XMUT220 Signals and Systems

Demo 7: z-Transforms

Section A. Z-transform in Matlab

In this section, there are several basic exercises for working with and manipulating the z-transform in MATLAB e.g. z-transform with `ztrans()` function, inverse z-transform with `iztrans()` function and `deconv()` function, and partial fraction expansion with `residuez()` function.

Exercise 1 – Z-transform and Inverse Z-transform of functions

Z-transform is defined as:

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} \qquad \text{or} \qquad X(z) = Z[x(n)]$$

The inverse z-transform is denoted by:

$$x(n) = Z^{-1}[x(z)]$$

MATLAB Symbolic Toolbox gives the z-transform of a function:

$$x[n] = \left(\frac{1}{4^n}\right) u(n) \qquad \text{and} \qquad X(z) = \frac{2z}{2z-1}$$

The MATLAB programs for obtaining z-transform and inverse z-transform of the z-transform equations as given above are:

MATLAB code:

```
% demo71a.m
% z-transfer

syms z n

ztrans(1/4^n)
```

```
% demo71b.m
% inverse z-transfer

syms z n

iztrans(2*z/(2*z-1))
```

The results of the simulation are given below.

```
ans =

z/(z - 1/4)
```

```
ans =

(1/2)^n
```

Exercise 2 – Inverse Z-transform of Difference Equation

Let us consider the difference equation of a discrete-time system:

$$6y(n) - 5y(n-1) + y(n-2) = \frac{1}{4^n} \quad n \geq 0$$

And

$$y(n-1) = 1, y(n-2) = 0$$

Taking the z-transform:

$$6Y(z) - 5\{z^{-1}Y(z) + y(-1)\} + \{z^{-2}Y(z) + z^{-1}y(-) + y(n-2)\} = \frac{4z}{4z-1}$$

Gather the coefficients of the same variables.

$$(6 - 5z^{-1} + z^{-2})Y(z) = \frac{4z}{4z-1} - z^{-1} + 5$$

Rearrange the above equation

$$Y(z) = \left(\frac{1}{6 - 5z^{-1} + z^{-2}}\right)\left(\frac{4z}{4z-1} - z^{-1} + 5\right)$$

The MATLAB program for finding the discrete-time system differential equation of the inverse z-transform as given above is:

MATLAB code:

```
% demo72.m
% Inverse z-transform of a given z-transform

syms z n;

iztrans((4 * z/(4 * z - 1) - z^ - 1 + 5)/(6 - 5 * z^ - 1 + z^ - 2))
```

The output of the simulation is given as follows.

```
ans =

(5*(1/2)^n)/2 - 2*(1/3)^n + (1/4)^n/2
```

Therefore, the inverse z-transform of the given difference equation is:

$$y(n) = \frac{5}{2}\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n + \frac{1}{2}\left(\frac{1}{4}\right)^n$$

Exercise 3 – Inverse Z-transform with `deconv()` Function

In MATLAB, `deconv()` function is used to compute inverse z transform. It provides deconvolution and polynomial division of a given z-transform. In this exercise we have several examples to determine the values of `x(n)` for few samples.

Syntax:

```
[q,r]=deconv(x,h)
```

Description:

If `x` and `h` are vectors of polynomial coefficients, convolving them is equivalent to multiplying the two polynomials, and deconvolution a polynomial division. The result of dividing `h` by `x` is quotient `q` and remainder `r`.

`[q,r]=deconv(x,h)` deconvolves vector `x` out of vector `h`, using long division. The quotient is returned in vector `q` and the remainder in vector `r` such that `v=conv(u,q)+r`

Examples

If `x=[1234]` and `h=[10 20 30]`, the convolution is:

```
c=conv(x,h)
```

The output of the simulation is given as follows.

```
c = 10 40 100 150 170 120
```

Now, use deconvolution to recover `h`:

```
[q,r]=deconv(c,x)
```

The output of the simulation is given as follows.

```
q = 10 20 30

r = 0 0 0 0 0 0
```

This gives a quotient equal to vector as shown above and a zero remainder.

Find the first five terms of the inverse z-transform of:

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 2z^{-1} + 4z^{-2}}$$

The MATLAB program finding the first five terms of the inverse z-transform given above is:

MATLAB code:

```
% demo73.m
% Find first five terms of given z-transform

b=[1 2 1];
a=[1 -2 4];
n=5;

b=[b zeros(1,n-1)];
[x,r]=deconv(b,a);
disp(x)
```

The output of the simulation is given as follows.

```
     1     4     5    -6   -32
```

Therefore, the first five terms of the given z-transform are given below:

$$x(0) = 1, \quad x(1) = 4, \quad x(2) = 5, \quad x(3) = -6, \quad x(4) = -32$$

Hence, discrete-time sequence of the given inverse z-transform is:

$$x(n) + 4x(n-1) + 5x(n-2) - 6x(n-3) - 32x(n-4)$$

Exercise 4 – Partial Fraction Expansion with MATLAB

Partial fraction expansion is performed with the `residuez()` function in MATLAB. The function may be used to perform partial fraction expansion of a z-transform:

$$X(z) = \frac{N(z)}{D(z)} = \frac{h_0 + h_1 z^{-1} + \cdots + h_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

The partial fraction expansion has the form:

$$\frac{N(z)}{D(z)} = \frac{r_1}{1 - p_1 z^{-1}} + \cdots + \frac{r_N}{1 - p_N z^{-1}} + k_1 + k_2 z^{-1} + \cdots + k_{M-N-1} z^{-(M-N)}$$

Find the partial fraction expansion of the following z-transform:

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 2z^{-1} + 4z^{-2}}$$

The MATLAB program for finding the partial fraction expansion of the z-transform given above is:

MATLAB code:

```
% demo74.m
% Find partial fraction of z-transform

[r,p,k]=residuez([1,2,1],[1,-2,4])
```

The output of the simulation is given as follows.

```
r =

   0.3750 - 0.9382i
   0.3750 + 0.9382i


p =

   1.0000 + 1.7321i
   1.0000 - 1.7321i


k =

    0.2500
```

This tells us that the partial fraction of the given z-transform equation is:

$$X(z) = \frac{0.375 - j0.9382}{z - (1 + j1.7321)} + \frac{0.375 + j0.9382}{z - (1 - j1.7321)} + 0.25$$

Section B. Application of Z-Transform in Frequency Response Analysis

In this section, several exercises for application of z-transform in frequency response analysis are outlined. Several functions which are used in MATLAB are typically: `zplane()` and `freqz()` functions.

Exercise 5 - Pole-Zero Diagram in MATLAB

The `zplane()` function in MATLAB can display the pole-zero diagram of z-transform.

Given the following discrete-time system represented with the following equation:

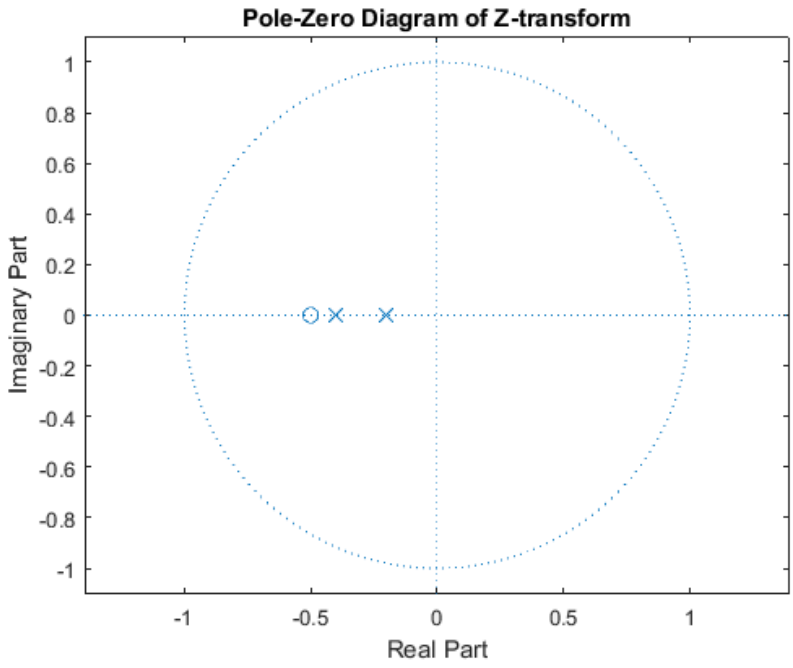
$$H(z) = \frac{z^{-1} + \frac{1}{2}z^{-2}}{1 + \frac{3}{5}z^{-1} + \frac{2}{25}z^{-2}}$$

The MATLAB program for creating a pole-zero diagram of the z-transform given above is:

MATLAB code:

```
% demo75.m
% Plot pole zero diagram of z-transform

b=[0 1 1/2];
a=[1 3/5 2/25];

zplane(b,a);

title('Pole-Zero Diagram of Z-transform')
```

The output of the simulation is given as follows.



Exercise 6 – Roots and Pole Zero Diagram of Z-transform

The `zplane` command function in MATLAB computes and display the pole-zero diagram of z-function. The command function is:

```
zplane(b,a)
```

To display the poles values, use `root(a)` and to display the zero value, use `root(b)`.

Given the following z-transform equation:

$$X(z) = \frac{z^{-2} + z^{-1}}{1 - 2z^{-1} + 3z^{-2}}$$

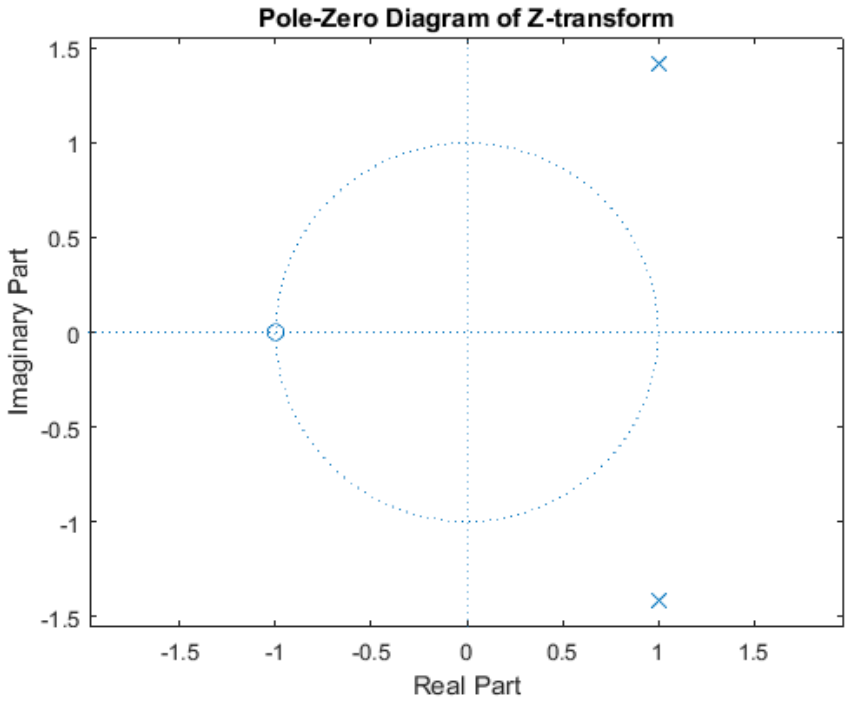
The MATLAB program for determining roots and creating pole-zero diagram of the equation of the given discrete-time system shown above is:

MATLAB Code:

```
% demo76.m
% Determine roots and plot pole zero diagram of z-transform

b=[0 1 1 ];
a=[1 -2 +3];

roots(a);
roots(b);

zplane(b,a);

title('Pole-Zero Diagram of Z-transform')
```

The output of the simulation is given as follows.

```
ans =
1.0000 + 1.4142i
1.0000 - 1.4142i

ans=
-1
```



Exercise 7 - Frequency Response of z-Transform

The `freqz()` function in MATLAB computes and display the frequency response of given z-transform of the function:

$$freqz(b, a, F_s)$$

Where: b = coefficient of numerator, a = coefficient of denominator, and F_s = sampling frequency.

Given the following equation of a discrete-time system, simulate the frequency response of the system in MATLAB.

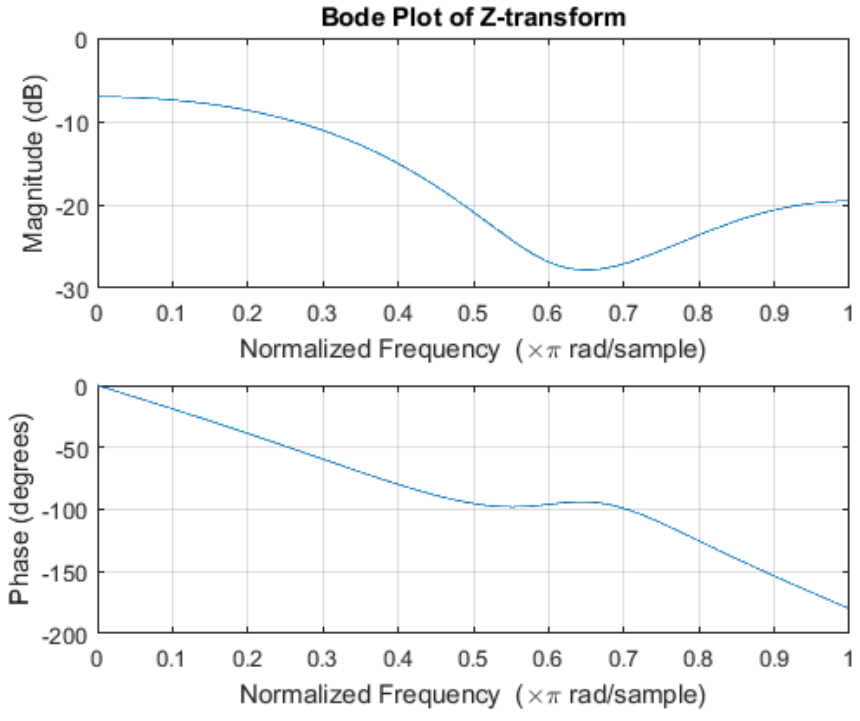
$$H(z) = \frac{2z^{-4} + 5z^{-3} + 9z^{-2} + 5z^{-1} + 3}{5z^{-4} + 45z^{-3} + 2z^{-2} + z^{-1} + 1}$$

The MATLAB program below simulates the frequency response of the given z-transform function.

MATLAB Code:

```
% demo77.m
% Create Bode plot of z-transform

b=[2 5 9 5 3];
a=[5 45 2 1 1];

freqz(b,a);
title('Bode plot of z-transform')
```

The magnitude and phase plots of frequency response simulation is shown in the figure below.



Exercise 8 – Plotting Frequency Response of Digital Filter

Plot in MATLAB the magnitude and phase of the frequency response of a digital filter with differential equation as shown below using `freqz()` function:

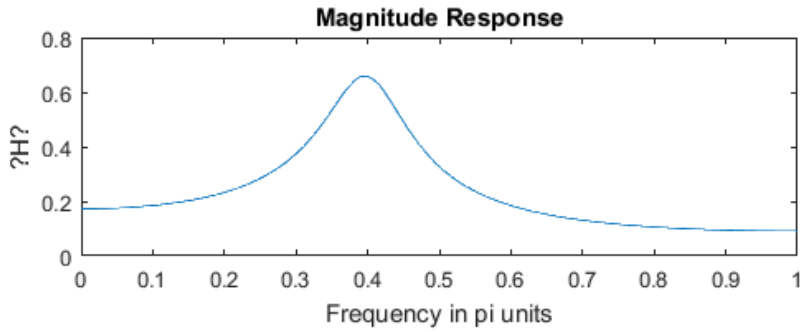
$$Y(n) = 0.2x(n) + 0.52y(n-1) - 0.68y(n-2)$$

The following MATLAB program plots the Bode plot of the given digital filter.

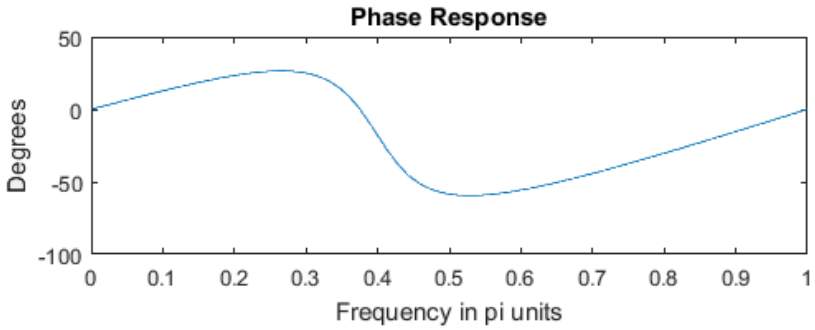
MATLAB Code:

```
% demo78.m
% Create Bode plot of z-transform

b=[0.2];
a=[1,-0.52,0.68];
w=[0:1:500]*pi/500;

H=freqz(b,a,w);

magH=abs(H);
phaH=angle(H)*180/pi;

subplot(2,1,1);
plot(w/pi,magH);
title('Magnitude Response');
xlabel('Frequency in pi units');
ylabel('|H|');

subplot(2,1,2);
plot(w/pi,phaH);
title('Phase Response');
xlabel('Frequency in pi units');
ylabel('Degrees');
```

The result of frequency response simulation in Matlab is shown in the figure below. The magnitude plot of the frequency response is shown in the figure below.



The phase plot of the frequency response in shown in the figure below.


Phase Response
Degrees
50
0
-50
-100
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Frequency in pi units