three-phase system, the source has an abc sequence, is Y-connected, and $V_{an} = 120\angle 20°$ V. The source feeds two loads, both of which are Y-connected. The impedance of load 1 is $8 + j6\ \Omega/\phi$. The complex power for the a-phase of load 2 is $600\angle 36°$ VA. Find the total complex power supplied by the source. (Answer: $6120\angle 36.61°$ VA) [14 marks]

Solution

The a-phase equivalent circuit is as shown in the figure below.



The current I_1 in the circuit is:

$$I_1 = \frac{V_{an}}{Z_1} = \frac{120\angle 20°}{8 + j6} = 12\angle -16.87° \text{ A (rms)}$$

The conjugate current I_2^* in the circuit is:

$$I_2^* = \frac{P_{an}}{V_{an}} = \frac{600\angle 36°}{120\angle 20°} = 5\angle 16° \text{ A (rms)}$$

The total current I in the circuit is:

$$I = I_1 + I_2 = (12\angle -16.87°) + (5\angle 16°) = 17\angle -16.61° \text{ A (rms)}$$

The a-phase complex power at the source is:

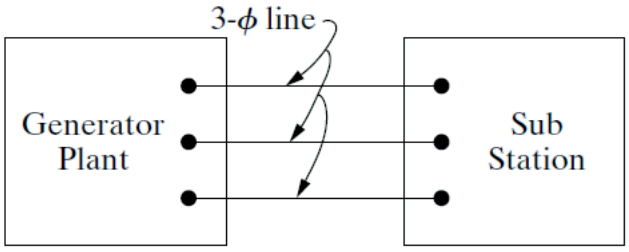
$$S_a = VI^* = (120\angle 20°)(17\angle 16.61°) = 2040\angle 36.61° \text{ VA}$$

The total complex power at the source is:

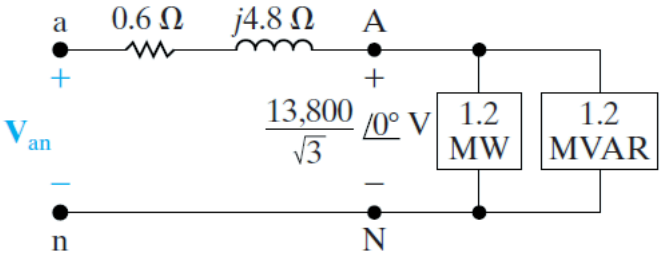
$$S_T = 2S_a = 3(2040\angle 36.61°) = 6120\angle 36.61° \text{ VA}$$

F. Electricity Distribution and Tariffs

1. The circuit shown in the figure below represents a substation that is a part of the national electricity grid. Assuming the system is balanced, the line-to-line voltage at the substation is 13.8 kV, the phase impedance of the distribution line is 0.6 + $j4.8\ \Omega$, and the load at the substation at 3 PM on a hot, humid day in January is 3.6 MW and 3.6 magnetising MVAr.



Using the line-to-neutral voltage at the substation as a reference, the single-phase equivalent circuit for the system above is shown in the figure below.



a. Determine the line current at the substation. (Answer: $150.61 - j150.61$ A) [2 marks]

b. Calculate the line to neutral voltage at the generating plant. (Answer: 15.248.11 V) [4 marks]

c. Assuming that an allowable voltage level tolerance is ± 5.8%, thus a nominal rms voltage of 120 V could range from to 113 V to 127 V. Does the magnitude of the line voltage at the generating plant cause problems for the customer? (Answer: No) [2 marks]

d. If compensating capacitors strategically located on the system could be used to support voltage levels, determine the line-to-line voltage at the substation now. (Answer: 14,012.58 V) [6 marks]

e. Calculate the total line loss in kW before and after the capacitors are connected to the substation bus. (Answer: 40.83 kW) [4 marks]

f. If the load is turned on for 10,000 hours each year and the electricity is charged at flat rate of 2.5 cents per kWh, calculate how much money that you are saving after capacitors are connected in a year. (Answer: $ 10,207.50) [4 marks]

Solution

a. The line current can be calculated from the expression for the complex power at the substation. Thus,

$$\left(\frac{13{,}800}{\sqrt{3}}\right) I_{aA}^{*} = (1.2 + j1.2)10^{6}$$

It follows that:

$$I_{aA}^{*} = 150.61 + j150.61 \text{ A}$$

Or

$$I_{aA} = 150.61 - j150.61 \text{ A}$$

b. The line-to-neutral voltage at the generating plant is:

$$V_{an} = \left(\frac{13{,}800}{\sqrt{3}}\right)\angle 0° + (0.6 + j4.8)(150.61 - j150.61)$$

$$= 8780.74 + j632.58 = 8803.50\angle 4.12° \text{ V}$$

Therefore, the magnitude of the line voltage at the generating plant is:

$$|V_{ab}| = \sqrt{3}(8803.50) = 15.248.11 \text{ V}$$

c. Assuming the utility is required to keep the voltage level within ± 5.8 % of the nominal value. This means the magnitude of the line-to-line voltage at the power plant should not exceed 14.6 kV nor be less than 13 kV. Therefore, in this case the magnitude of the line voltage at the generating plant could cause problems for customers.

d. When the magnetizing vars are supplied by a capacitor bank connected to the substation bus, the line current becomes:

$$I_{aA} = 150.61 + j0 \text{ A}$$

Therefore, the voltage at the generating plant necessary to maintain a line-to-line voltage of at the substation is:

$$V_{an} = \left(\frac{13{,}800}{\sqrt{3}}\right)\angle 0° + (0.6 + j4.8)(150.61 + j0)$$

$$= 8057.80 + j722.94 = 8090.17\angle 5.13° \text{ V}$$

Hence

$$|V_{ab}| = \sqrt{3}(8090.17) = 14{,}012.58 \text{ V}$$

This voltage level falls within the allowable range of 13 kV to 14.6 kV.

e. Before the capacitors are added the total line loss is:

$$P_L = 3(150.61 + j150.61)^2(0.6) = 81.66 \text{ kW}$$

After the capacitors are added the total line loss is:

$$P_L = 3(150.61)^2(0.6) = 40.83 \text{ kW}$$

Note that adding the capacitors to control the voltage level also reduces the amount of power loss in the lines, which in this example is cut in half.

f. The total annual cost saving for the electricity bill after the capacitor is added is:

Total Saving = (81.66 kW – 40.83 kW)(10,000 Hrs)($ 0.025/kWh) = $ 10,207.50

2. A consumer has a maximum demand of 200 kW at 40% load factor. If the tariff is $ 100 per kW of maximum demand plus 10 cents per kWh, find the overall cost per kWh. (Answer: 12.85 cents) [6 marks]

Solution

The amount of electricity unit consumed in a year is:

Units consumed/year = Max. demand × L.F. × Hours/Year = (200)(0·4)(8760) = 700,800 kWh

The annual charges are:

Annual charges = Annual M.D. charges + Annual energy charges

= (100 × 200 + 0·1 × 700,800) = $ 90,080

The overall cost per kWh consumed is:

$$\text{Overall cost/kWh} = \frac{90{,}080}{700{,}800} = 0.1285 = 12.85 \text{ cents}$$

3. The following two tariffs are offered:

(a) $ 100 plus 15 cents per unit.

(b) A flat rate of 30 cents per unit.

At what consumption is first tariff economical? (Answer: 66.67 units) [4 marks]

Solution

Let x be the number of units at which charges due to both tariffs become equal. Then,

$$100 + 0.15x = 0.3x$$

or

$$0.15x = 100$$

Thus

$$x = \frac{100}{0.15} = 666.67 \text{ units}$$

Therefore, tariff (a) is economical if consumption is more than 666.67 units.

4. Consider the following monthly bill for a fairly large industrial plant:

   - Real power consumption: 146.5 MWh.
   - Peak real power demand: 241 kW.
   - Real power demand: 233 kW.
   - Apparent power demand: 250 kVA.

   The rate schedule provided by the local power authority is the following:

   - Energy:

     First 450 kWh @ 22.3 cent/kWh.

     Next 12 MWh @ 17.1 cent/kWh.

     Additional kWh @ 8.9 cent/kWh.

   - Power:

     First 240 kW @ free.

     Additional kW @ $ 12.05/kW.

   Note that this rate schedule has an energy cost breakdown and a power breakdown. This second fee is the one sensitive to the overall power factor of the plant.

   a. Determine the total electric bill for the month. (Answer: $ 14,056.10) [4 marks]

   b. Calculate the power factor of the load for the month. (Answer: 0.932) [2 marks]

   c. Describe the significant of the power factor calculated in part (b) towards the electric bill for the month for the company. [4 marks]

   d. If the kVA demand of the bill were 388 kVA with the kW demand staying at 233 kW, calculate the total electric bill for the month. (Answer: $ 15,371.96) [10 marks]

   e. Calculate the power factor as per condition in part (d) and describe whether this could affect the performance of the electricity distribution of the plant. What is the solution if it is a problem? (Answer: 0.6) [6 marks]

Solution

a. The electric bill for the month is then calculated as follows:

$$\text{Total Cost} = (450 \text{ kWh})(22.3¢/\text{kWh}) + (12 \text{ MWh})(17.1¢/\text{kWh}) + [146.2 \text{ MWh} - (12 \text{ MWh} + 450 \text{ kWh})](8.9¢/\text{kWh})$$

$$= \$\,100.35 + \$\,2052.00 + \$\,11{,}903.75$$
$$= \$\,14{,}056.10$$

b. Before examining the effect of the power fee structure, we can find the overall power factor of the load for the month with the following ratio taken from the monthly statement:

$$pf = \frac{P}{S} = \frac{233\ \text{kW}}{250\ \text{kVA}} = 0.932$$

c. Since the power factor is larger than 0.9, the chances are that there will not be a surcharge or that the surcharge will be minimal.

When the power component of the bill is determined, the kVA demand is multiplied by 0.9 to determine a kW level at this power factor. This kW level is compared to the metered level, and the consumer pays for the higher level.

In this case, if we multiply the 250 kVA by 0.9, we obtain 225 kW which is slightly less than the metered level of 233 kW. However, both levels are less than the free level of 240 kW, so there is no additional charge for the power component. The total bill remains at $ 14,056.10.

d. If the kVA demand of the bill were 388 kVA with the kW demand staying at 233 kW, the situation would change because 0.9 times 388 kVA would result in 349.2 kW which is much greater than the metered 233 kW. The 349.2 kW would then be used to determine the bill as follows:

$$P_{extra} = 349.2\ \text{kW} - 240\ \text{kW} = 109.2\ \text{kW}$$

Thus

$$\text{Extra Cost} = (109.2\ \text{kW})(\$\,12.05/\text{kW}) = \$\,1315.86$$

which is significant.

The total bill can then be determined as follows:

$$\text{Total Cost} = \$\,14{,}056.10 + \$\,1{,}315.86 = \$\,15{,}371.96$$

e. Thus, the power factor of the load dropped to:

pf = 233 kW/388 kVA = 0.6

This would put an unnecessary additional load on the power plant. It is certainly time to consider the power-factor-correction option.
It is not uncommon to see large capacitors sitting at the point where power enters a large industrial plant to perform a needed level of power-factor correction.