

Demo 7 – Z-Transforms

XMUT220 Signals and Systems

Overview

1. Z-transforms and inverse Z-transform of functions.
2. Inverse z-transform of difference equation.
3. Inverse z-transform with `deconv()` function.
4. Partial fraction expansion with MATLAB.
5. Pole-zero diagram in MATLAB.
6. Roots and pole zero diagram of z-transform.
7. Frequency response of z-transform.
8. Plotting frequency response of digital filter.

Exercise 1 – Z-transform and Inverse Z-transform of functions

Z-transform is defined as:

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} \qquad \text{or} \qquad X(z) = Z[x(n)]$$

The inverse z-transform is denoted by:

$$x(n) = Z^{-1}[x(z)]$$

MATLAB Symbolic Toolbox gives the z-transform of a function:

$$x[n] = \frac{1}{4^n} u(n) \qquad \text{and} \qquad X(z) = \frac{2z}{2z-1}$$

The MATLAB programs for obtaining z-transform and inverse z-transform of the z-transform equations as given above are:

MATLAB code:

```
% demo71a.m
% z-transfer

syms z n
ztrans(1/4^n)
```

```
% demo71b.m
% inverse z-transfer

syms z n
iztrans(2*z/(2*z-1))
```

The results of the simulation are given below.

```
ans =

z/(z - 1/4)
```

```
ans =

(1/2)^n
```

Exercise 2 – Inverse Z-transform of Difference Equation

Let us consider the difference equation of a discrete-time system:

$$6y(n) - 5y(n-1) + y(n-2) = \frac{1}{4^n} \quad n \geq 0 \quad \text{and} \quad y(n-1) = 1, y(n-2) = 0$$

Taking the z-transform:

$$6Y(z) - 5\{z^{-1}Y(z) + y(-1)\} + \{z^{-2}Y(z) + z^{-1}y(-) + y(n-2)\} = \frac{4z}{4z-1}$$

Gather the coefficients of the same variables.

$$(6 - 5z^{-1} + z^{-2})Y(z) = \frac{4z}{4z-1} - z^{-1} + 5$$

Rearrange the above equation

$$Y(z) = \left(\frac{1}{6 - 5z^{-1} + z^{-2}}\right)\left(\frac{4z}{4z-1} - z^{-1} + 5\right)$$

The MATLAB program for finding the discrete-time system differential equation of the inverse z-transform as given above is:

MATLAB code:

```
% demo72.m
% Inverse z-transform of a given z-transform

syms z n;
```

$$\text{iztrans}\big((4*\text{z}/(4*\text{z}-1)-\text{z}\text{\^{}}-1+5)/(6-5*\text{z}\text{\^{}}-1+\text{z}\text{\^{}}-2)\big)$$

The output of the simulation is given as follows.

```
ans =

(5*(1/2)^n)/2 - 2*(1/3)^n + (1/4)^n/2
```

Therefore, the inverse z-transform of the given difference equation is:

$$y(n) = 5/2(1/2)^n - 2(1/3)^n + 1/2(1/4)^n$$

Exercise 3 – Inverse Z-transform with `deconv()` Function

In MATLAB, `deconv()` function is used to compute inverse z transform. It provides deconvolution and polynomial division of a given z-transform. In this exercise we have several examples to determine the values of `x(n)` for few samples.

Syntax:

```
[q,r]=deconv(x,h)
```

Description:

If `x` and `h` are vectors of polynomial coefficients, convolving them is equivalent to multiplying the two polynomials, and deconvolution a polynomial division. The result of dividing `h` by `x` is quotient `q` and remainder `r`.

`[q,r]=deconv(x,h)` deconvolves vector `x` out of vector `h`, using long division. The quotient is returned in vector `q` and the remainder in vector `r` such that `v=conv(u,q)+r`

Examples

If `x=[1234]` and `h=[10 20 30]`, the convolution is:

```
c=conv(x,h)
```

The output of the simulation is given as follows.

```
c = 10 40 100 150 170 120
```

Now, use deconvolution to recover `h`:

```
[q,r]=deconv(c,x)
```

The output of the simulation is given as follows.

```
q = 10 20 30

r = 0 0 0 0 0 0
```

This gives a quotient equal to vector as shown above and a zero remainder.

Find the first five terms of the inverse z-transform of:

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 2z^{-1} + 4z^{-2}}$$

The MATLAB program finding the first five terms of the inverse z-transform given above is:

MATLAB code:

```
% demo73.m
% Find first five terms of given z-transform

b=[1 2 1];
a=[1 -2 4];
n=5;

b=[b zeros(1,n-1)];
[x,r]=deconv(b,a);
disp(x)
```

The output of the simulation is given as follows.

```
1       4       5      -6     -32
```

Therefore, the first five terms of the given z-transform are given below:

$$x(0) = 1, x(1) = 4, x(2) = 5, x(3) = -6, x(4) = -32$$

Hence, discrete-time sequence of the given inverse z-transform is:

$$x(n) + 4x(n-1) + 5x(n-2) - 6x(n-3) - 32x(n-4)$$

Exercise 4 – Partial Fraction Expansion with MATLAB

Partial fraction expansion is performed with the `residuez()` function in MATLAB. The function may be used to perform partial fraction expansion of a z-transform:

$$X(z) = \frac{N(z)}{D(z)} = \frac{h_0 + h_1 z^{-1} + \cdots + h_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

The partial fraction expansion has the form:

$$\frac{N(z)}{D(z)} = \frac{r_1}{1 - p_1 z^{-1}} + \cdots + \frac{r_N}{1 - p_N z^{-1}} + k_1 + k_2 z^{-1} + \cdots + k_{M-N-1} z^{-(M-N)}$$

Find the partial fraction expansion of the following z-transform:

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 2z^{-1} + 4z^{-2}}$$

The MATLAB program for finding the partial fraction expansion of the z-transform given above is:

MATLAB code:

```
% demo74.m
% Find partial fraction of z-transform

[r,p,k]=residuez([1,2,1],[1,-2,4])
```

The output of the simulation is given as follows.

```
r =

   0.3750 - 0.9382i
   0.3750 + 0.9382i


p =

   1.0000 + 1.7321i
   1.0000 - 1.7321i


k =

    0.2500
```

This tells us that the partial fraction of the given z-transform equation is:

$$X(z) = \frac{0.375 - j0.9382}{z - (1 + j1.7321)} + \frac{0.375 + j0.9382}{z - (1 - j1.7321)} + 0.25$$

Exercise 5 - Pole-Zero Diagram in MATLAB

The `zplane()` function in MATLAB can display the pole-zero diagram of z-transform.

Given the following discrete-time system represented with the following equation:

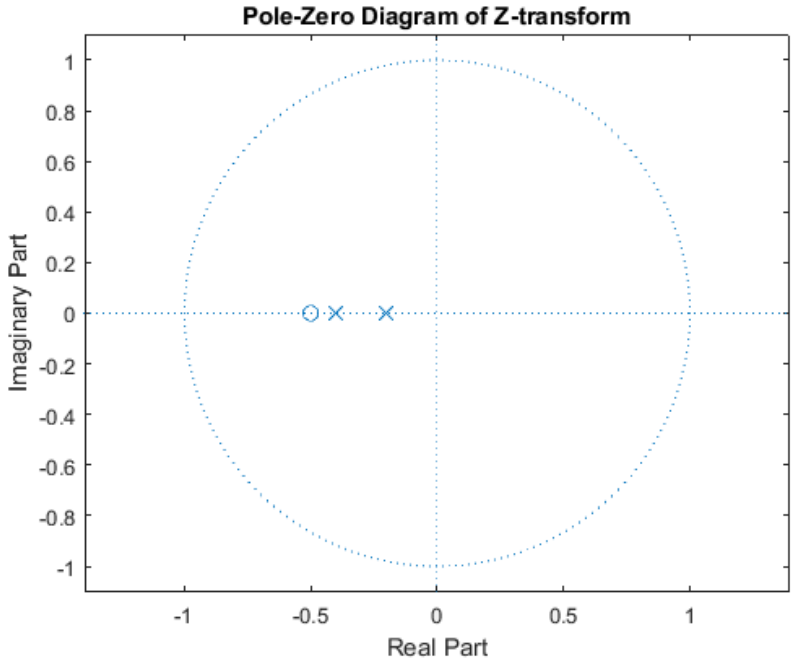
$$H(z) = \frac{z^{-1} + \frac{1}{2}z^{-2}}{1 + \frac{3}{5}z^{-1} + \frac{2}{25}z^{-2}}$$

The MATLAB program for creating pole-zero diagram of the z-transform given above is:

MATLAB code:

```
% demo75.m
% Plot pole zero diagram of z-transform

b=[0 1 1/2];
a=[1 3/5 2/25];

zplane(b,a);

title('Pole-Zero Diagram of Z-transform')
```

The output of the simulation is given as follows.



Exercise 6 – Roots and Pole Zero Diagram of Z-transform

The `zplane` command function in MATLAB computes and display the pole-zero diagram of z-function. The command function is:

$$\text{zplane}(b, a)$$

To display the poles values, use `root(a)` and to display the zero value, use `root(b)`.

Given the following z-transform equation:

$$X(z) = (z^{-2} + z^{-1})/(1 - 2z^{-1} + 3z^{-2})$$

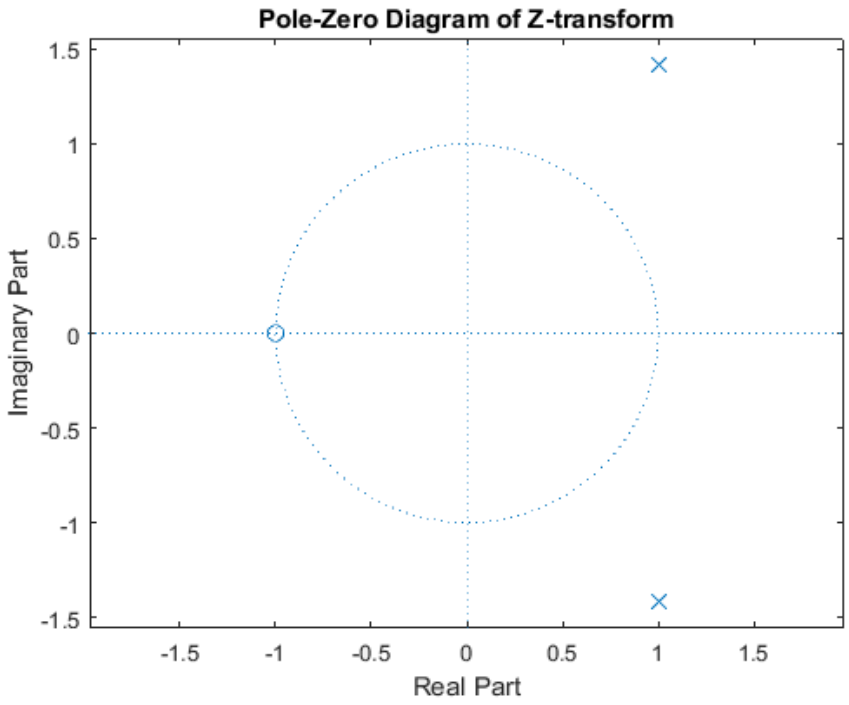
The MATLAB program for determining roots and creating pole-zero diagram of the equation of the given discrete-time system shown above is:

MATLAB Code:

```
% demo76.m
% Determine roots and plot pole zero diagram of z-transform

b=[0 1 1 ];
a=[1 -2 +3];

roots(a);
roots(b);

zplane(b,a);

title('Pole-Zero Diagram of Z-transform')
```

The output of the simulation is given as follows.

```
ans =
1.0000 + 1.4142i
1.0000 - 1.4142i

ans=
-1
```



Exercise 7 - Frequency Response of Z-Transform

The `freqz()` function in MATLAB computes and display the frequency response of given z-transform of the function:

$$\text{freqz}(b, a, Fs)$$

Where: b= coefficient of numerator, a= coefficient of denominator, and Fs= sampling frequency.

Given the following equation of a discrete-time system, simulate the frequency response of the system in MATLAB.

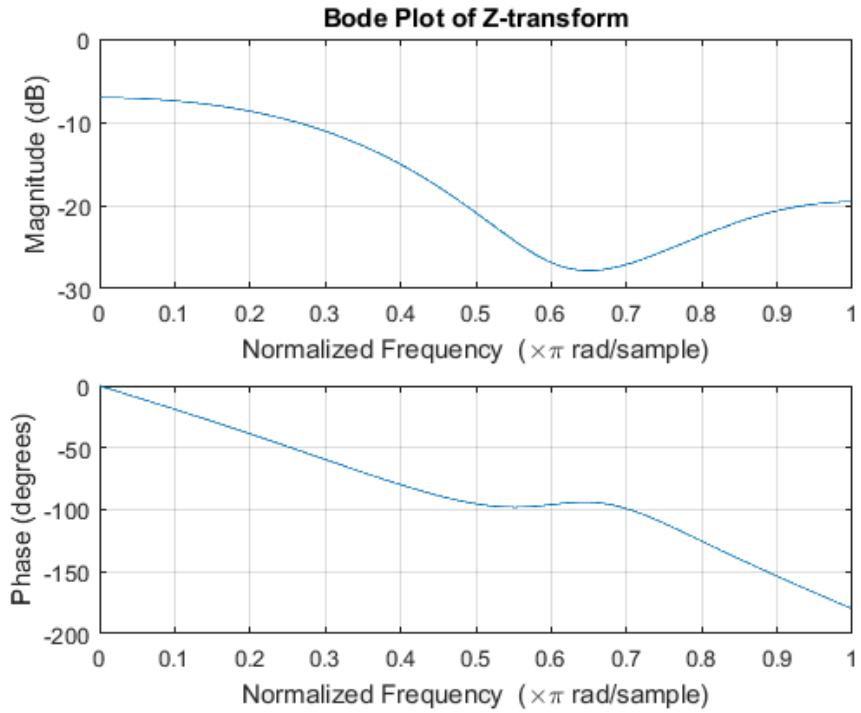
$$H(z) = \frac{2z^{-4} + 5z^{-3} + 9z^{-2} + 5z^{-1} + 3}{5z^{-4} + 45z^{-3} + 2z^{-2} + z^{-1} + 1}$$

The following MATLAB program computes and plots the frequency response of the given z-transform function.

MATLAB Code:

```
% demo77.m
% Create Bode plot of z-transform

b=[2 5 9 5 3];
a=[5 45 2 1 1];

freqz(b,a);

title('Bode plot of z-transform')
```

The magnitude and phase plots result of frequency response simulation is shown in the figure below.



Exercise 8 – Plotting Frequency Response of Digital Filter

Plot in MATLAB the magnitude and phase of the frequency response of the given digital filter with differential equation as shown below using `freqz()` function:

$$Y(n) = 0.2x(n) + 0.52y(n-1) - 0.68y(n-2)$$

The following MATLAB program plots the Bode plot of the given digital filter.

MATLAB Code:

```
% demo78.m
% Create Bode plot of z-transform

b=[0.2];
a=[1,-0.52,0.68];
w=[0:1:500]*pi/500;

H=freqz(b,a,w);

magH=abs(H);
phaH=angle(H)*180/pi;
```

```
subplot(2,1,1);
plot(w/pi,magH);

title('Magnitude Response');
xlabel('Frequency in pi units');
ylabel('|H|');

subplot(2,1,2);
plot(w/pi,phaH);

title('Phase Response');
xlabel('Frequency in pi units');
ylabel('Degrees');
```

The magnitude and phase responses of simulation is shown in the figure below.

