

Time and Frequency Characteristics (Time Response Analysis)

XMUT220 Signals and Systems

Topics

Introduction to the concepts and parameters used for time-domain analysis.

1st order system:

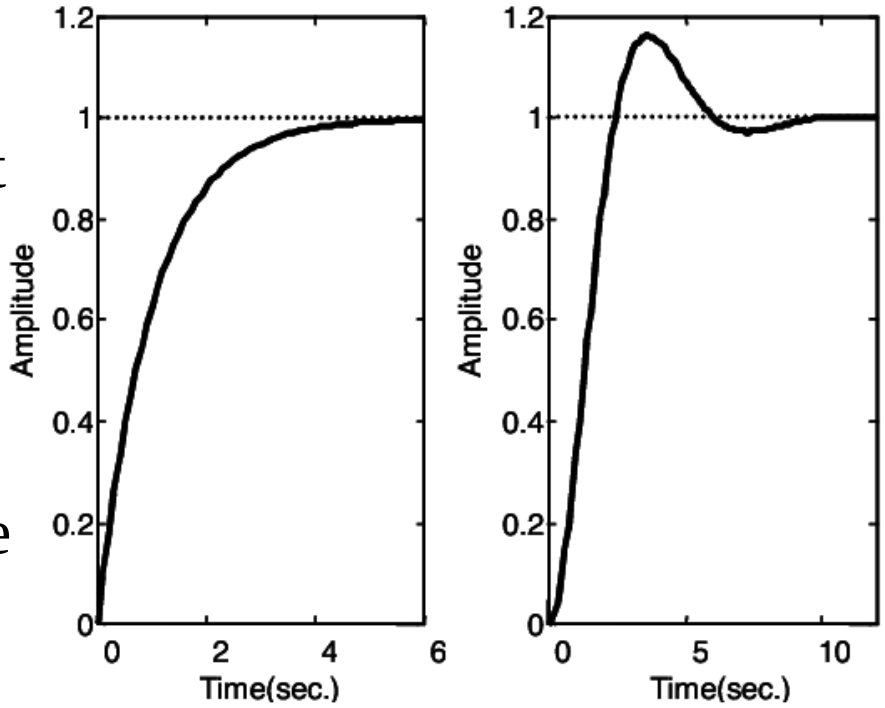
- Time constant (τ)
- Rise time (T_r)
- Settling time (T_s)

2nd order system:

- Natural frequency (ω_n)
- Damping ratio (ζ)
- Rise time (T_r)
- Settling time (T_s)
- Time-to-peak (T_p)
- Percentage overshoot ($\%OS$)
- Steady-state error ($e(\infty)$)

Time Domain Analysis

- It is the response of a system to an input in the time domain that determine the characteristics of the systems.
- Several quantitative parameters are used to characterise the time response of a system.



- We evaluate the time response of the systems in terms of transient response (e.g. transition from one condition to another after given an input) of the first order- and second-order systems.
- Higher order systems could be approximated as any combinations of first-order, second-order systems, etc.

First-Order System

- A first-order system is defined as a type of system whose input-output relationship (also known as a transfer function) is a first-order differential equation.
- The first-order system above can be represented with the following equation:

$$\tau\left[\frac{dy(t)}{dt}\right] + y(t) = x(t)$$

Where: t is the time constant of the system that controls the rate at which the first-order system responds.

- The expression used for the given first-order system is:

$$H(\omega) = \frac{1}{j\omega\tau + 1}$$

First-Order Time Response

- A first-order system may be written as:

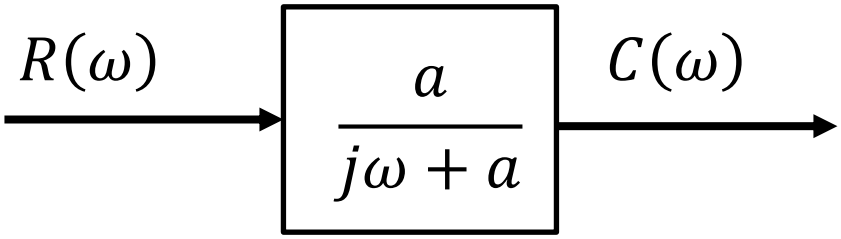
$$\frac{C(\omega)}{R(\omega)} = \frac{K}{j\omega\tau + 1} \quad \text{or} \quad \frac{C(\omega)}{R(\omega)} = \frac{a}{j\omega + a}$$

  Where: $K = 1$ and $\tau = 1/a$

- These time constant (τ) and system gain (K) are often used in the time response analysis of first-order systems.
- Other parameters for analysis of first-order system are typically rise time (T_r) and settling time (T_s).
- For second-order system, these parameters are also considered, together with other parameters specific for second order system e.g. damping ratio, natural frequency, time-to-peak, etc.

First-Order Time Response

- For a first-order system given below, derive its time response parameters.



$G(\omega)$

- The transfer function of the first-order system given above is:

$$G(\omega) = \frac{C(\omega)}{R(\omega)} = \frac{a}{j\omega + a}$$

- Rearrange the equation given above:

$$C(\omega) = R(\omega)\left(\frac{a}{j\omega + a}\right)$$

First-Order Time Response

- For a unit step, $R(\omega) = u(t) = 1/j\omega$, the equation becomes:

$$C(\omega) = \left(\frac{1}{j\omega}\right)\left(\frac{a}{j\omega + a}\right)$$

- Applying a partial fraction to the equation above:

$$C(\omega) = \left(\frac{1}{j\omega}\right) - \left(\frac{1}{j\omega + a}\right)$$

- By using inverse transform from Fourier transform table, we obtain the response of first-order system in time domain:

$$c(t) = c_f(t) + c_n(t)$$

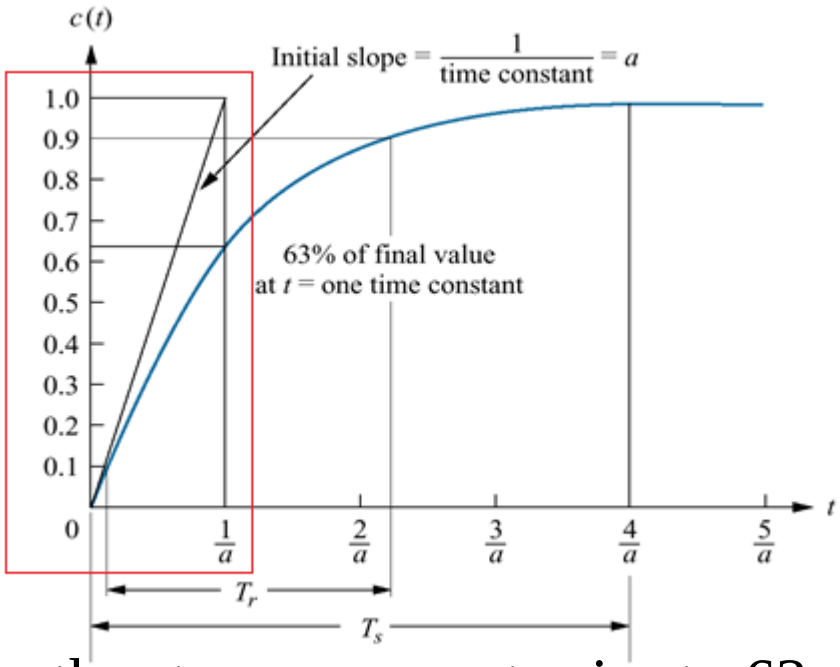
Or

$$c(t) = 1 - e^{-at}$$

Time Constant

- Time constant (τ) is derived from the time for e^{-at} to decay to 37% of its final value.

$$\tau_{decay} = e^{-at}\Big|_{t=1/a}$$
$$= e^{-1} = 0.37$$



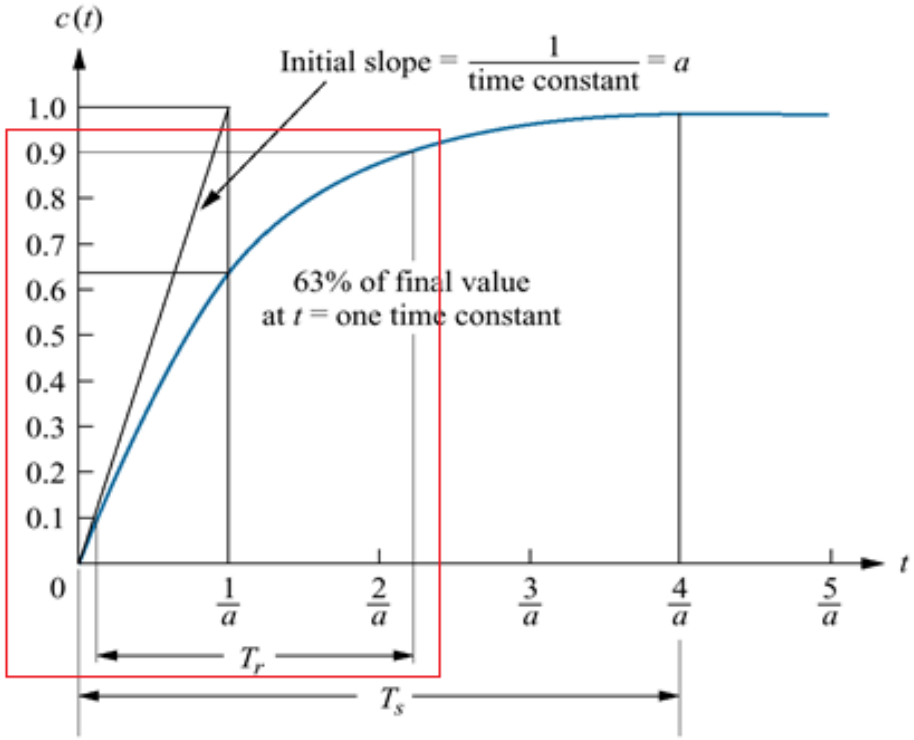
- It is actually defined as the time for the step response to rise to 63 % of its final value.
- Referring to the equation of the first-order system equation, this is achieved when $t = 1/a$.

$$\tau = \tau_{increase} = c(t)\Big|_{t=1/a} = 1 - e^{-at}\Big|_{t=1/a} = 1 - 0.37 = 0.63$$

Rise Time

- Rise time (T_r) is defined as the time for the response to go from 0.1 to 0.9 of its final value:

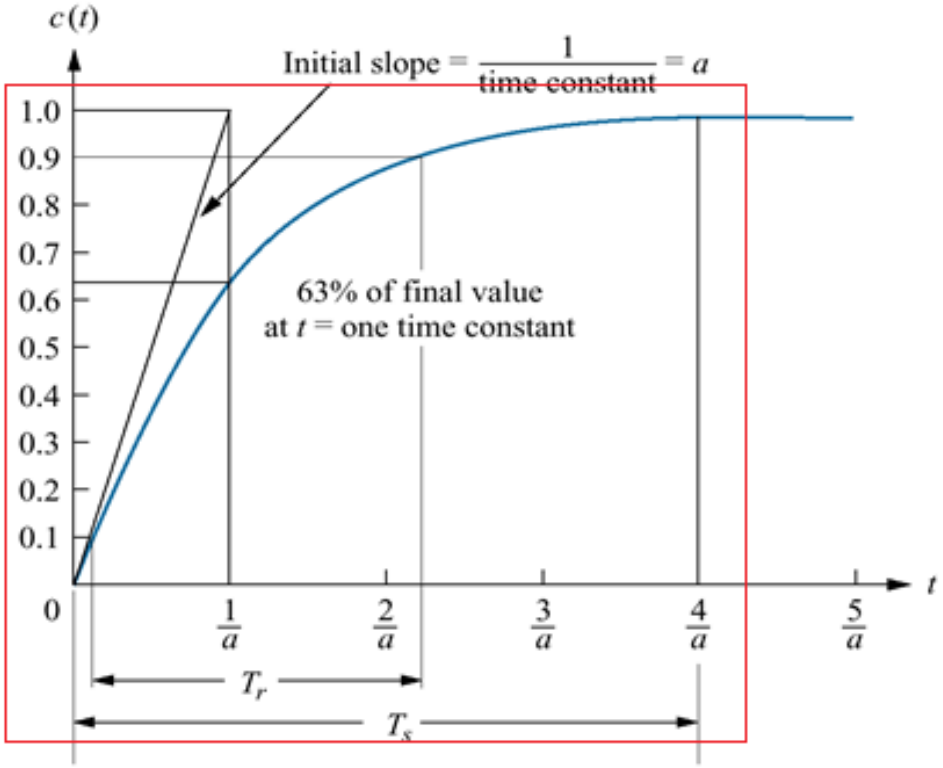
$$T_r = \ln 0.9 - \ln 0.1 = \frac{2.3}{a} - \frac{0.1}{a} = \frac{2.2}{a} \quad \text{or} \quad 2.2\tau$$



Settling Time

- Settling time (T_s) is defined as the time for the response to reach, and stay within, 2% of its final value (e.g. 2% is standard).

$$T_s = \ln 1 - \ln 0.02 = \frac{3.9}{a} \quad \text{or} \quad 3.9\tau$$



Example of First Order System Analysis

You are given the following first-order systems:

$$G(\omega) = \frac{5}{j\omega + 5}$$

a. Calculate its time constant, rise time, and settling time. [6 marks]

b. Simulate the transient response of the system in MATLAB when it is given a step input. [4 marks]

Example of First Order System Analysis

a. For the given system:

$$G(\omega) = \frac{5}{(j\omega + 5)} = \frac{1}{j\omega/5 + 1}$$

Time constant:

$$\tau = \frac{1}{5} = 0.2 \text{ second}$$

Rise time:

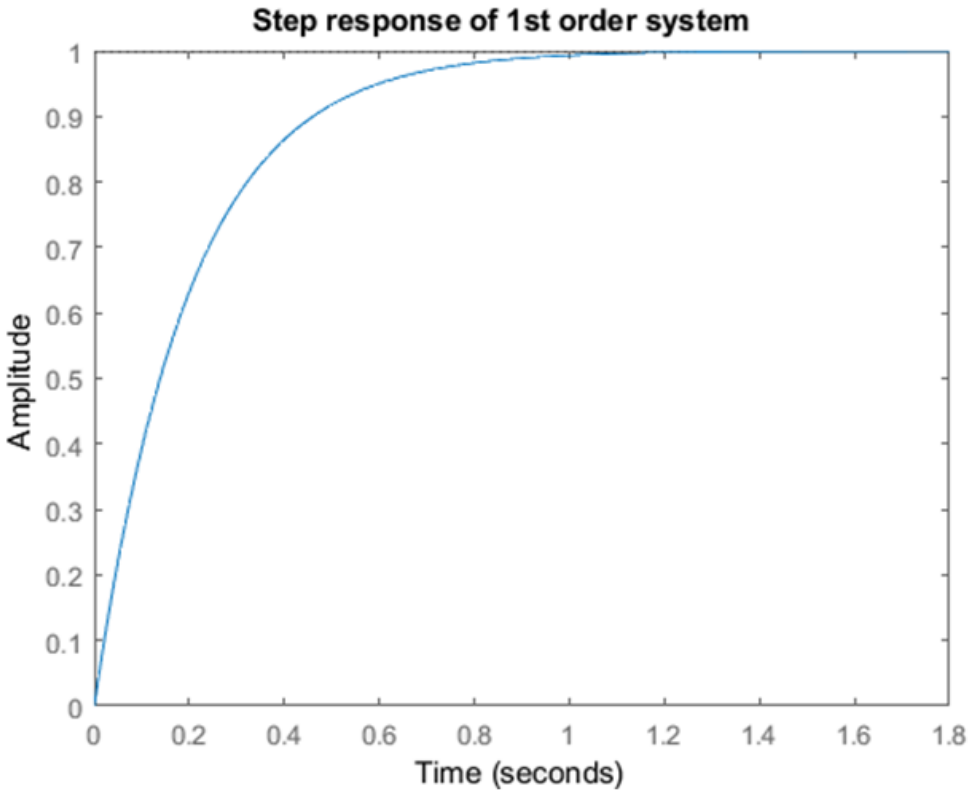
$$T_r = 2.2\tau = \frac{2.2}{a} = \frac{2.2}{5} = 0.44 \text{ second}$$

Settling time:

$$T_s = 4\tau = \frac{4}{a} = \frac{4}{5} = 0.8 \text{ second}$$

Example of First Order System Analysis

b. The transient response of the system is as shown below.



From the plot, it seems the system has τ = 0.22 second, T_r = 0.44 second and T_s = 0.8 second.

Second-Order System

- Second-order system is a type of system whose input-output relationship is a second-order differential equation.
- A typical standardized linear constant-coefficient differential equation for a second-order system is:

$$\frac{d^2y(t)}{dt^2} + (2\zeta\omega_n)\frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

- The second-order system given above is typically represented with a standardized transfer function equation:

$$\frac{Y(\omega)}{X(\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

Second-Order Standardised Form

- Given a standardised form of second-order equation:

$$G(\omega) = \frac{(\omega_n)^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + (\omega_n)^2}$$

- Normalising the equation given above:

$$G(\omega) = \frac{1}{1 + \left(\frac{2\zeta}{\omega_n}\right)j\omega + \left(\frac{j\omega}{\omega_n}\right)^2}$$

- The characteristic equation of the equation given above is the equation in the denominator of the equation:

$$1 + \left(\frac{2\zeta}{\omega_n}\right)j\omega + \left(\frac{j\omega}{\omega_n}\right)^2$$

Second-Order Standardised Form

- The standard equation for second-order equation is:

$$\text{root}_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Referring to the characteristic equation of the transfer function equation above, roots of the equation are:

$$\text{root}_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\frac{2\zeta}{\omega_n} \pm \sqrt{\left(\frac{2\zeta}{\omega_n}\right)^2 - \frac{4}{\omega_n^2}}}{2\left(\frac{1}{\omega_n}\right)^2}$$

- Thus, the roots of equation are:

$$\text{root}_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - (\zeta)^2} = \sigma \pm j\omega$$

Second-Order Time Response

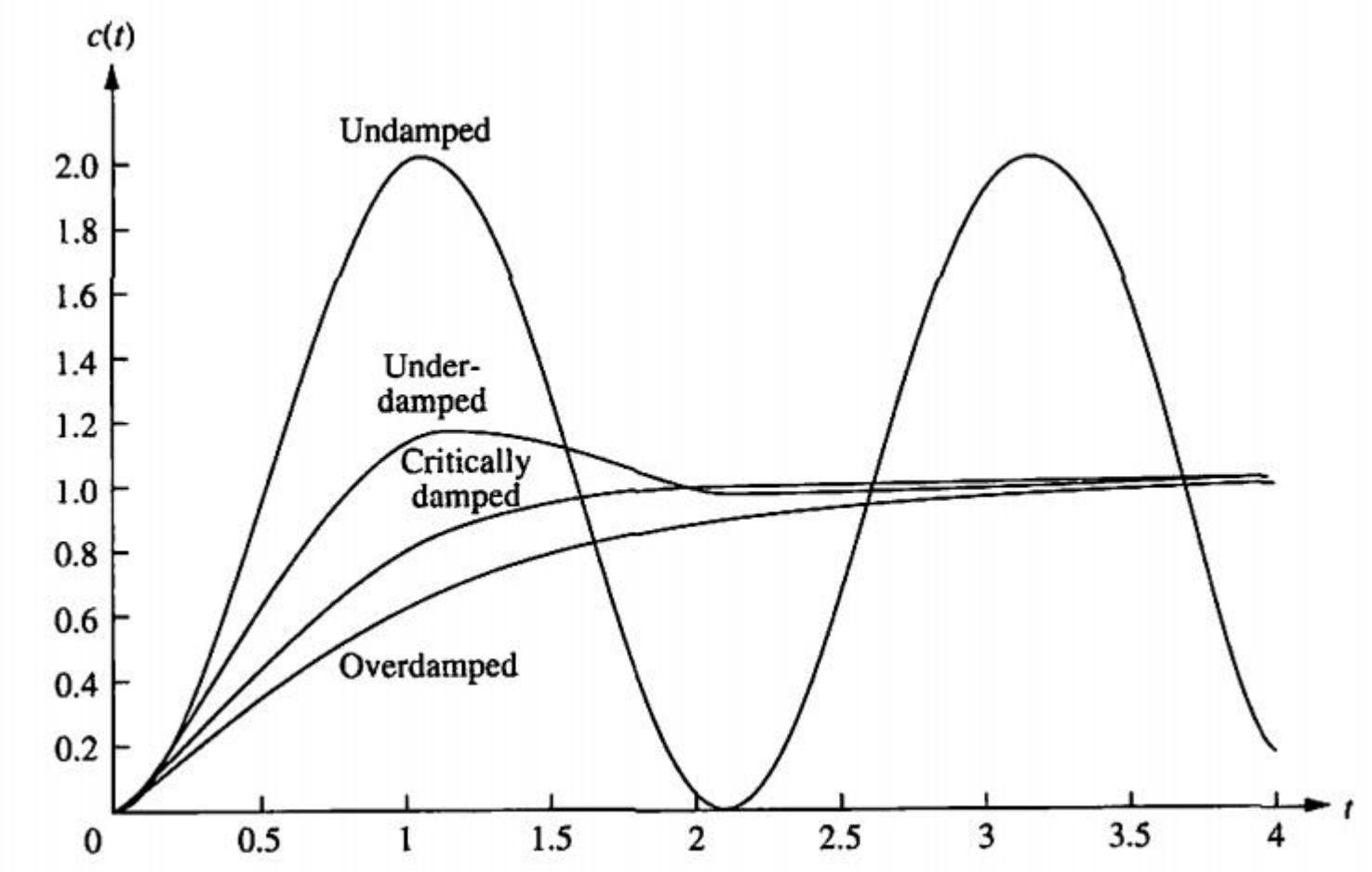
Roots	Time Response	Type of Response
$-\sigma_1, -\sigma_2$	$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$	Overdamped
$-\sigma_d$	$c(t) = K_1 e^{-\sigma_d t} + K_2 t e^{-\sigma_d t}$	Critically damped
$-\sigma_d \pm j\omega_d$	$c(t) = A\, e^{-\sigma_d t}\cos(\omega_d t - \phi)$	Underdamped
$\pm j\omega_d$	$c(t) = A\cos(\omega_d t - \phi)$	Undamped

Note: σ_d is repeated roots and $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

- Overdamped – heavily damped transient response.
- Critically damped – critically damped transient response.
- Underdamped – lightly damped transient response.
- Undamped – no damping, constant amplitude oscillation transient response.

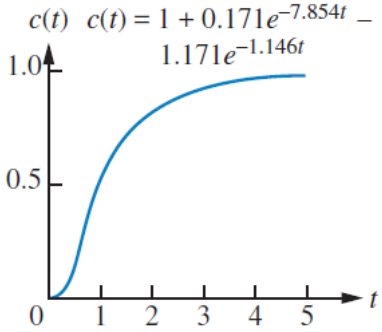
Roots of Equation and Time Response

- The graph below shows the various transient responses of the second-order system based on roots of equation.

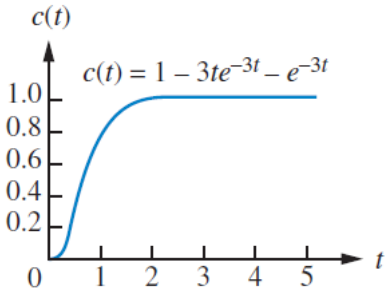


Second-Order Time Response

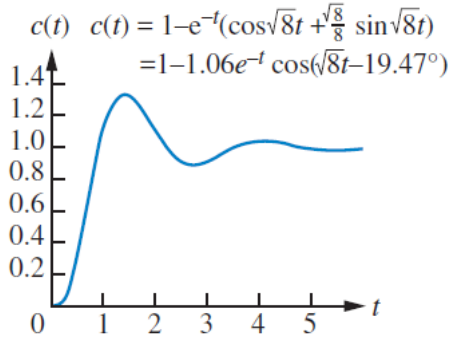
- The graphs below show a number of the transient responses of several second-order systems.



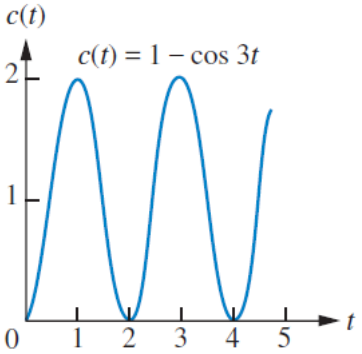
Overdamped response



Critically damped response



Underdamped response



Undamped response

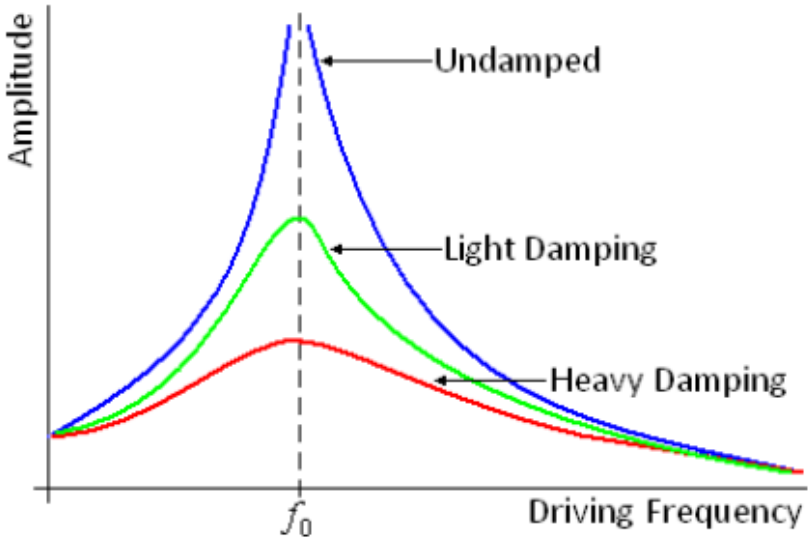
Natural Frequency

- Natural frequency (ω_n) is the frequency when there is no damping in the system (a = 0 in this case).

- We obtain:

$$\frac{C(\omega)}{R(\omega)} = \frac{b}{(j\omega)^2 + b}$$

with $\text{roots}_{1,2} = \pm j\sqrt{b}$



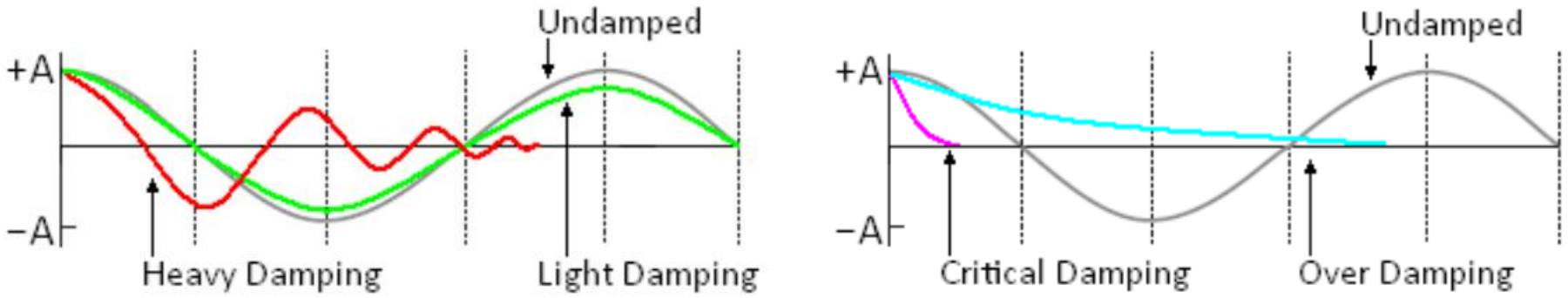
- We know that $\omega = \sqrt{b} = \omega_n$ as a complex number.
- So, the frequency of oscillation $\omega_n = \sigma + j\omega$, which is termed the natural frequency.

$$y(t) = y_{ss} + (y_0 - y_{ss})e^{-at}\cos(\omega_n t)$$

Natural Frequency

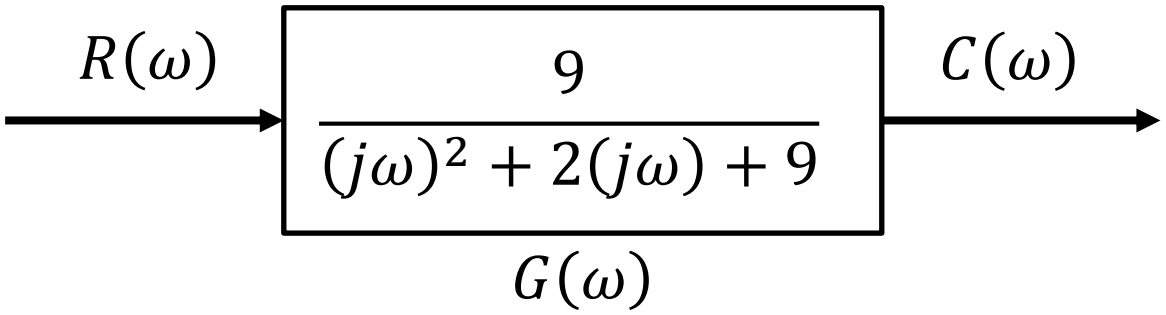
Furthermore, for a given second-order system, the natural response and forced response are related through the damping ratio as follow:

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency}} = \frac{\sigma}{\omega_n}$$



Example of Second Order Time Response

Given a second-order system below.



Determine:

a. Roots of the characteristic equation of the system. [4 marks]

b. Sketch the transient response of the system. [4 marks]

Example of Second Order Time Response

Solution

a. Second-order step response generated by complex roots:

$$\frac{C(\omega)}{R(\omega)} = \frac{9}{(j\omega)^2 + 2(j\omega) + 9}$$

Consider the characteristic equation of to give roots:

$$\text{roots}_{1,2} = -1 \pm j\sqrt{8}$$

- The real part generates: e^{-t}
- The complex part generates: $k\cos\left(\sqrt{8}t - \phi\right)$.

Where: t = time, k = amplitude, and ϕ = phase shift.

Example of Second Order Time Response

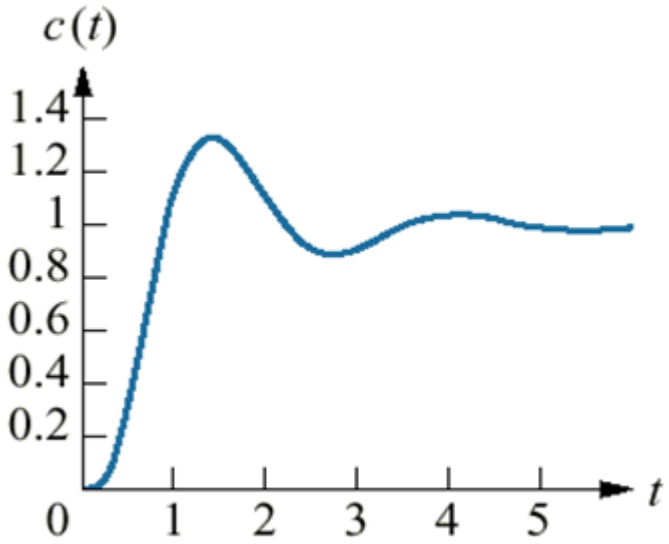
The response is a decaying sinusoidal oscillation given by:

$$c(t) = 1 - e^{-t}\left(\cos\sqrt{8}t + \frac{\sqrt{8}}{8}\sin\sqrt{8}t\right)$$

$$= 1 - 1.06e^{-t}\cos\left(\sqrt{8}t - -19.47°\right)$$

b. The transient response of the system is shown below.

The response of the system is found to be lightly damped (underdamped) response.



Normalise Transfer Function

- Consider a second order system:

$$\frac{C(\omega)}{R(\omega)} = \frac{k}{a'j\omega^2 + b'j\omega + c'}$$

- For determining damping of the second order system, it is important to normalise the equation above into a unity coefficient form in front of the $(j\omega)^2$ term and k such that:

$$\frac{C(\omega)}{R(\omega)} = \frac{b}{(j\omega)^2 + a(j\omega) + b}$$

- Note that the constants a`, $b$`, c` are not equivalent to a, b.

Example of Normalise Transfer Function

Given a second order system with following transfer function equation:

$$G(\omega) = \frac{10}{3j\omega^2 + 2j\omega + 5}$$

a. Normalise the transfer function equation. [2 marks]

b. Determine the gain at DC, natural frequency ω_n, and damping ratio ζ of the system. [6 marks]

Example of Normalise Transfer Function

Solution

a. The normalised transfer function equation is:

$$G(\omega) = \left(\frac{10}{5}\right)\left[\frac{\left(\frac{\sqrt{5}}{\sqrt{3}}\right)^2}{(j\omega)^2 + \left(\frac{2}{3}\right)(j\omega) + \left(\frac{\sqrt{5}}{\sqrt{3}}\right)^2}\right]$$

b. Knowing the normalised transfer function, we can obtain the gain of the system at DC ($\omega = 0$) that is 10/5, natural frequency (ω_n) is $\sqrt{5/3}$, and damping ratio (ζ) is:

$$2\zeta\omega_n = \frac{2}{3}$$

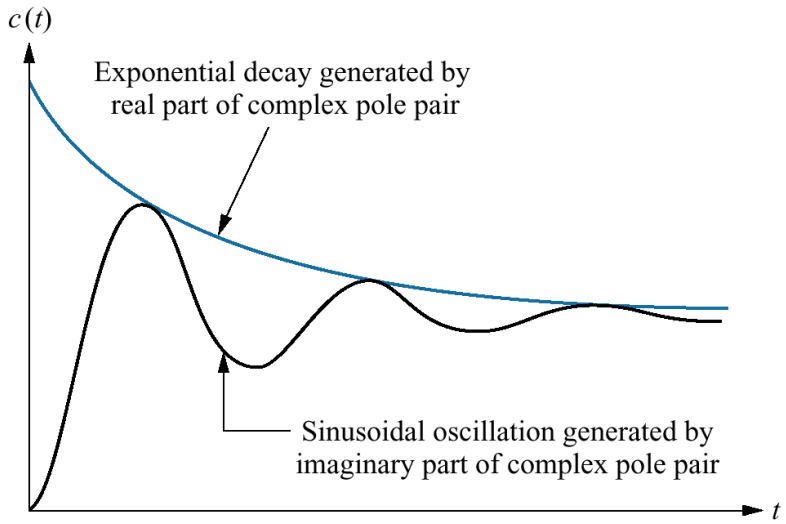
With $\omega_n = \sqrt{5/3}$, thus $\zeta = 1/3\sqrt{3}$.

Exponential Decay Frequency

- Exponential decay frequency determines how fast the oscillation in underdamped response to die down.
- Considering an underdamped second order system

$$\frac{C(\omega)}{R(\omega)} = \frac{b}{(j\omega)^2 + a(j\omega) + b}$$

- The real part of root = $\sigma + j\omega$, is $\sigma = -a/2$, where $|\sigma|$ is termed the exponential decay frequency.



Damping Ratio

- The damping ratio is defined as a measure describing how oscillations in a system decay after a disturbance.
- It is equated as the ratio of exponential decay frequency with natural frequency.

$$\zeta = \frac{\left(\frac{a}{2}\right)}{b} = \frac{|\sigma|}{\omega_n}$$

Where:

- $|\sigma|$is the exponential decay frequency
- ω_n is the natural frequency

Damping Ratio

- Given equation for the standardized second-order system with damping ratio and natural frequency:

$$\frac{C(\omega)}{R(\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

- Consider the roots of the characteristic equation given above:

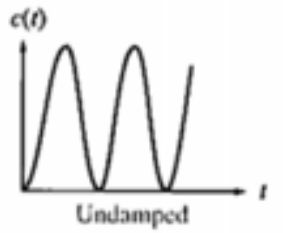
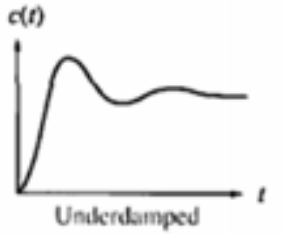
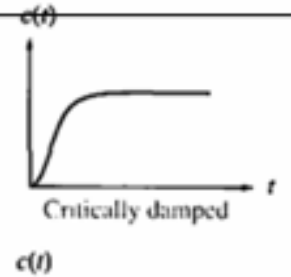
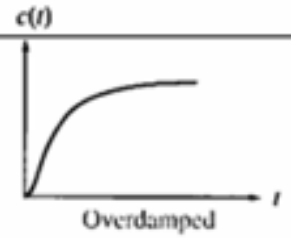
$$\text{roots}_{1,2} = \frac{-2\zeta\omega_n \pm 2\omega_n\sqrt{\zeta^2 - 1}}{2}$$

- So, the roots of the equation are:

$$\text{roots}_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = \sigma \pm j\omega$$

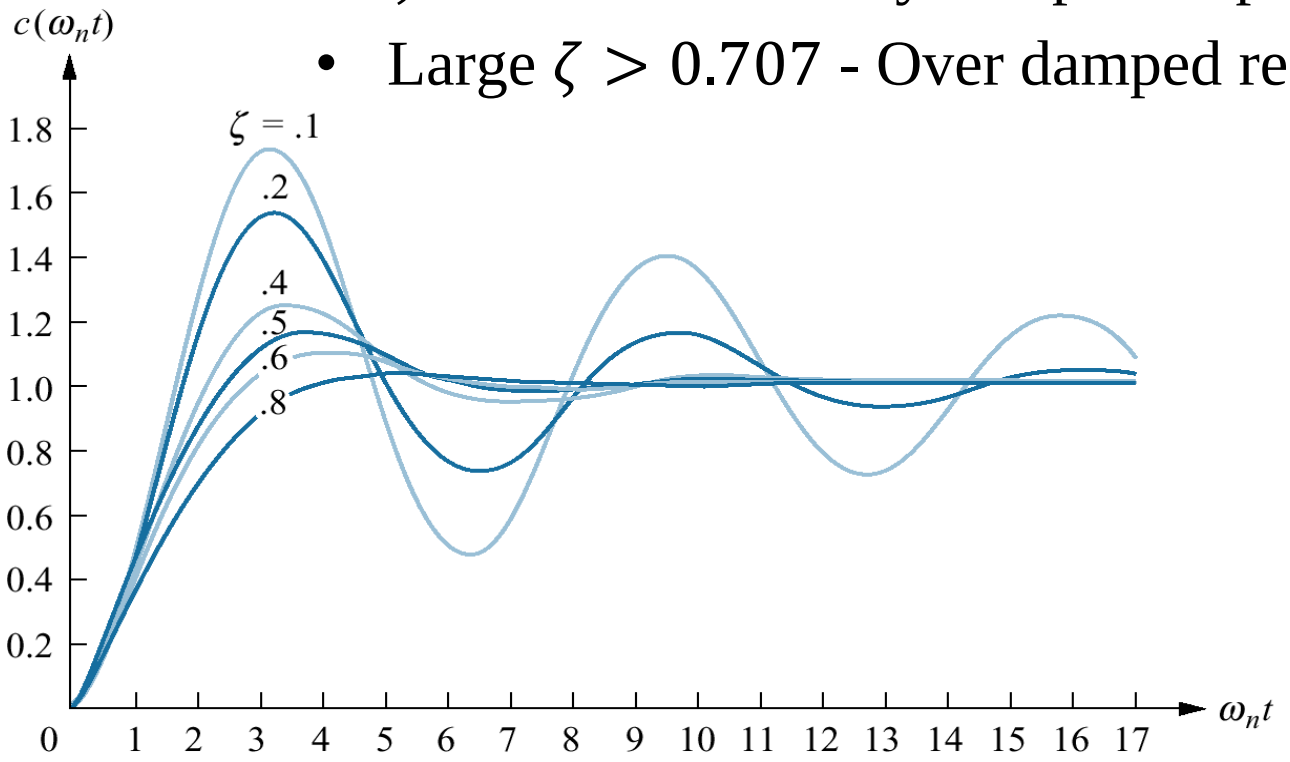
- Note: σ_d is repeated roots and $\omega_d = \omega_n\sqrt{1 - \zeta^2}$.

Damping Ratio

ζ	Roots	Step Response	Description
0	$\pm j\omega_d$		Constant amplitude oscillation.
$0 < \zeta < 1$	$-\sigma_d \pm j\omega_d$		Initial damped oscillation before settling down to steady state value.
$\zeta = 1$	$-\sigma_d$		Damped oscillation with value of damping ratio is equal to 1.
$\zeta > 1$	$-\sigma_1, -\sigma_2$		Damped oscillation that settles down to steady state value.

Damping Ratio and Time Response

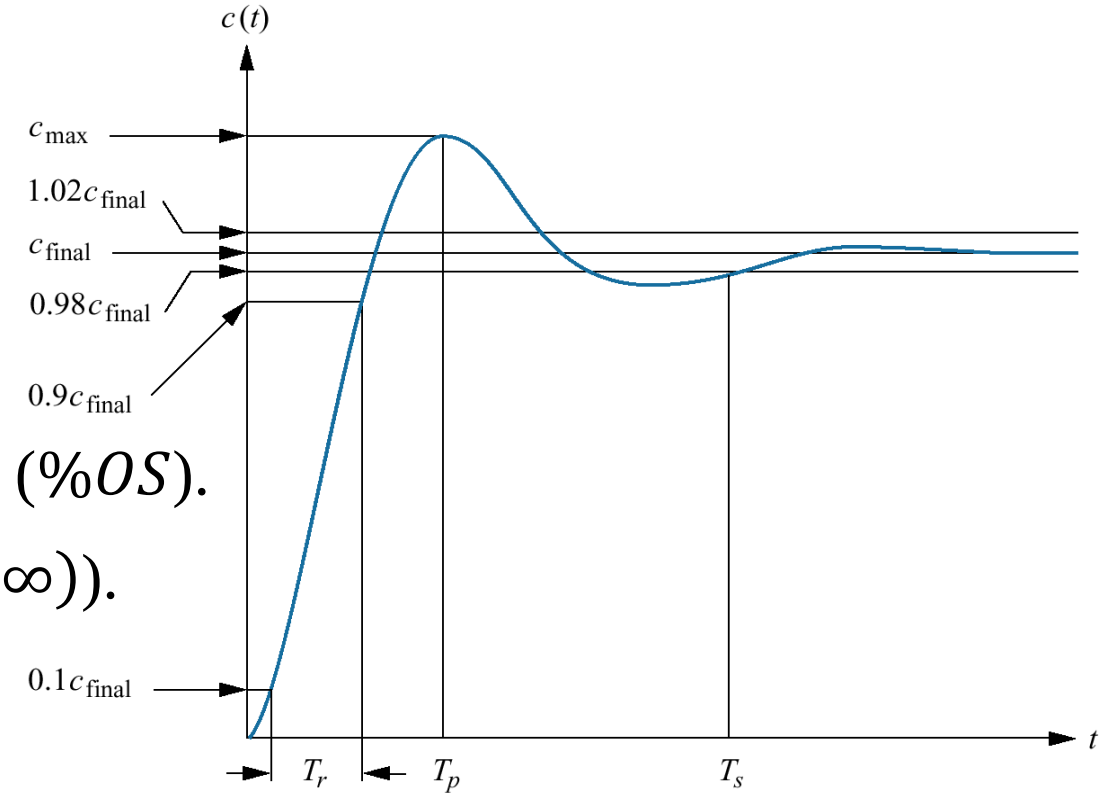
- Second-order underdamped responses for various damping ratio values.
  - Small $\zeta < 0.707$ - Under damped response.
  - $\zeta = 0.707$ - Critically damped response.
  - Large $\zeta > 0.707$ - Over damped response.



Second Order Time Response

Second-order underdamped response specifications.

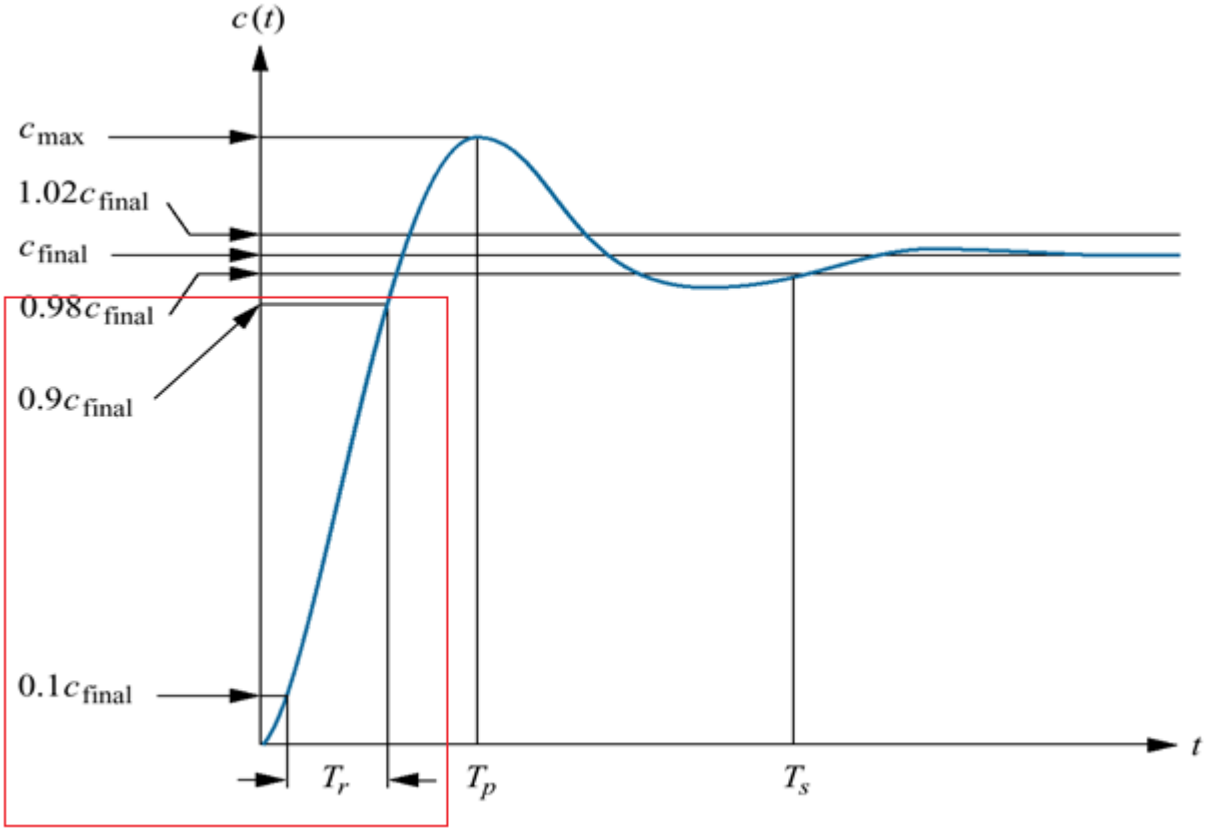
- We can not use equations for first order system – less accurate.
- We still would like to consider:
  - Rise time (T_r).
  - Settling time (T_s).
- But also:
  - Time-to-peak (T_p).
  - Percentage overshoot ($\%OS$).
  - Steady-state error ($e(\infty)$).



Rise Time

- Rise time (T_r) is defined as the time for the response to go from 0.1 to 0.9 of its final value.

$$T_r = \ln 0.9 - \ln 0.1 = \frac{2.3}{a} - \frac{0.1}{a} = \frac{2.2}{a} \quad \text{or } 2.2\tau$$



Rise Time

- The rise time equation for first order system is not accurate for second order system (especially underdamped system).
- As a result, the rise time (T_r) for a second order underdamped response is determined from.

$$T_r = \frac{(1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)}{\omega_n}$$

- The other rise time (T_r) for a second order underdamped response.

$$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}}$$

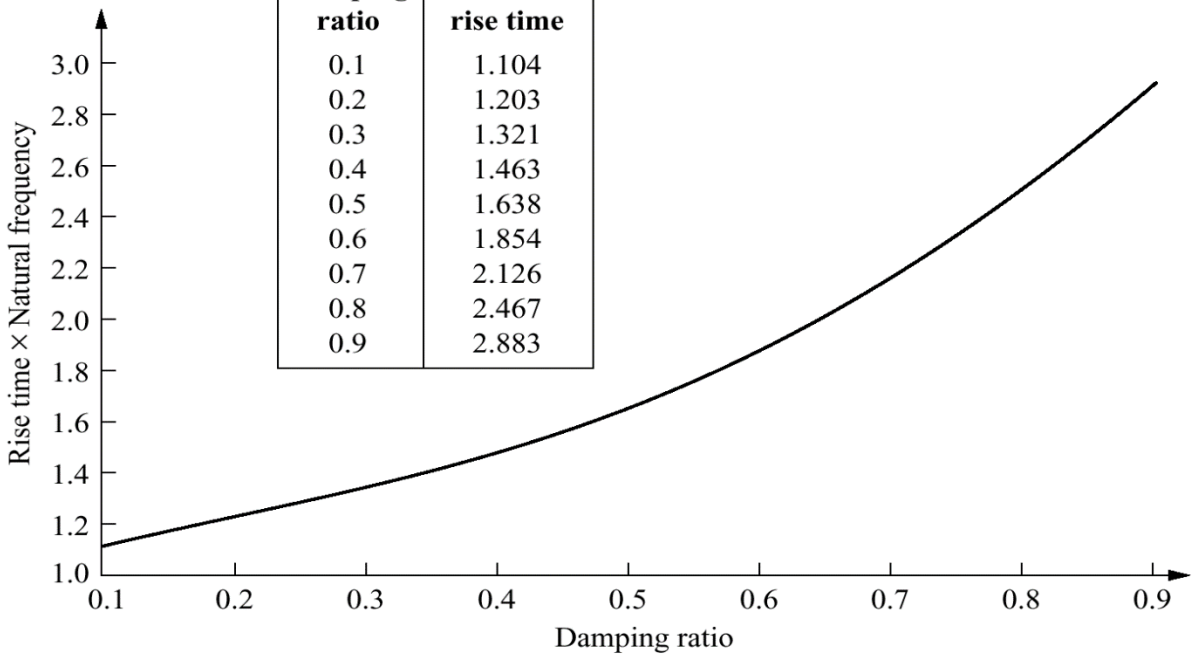
Where: $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

Rise Time

- Rise time can be determined graphically as well.
- The graph below is the normalised rise time ($T_r \times \omega_n$) vs. damping ratio (ζ) for a second order underdamped response.

$$t_r\omega_0 = 2.230\zeta^2 - 0.078\zeta + 1.12$$

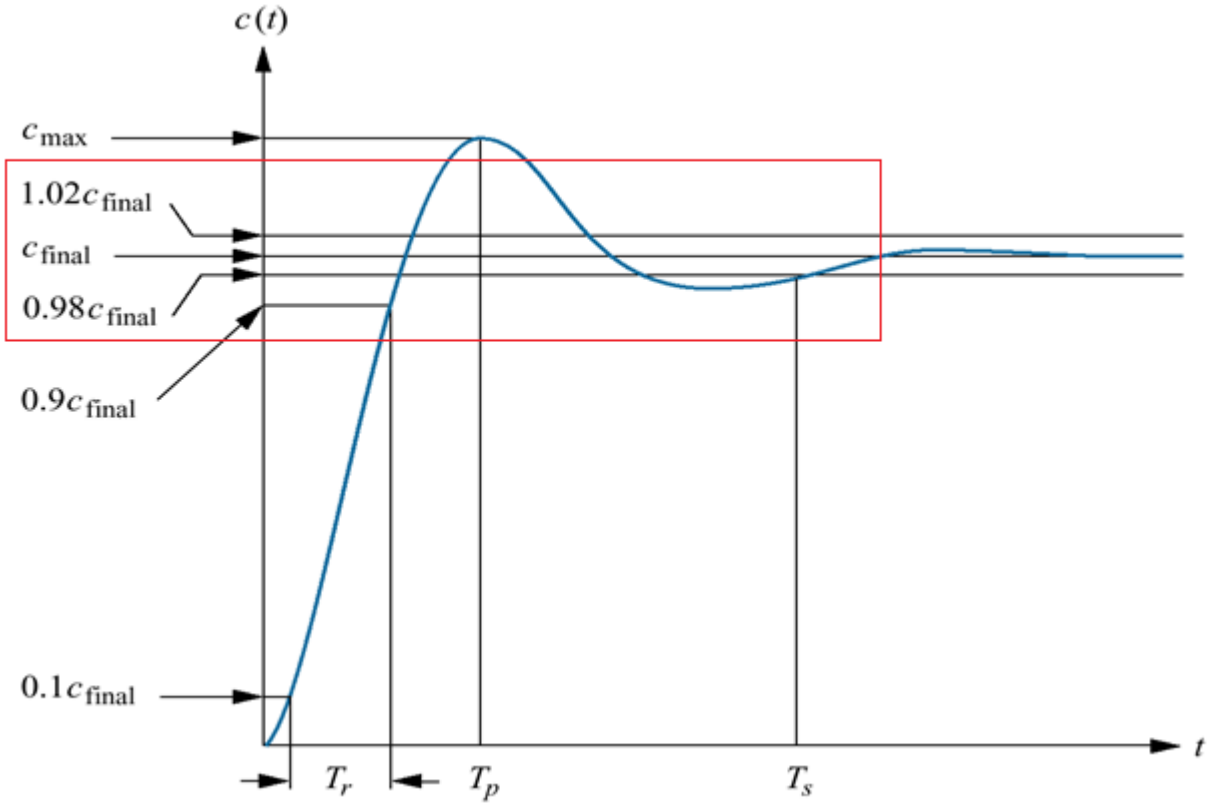
Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883



Settling Time

- Settling time (T_s) is defined as the time for the response to reach, and stay within, 2 % of its final value (e.g. 2 % is standard). This is just like in the first order response.

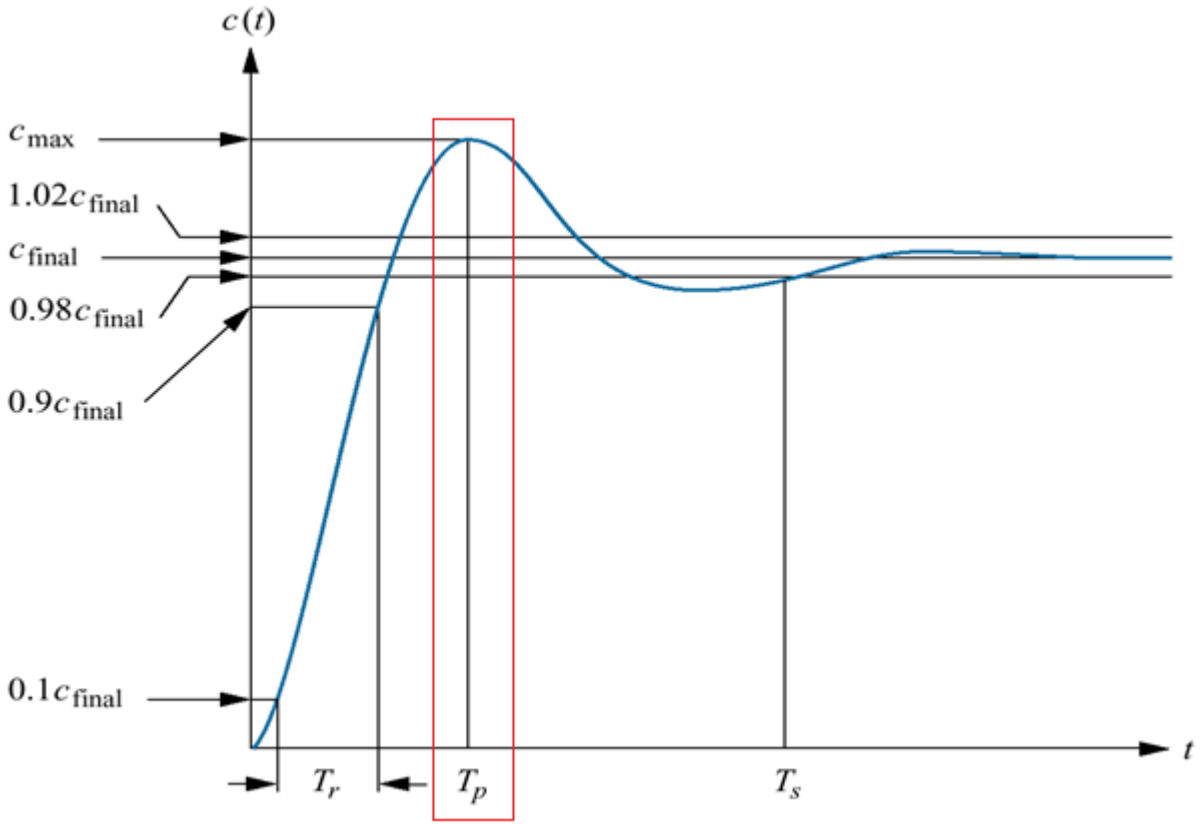
$$T_s = \ln 1 - \ln 0.02 = \frac{3.9}{a} \text{ or } 3.9\tau$$



Time-to-Peak

- Time-to-peak (T_p), it is the time required to reach the first, or maximum, peak.

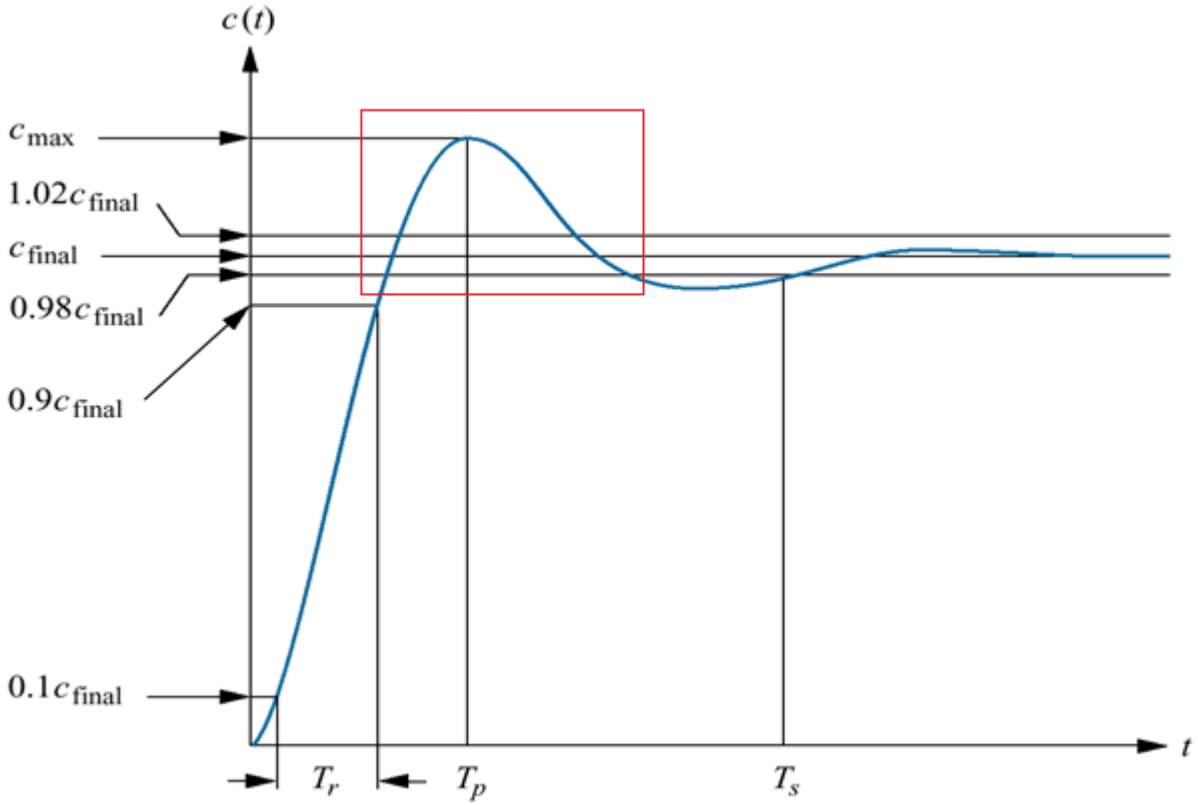
$$T_p = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} \qquad \text{where: } n = \text{nth peak}$$



Percentage Overshoot

- Percentage overshoot ($\%OS$) is the amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.

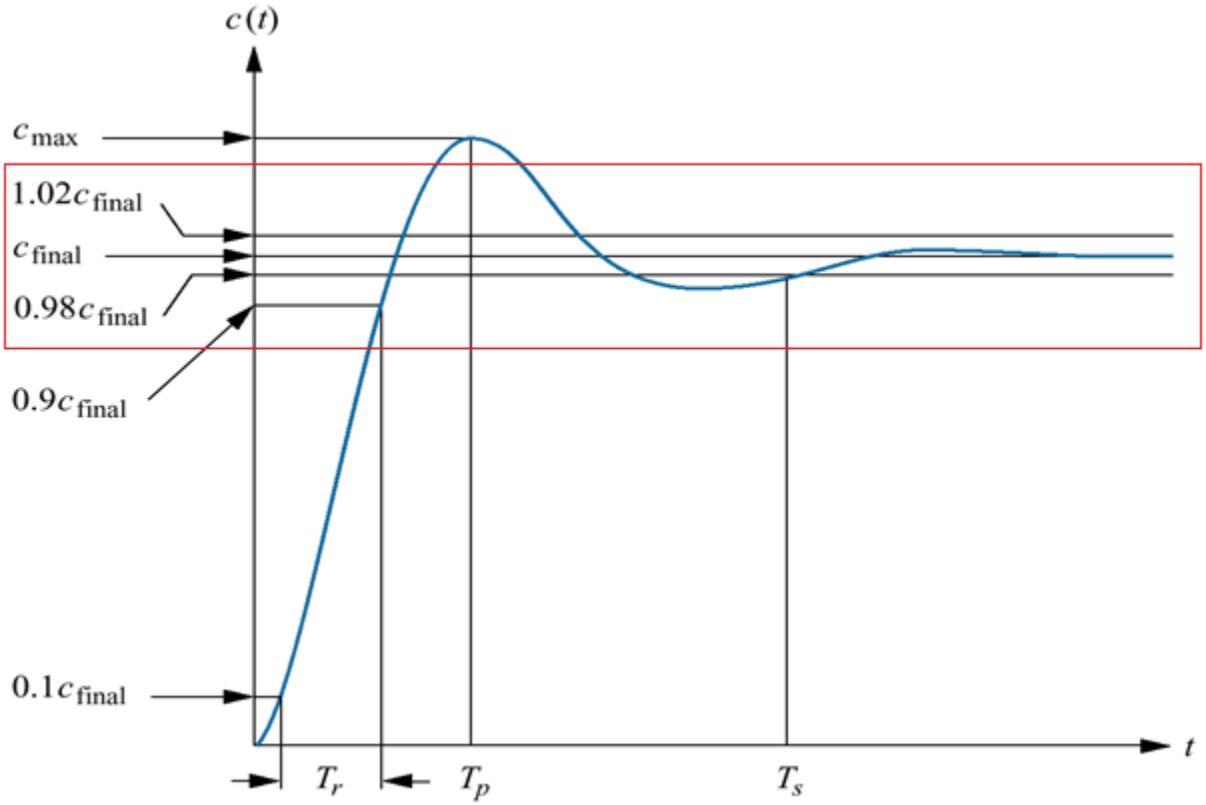
$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100\%$$



Steady-State Error

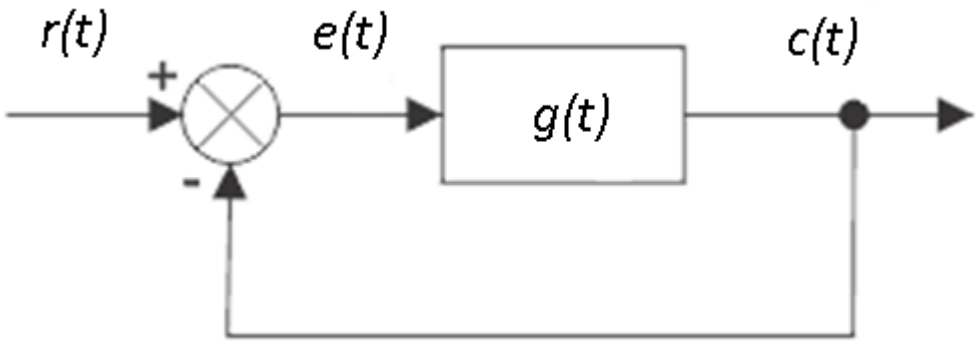
- Steady-state error ($e(\infty)$) is the difference between the input and intended output for a prescribed test input.

$$e(\infty) = \lim_{t \to \infty} [r(t) - c(t)]$$



Steady-State Error

- Steady-state error of a given a system with a feedback is determined from:



Where:

- $g(t)$ = system
- $r(t)$ = reference signal
- $e(t)$ = error signal
- $c(t)$ = output signal

Steady-State Output

- The equations from the system with feedback determined from:

$$e(t) = r(t) - c(t) \qquad (Eq.1)$$

And

$$c(t) = g(t)e(t) \qquad (Eq.2)$$

- Substituting $e(t)$ in equation (2) with equation (1), the steady-state output of the system is:

$$c(t) = g(t)[r(t) - c(t)]$$

- Rearrange the equation above for $c(t)$, it is now:

$$c(t) = \frac{g(t)r(t)}{[1 + g(t)]}$$

Steady-State Error

- Applying the final value theorem, when t approach infinity for steady-state condition, i.e. ω is approaching zero:

$$C(\omega) = \lim_{j\omega \to 0} \left[\frac{j\omega R(\omega) G(\omega)}{1 + G(\omega)} \right]$$

- Substituting $c(t)$ in equations (1) with (2), the steady-state error of the system is:

$$e(t) = \frac{r(t)}{[1 + g(t)]}$$

- Applying the final value theorem, the steady-state error is:

$$E(\omega) = \lim_{j\omega \to 0} \left[\frac{j\omega R(\omega)}{1 + G(\omega)} \right]$$

Example of Second-Order System Response

You are given the following second-order systems:

$$G(\omega) = \frac{16}{(j\omega)^2 + 3j\omega + 16}$$

a. Determine the transient responses of the systems based on the roots of equations. [6 marks]

b. Simulate the system in MATLAB. [4 marks]

c. Find the time response characteristics of the system in terms of damping ratio (ζ) and natural frequency (ω_n). [4 marks]

d. Determine the settling time (T_s), peak time (T_p), rise time (T_r), percentage overshoot ($\%OS$), and steady-state output and error. [12 marks]

Example of Second-Order System Response

Solution

a. For the given system its following transfer function is:

$$G(\omega) = \frac{16}{(j\omega)^2 + 3j\omega + 16}$$

Characteristic equation of the system is $(j\omega)^2 + 3j\omega + 16$.

Roots of the characteristic equation are:

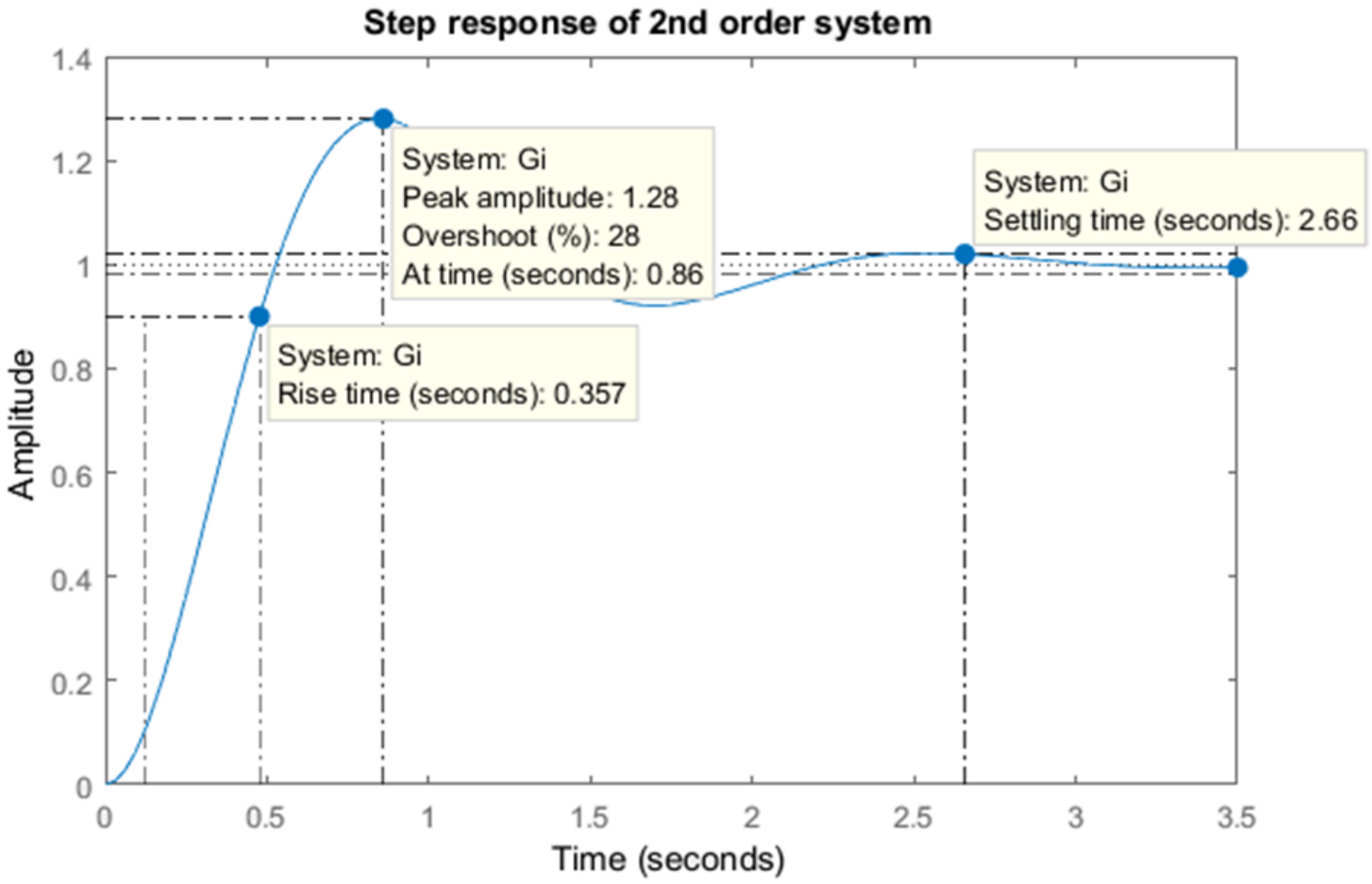
$$\text{root}_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{-3 \pm \sqrt{(3)^2 - 4(1)(16)}}{(2)(1)}$$

$$= -1.5 \pm j\frac{\sqrt{55}}{2}$$

Since the roots are a pair of complex numbers, the transient response of the system is underdamped response.

Example of Second-Order System Response

b. The result of simulation of the time response of the system in MATLAB is shown in the figure below.



Example of Second-Order System Response

c. Transfer function equation of the system is:

$$G(\omega) = \frac{16}{(j\omega)^2 + 3j\omega + 16}$$

Natural frequency is calculated from:

$$\omega_n^2 = 16 \text{ rad/sec}$$

Thus

$$\omega_n = 4$$

Damping ratio is calculate from:

$$2\zeta\omega_n = 3$$

Therefore

$$\zeta = \frac{3}{2\omega_n} = \frac{3}{2(4)} = 0.375$$

Example of Second-Order System Response

d. The peak time (first peak, n = 1) is:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{4\sqrt{1-(0.375)^2}} = 0.8472 \text{ second}$$

The percentage overshoot is:

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

$$= e^{-(0.375)\pi/\sqrt{1-(0.375)^2}} = 28.06\ \%$$

The steady-state output of the system is calculated from:

$$C(\omega) = \lim_{\omega \to 0} j\omega R(\omega) G(\omega)$$

Example of Second-Order System Response

For a step input of $R(\omega) = 1/j\omega$, it is:

$$C(\omega) = \lim_{\omega \to 0} R(\omega)G(\omega) = \lim_{\omega \to 0} (j\omega)\left(\frac{1}{j\omega}\right)\frac{16}{(j\omega)^2 + 3j\omega + 16} = 1$$

The steady-state error of the system is calculated from:

$$e(\omega) = \lim_{j\omega \to 0}\left[\frac{j\omega R(\omega)}{1 + G(\omega)}\right]$$

For a step input of $1/j\omega$, put the values into the equation, the steady-state error of the system is:

$$e(\omega) = \lim_{j\omega \to 0}\left[\frac{(j\omega)\left(\frac{1}{j\omega}\right)}{1 + \frac{16}{(j\omega)^2 + 3j\omega + 16}}\right] = 0$$