

Time & Frequency Characteristics (Frequency Response Analysis)

XMUT220 Signals and Systems

Topics

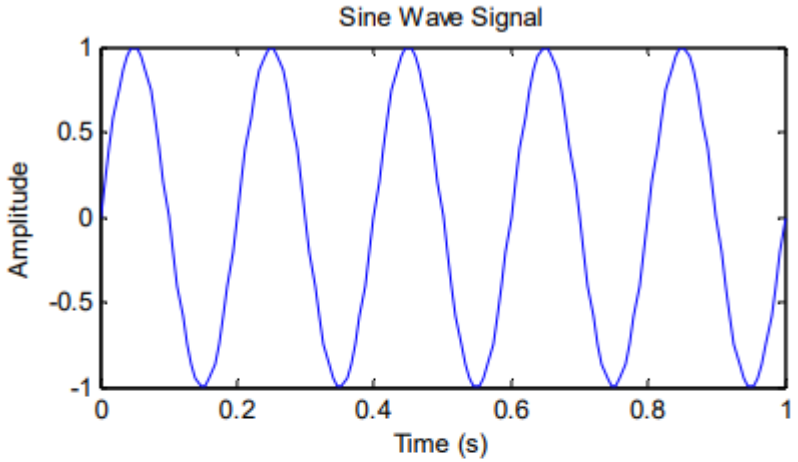
- Introduction to frequency response.
- Frequency response of continuous first order systems.
- Frequency response of discrete first order systems.
- Frequency response of continuous second order systems.
- Frequency response of discrete second order systems.

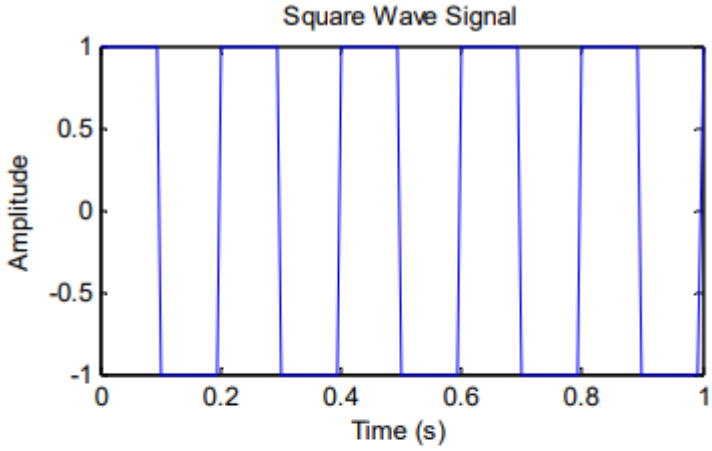
Introduction to Frequency Response

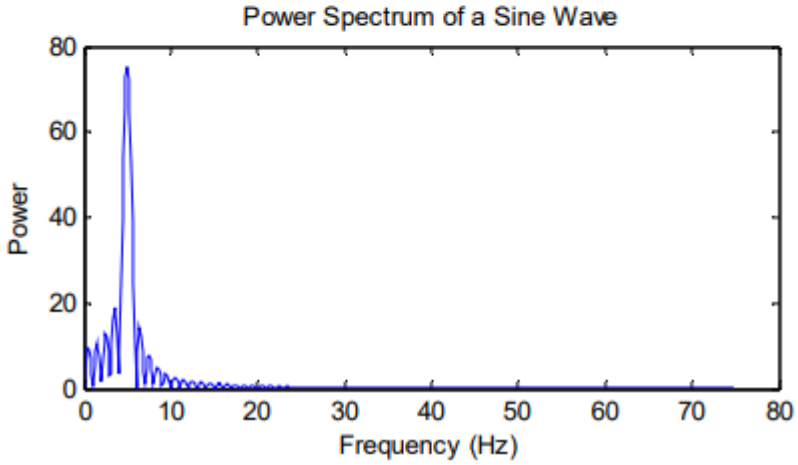
- Frequency response is the quantitative measure of the output of a system or device in response to a stimulus and is used to characterise the dynamics of the system in the frequency domain.
- It is a measure of magnitude and phase of the output as a function of frequency, in comparison to the input.
  - Frequency spectrum plot – magnitude (and phase) plot(s) over range of frequencies (note: we look its application in sampling and reconstruction).
  - Frequency response plot – magnitude and phase plots over a range of frequencies.

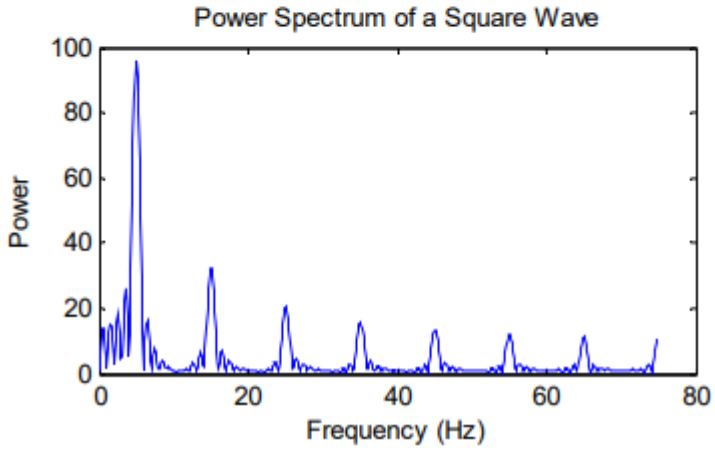
Introduction to Frequency Response

- Transient response and power spectrum plots of sine wave (left) and square wave (right).



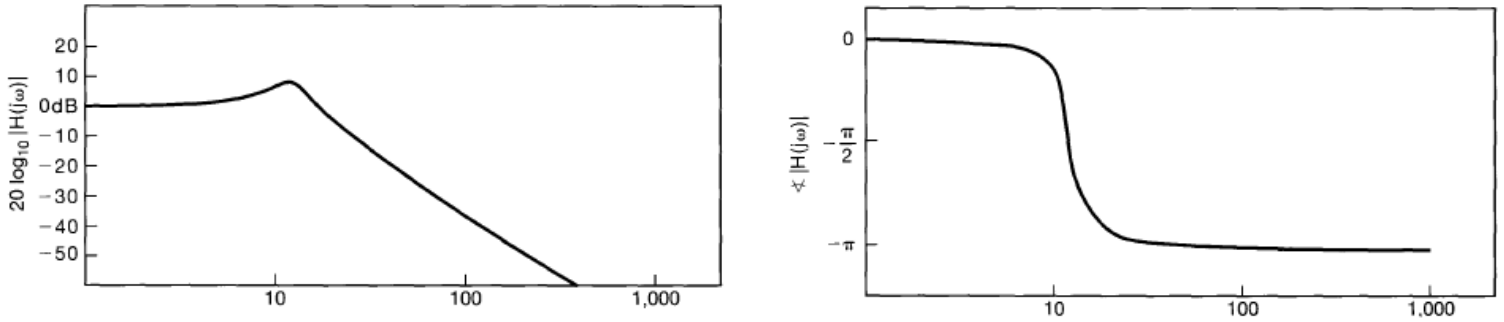




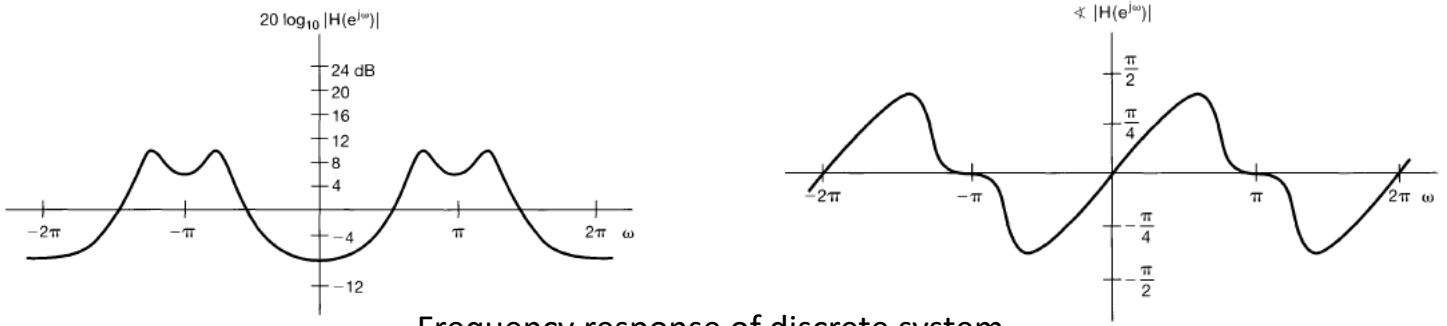


Introduction to Frequency Response

- The frequency response plot is characterized by the magnitude and the phase of the system's response.
- Magnitude is typically measured in decibels (dB) or as a decimal, and the phase is measured in radians or degrees, versus frequency in radians/sec or Hertz (Hz).



Frequency response of continuous system

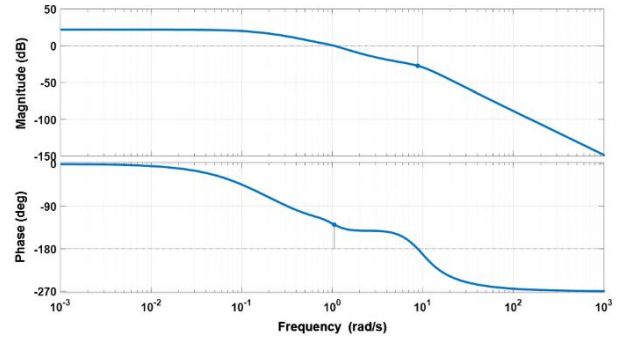


Frequency response of discrete system

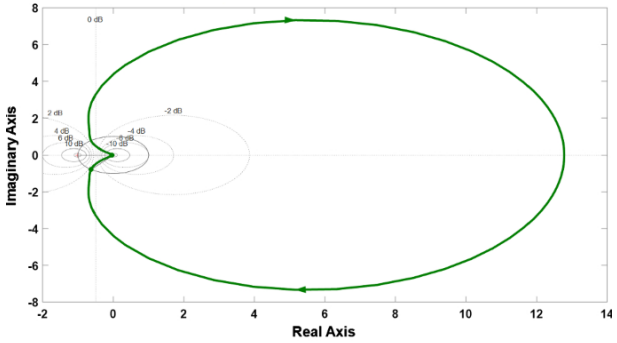
Introduction to Frequency Response

These frequency response analysis can be done through three different approaches:

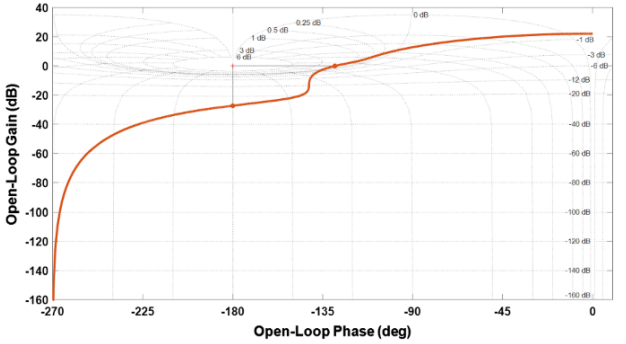
- Bode plot - magnitude and phase measurements on two rectangular plots as functions of frequency.
- Nyquist diagram - magnitude and phase angle on a single polar plot with frequency as a parameter.
- Nichols chart - magnitude and phase on a single rectangular plot with frequency as a parameter.



Bode Plot



Nyquist Diagram



Nichols Chart

Continuous 1st Order System

- The differential equation for a first-order system is often expressed in the form:

$$\tau\left[\frac{dy(t)}{dt}\right] + y(t) = x(t)$$

Where: the τ is the time constant of the system -> controls the rate at which the first-order system responds.

- The frequency response for the given first-order system is:

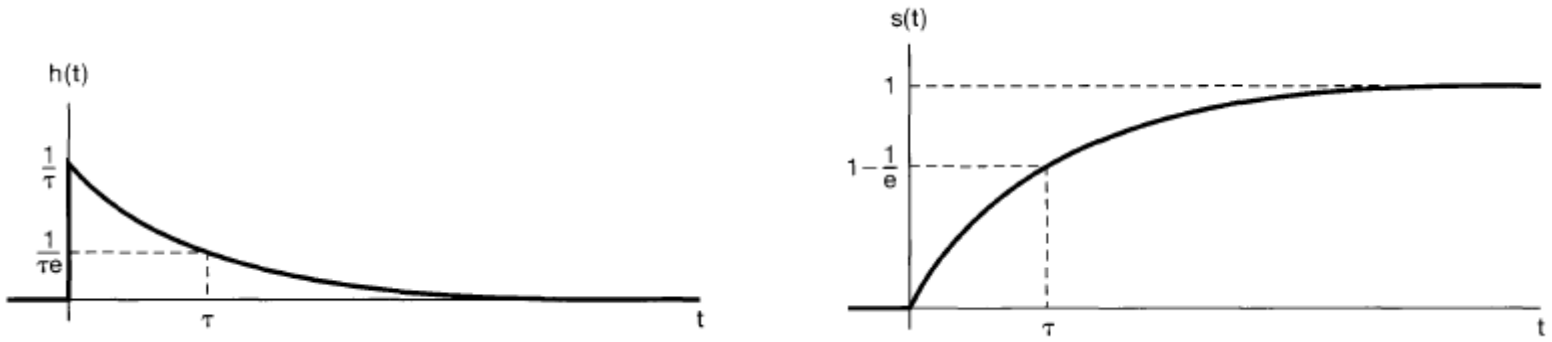
$$H(\omega) = \frac{1}{j\omega\tau + 1}$$

- Impulse response is (left) and step response is (right):

$$h(t) = \left(\frac{1}{\tau}\right)e^{\left(-\frac{t}{\tau}\right)}u(t) \qquad s(t) = h(t) * u(t) = \left[1 - e^{-\frac{t}{\tau}}\right]u(t)$$

Transient Response of Continuous 1st Order System

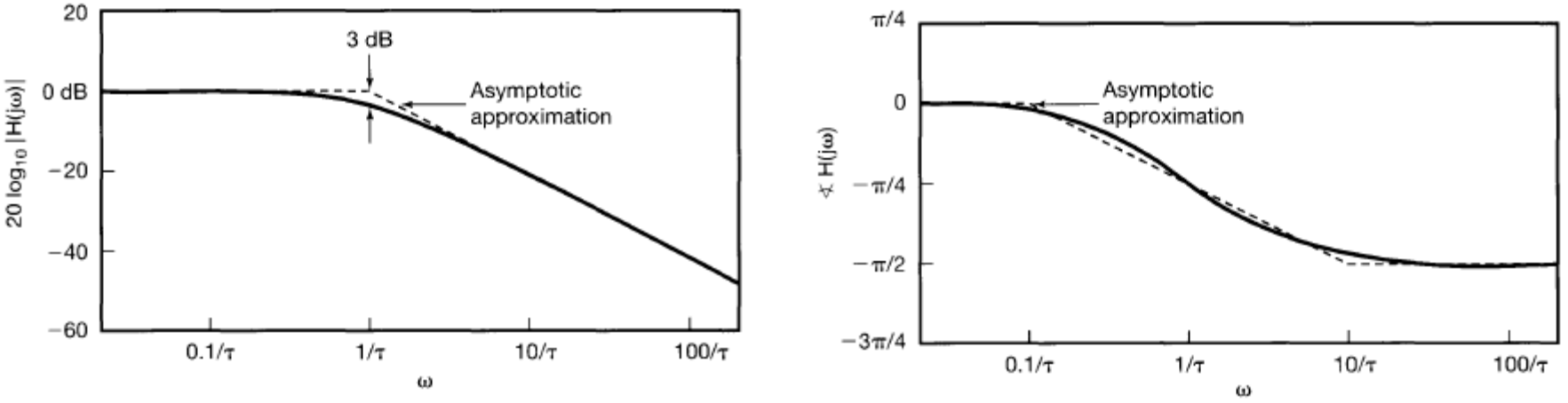
- At $t = \tau$ the impulse response has reached $1/e$ times its value at $t = 0$, and the step response is within $1/e$ of its final value.
- As τ is decreased, the impulse response decays more sharply and the rise time of the step response becomes shorter -> it rises more sharply toward its final value.
- The step response of a first-order system does not exhibit any ringing.



Transient response of 1st order system with impulse (left) and step inputs (right)

Frequency Response of Continuous 1st Order System

- Figures below depicts the Bode plot of the frequency response of the first order system.
- The x-axis of the graphs is in logarithmic frequency scale and its unit is radian/second.
- The y-axis of the magnitude plot is in dB (e.g. $20\log|H(\omega)|$) and the phase plot is in radian (e.g. $\measuredangle H(\omega)$).



Frequency response of 1st order system (magnitude plot (left) and phase plot (right))

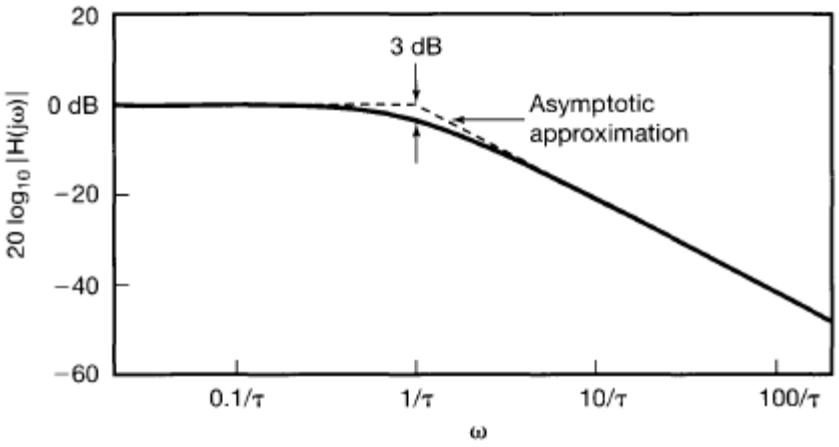
Magnitude Response of 1st Order System

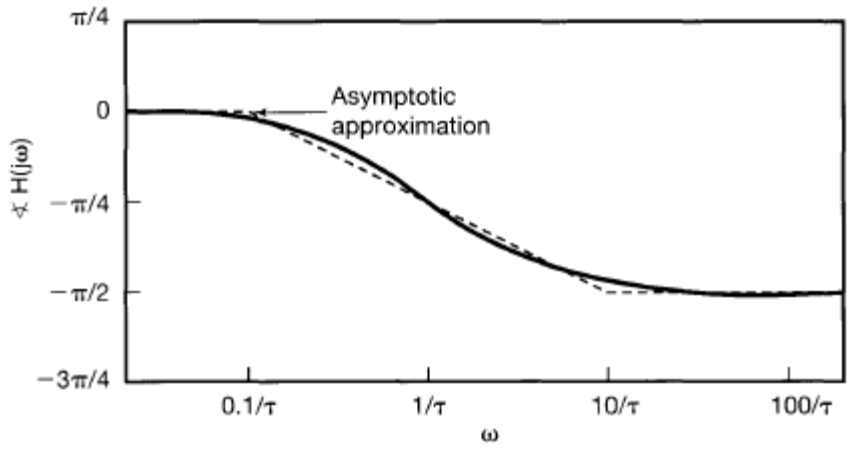
- The magnitude of the frequency response for given first order system:

$$20\log|H(\omega)| = 20\log\left|\frac{1}{\sqrt{(\omega\tau)^2+1}}\right|$$

$$= -10\log|(\omega\tau)^2+1|$$

- For the first-order system, the low- and high-frequency it





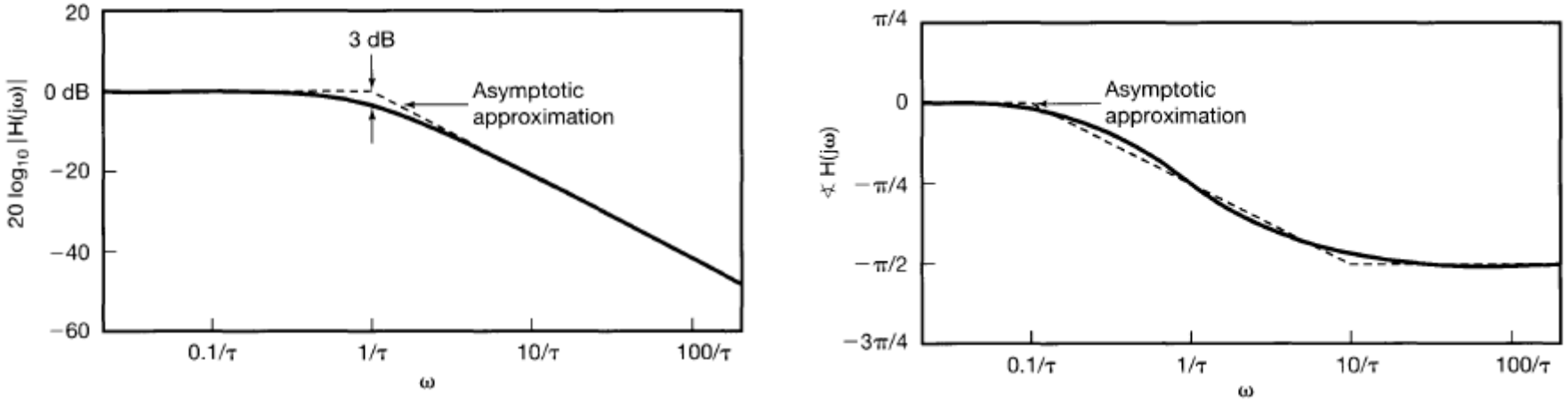
Frequency response of 1st order system (magnitude plot (left) and phase plot (right))

Magnitude Response of 1st Order System

- For $\omega\tau << 1$, the log magnitude is approximately zero. For $\omega \ll 1/\tau$

$$20\log|H(\omega)| \approx 0$$

- The low-frequency asymptote is just the 0-dB line



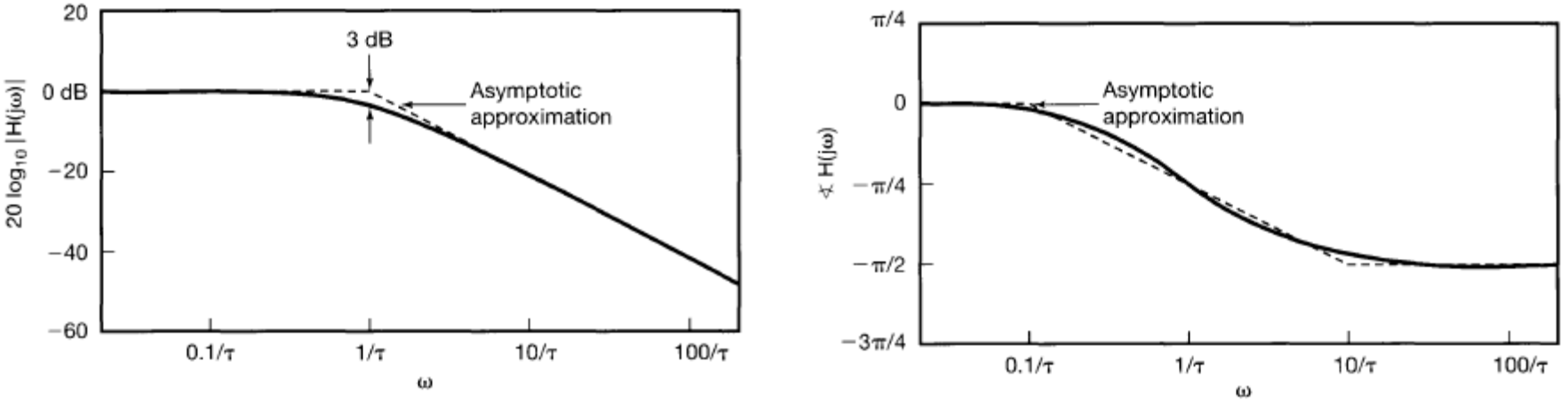
Frequency response of 1st order system (magnitude plot (left) and phase plot (right))

Magnitude Response of 1st Order System

- For $\omega\tau >> 1$, the log magnitude is approximately a linear function of log(ω). for $\omega \gg 1/\tau$

$$20\log|H(\omega)| \approx 20\log(\omega\tau) = -20\log(\omega) - 20\log(\tau)$$

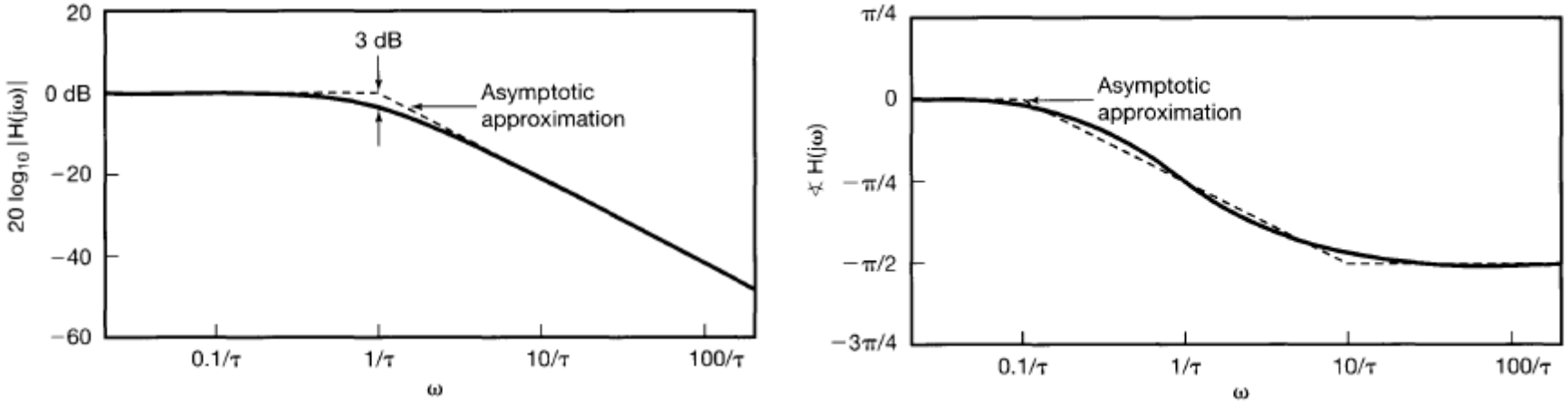
- The high-frequency asymptote corresponds to a decrease of 20 dB in $|H(\omega)|$ for every decade (i.e., factor of 10) in ω -> a "20-dB-per-decade" asymptote.



Frequency response of 1st order system (magnitude plot (left) and phase plot (right))

Magnitude of Continuous 1st Order System

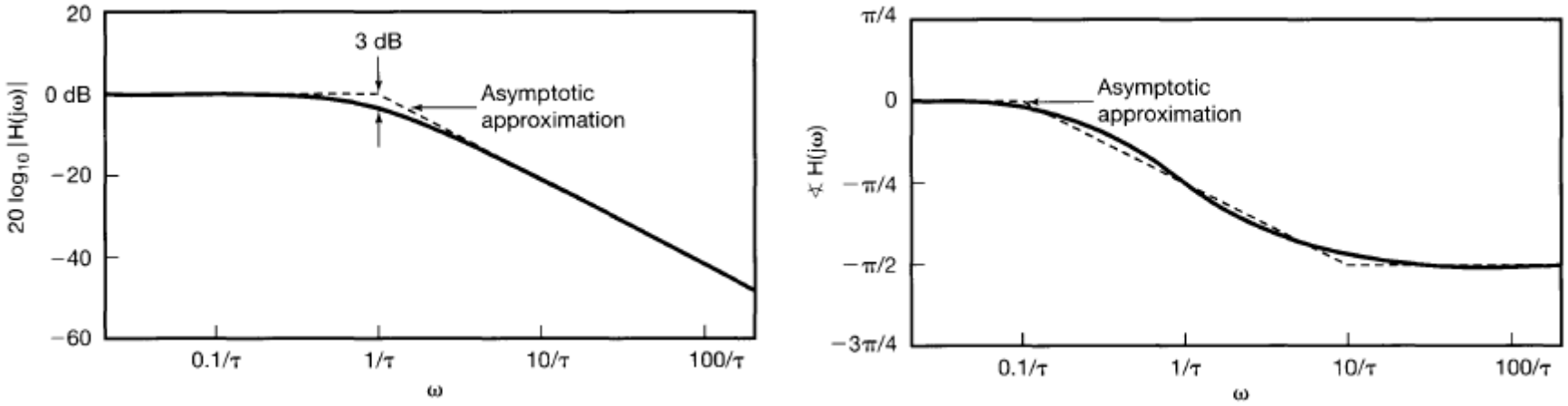
- The two asymptotic approximations are equal at the point $\log(\omega) = -\log(\tau)$, or equivalently, $\omega = 1/\tau$.
- Graphically, this means that the two straight-line asymptotes meet at $\omega = 1/\tau$, which suggests a straight-line approximation to the magnitude plot.



Frequency response of 1st order system (magnitude plot (left) and phase plot (right))

Magnitude of Continuous 1st Order System

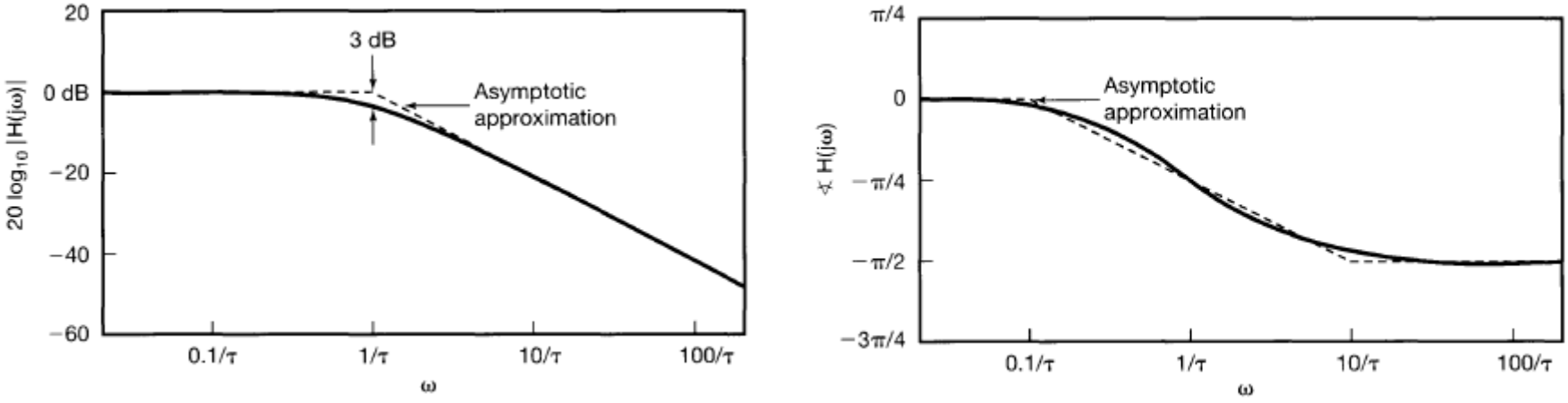
- The approximation to $20\log|H(\omega)|$ equals 0 for $\omega \leq= 1/\tau$ and $20\log(\omega\tau)$ for $\omega \geq= 1/\tau$.
- This approximation is also sketched (as a dashed line) in figure.



Frequency response of 1st order system (magnitude plot (left) and phase plot (right))

Break Point in Continuous 1st Order System

- The point at which the slope of the approximation changes is precisely $\omega = 1/\tau$ -> referred to as the break frequency.
- At $\omega = 1/\tau$, the two terms $(\omega\tau)^2$and 1 in the argument of the frequency response equation of the first order system are equal.



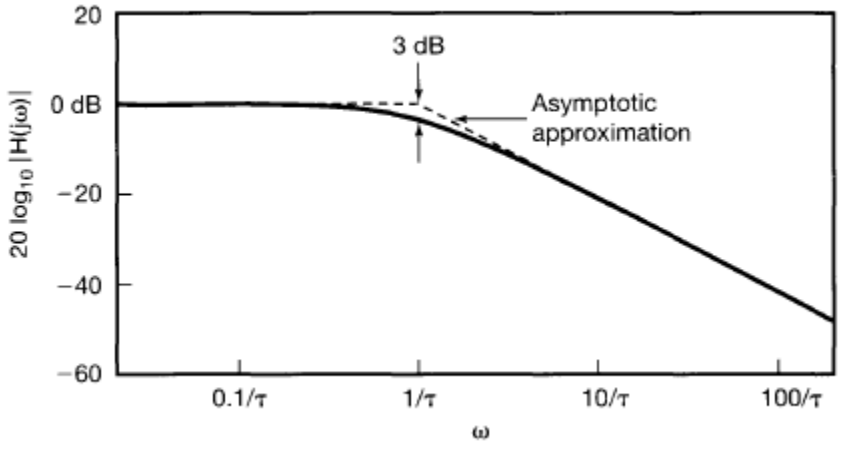
Frequency response of 1st order system (magnitude plot (left) and phase plot (right))

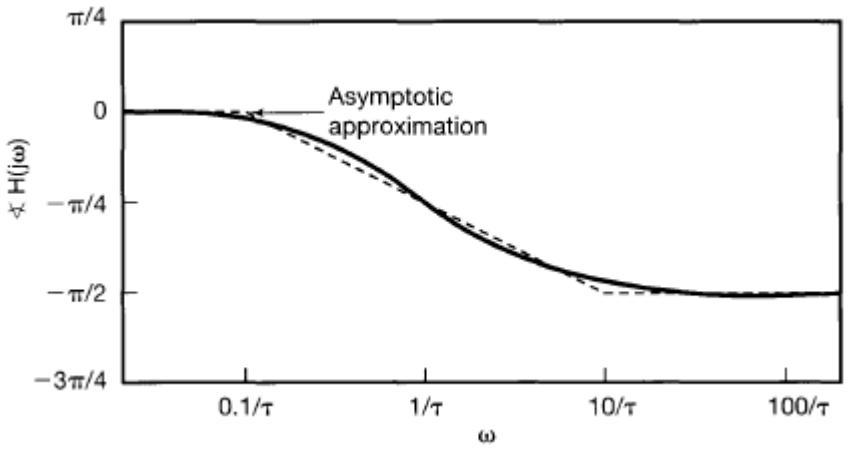
Break Point in Continuous 1st Order System

- Thus, at this point, the actual value of the magnitude is:

$$20\log\left|H\left(j\frac{1}{\tau}\right)\right| = -10\log(2) \cong -3 \text{ dB}$$

- Because of this, the point $\omega = 1/\tau$ is also called the 3-dB point.

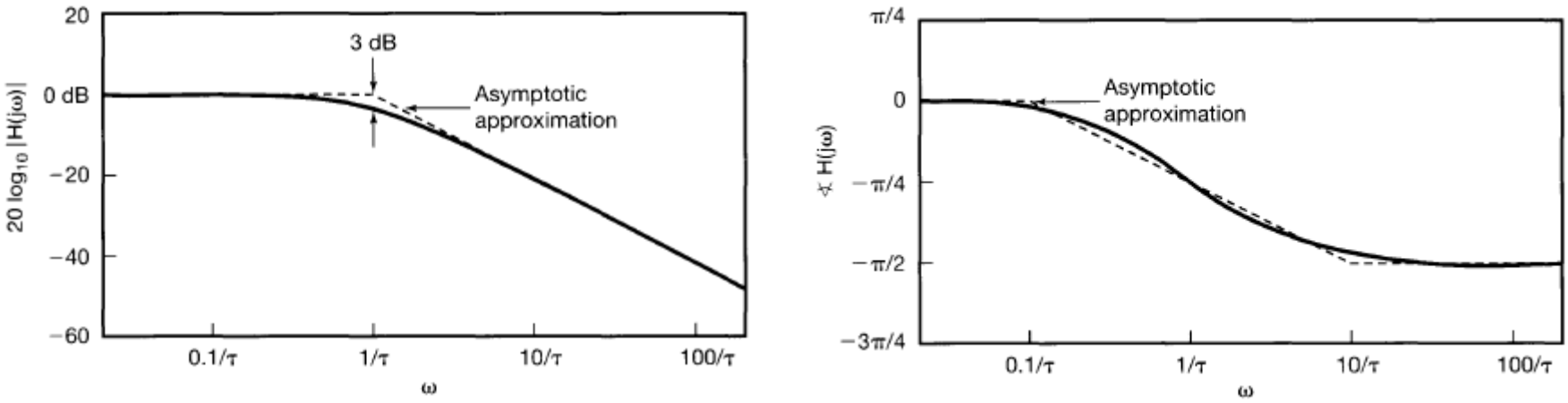




Frequency response of 1st order system (magnitude plot (left) and phase plot (right))

Break Point in Continuous 1st Order System

- From the figure, near the break frequency there is error in the straight-line approximate Bode plot.
- To obtain a more accurate sketch of the Bode plot, we need only modify the approximation near the break frequency.



Frequency response of 1st order system (magnitude plot (left) and phase plot (right))

Phase Response of 1st Order System

- The phase of frequency response of the system, $\measuredangle H(\omega)$ can be approximated as:

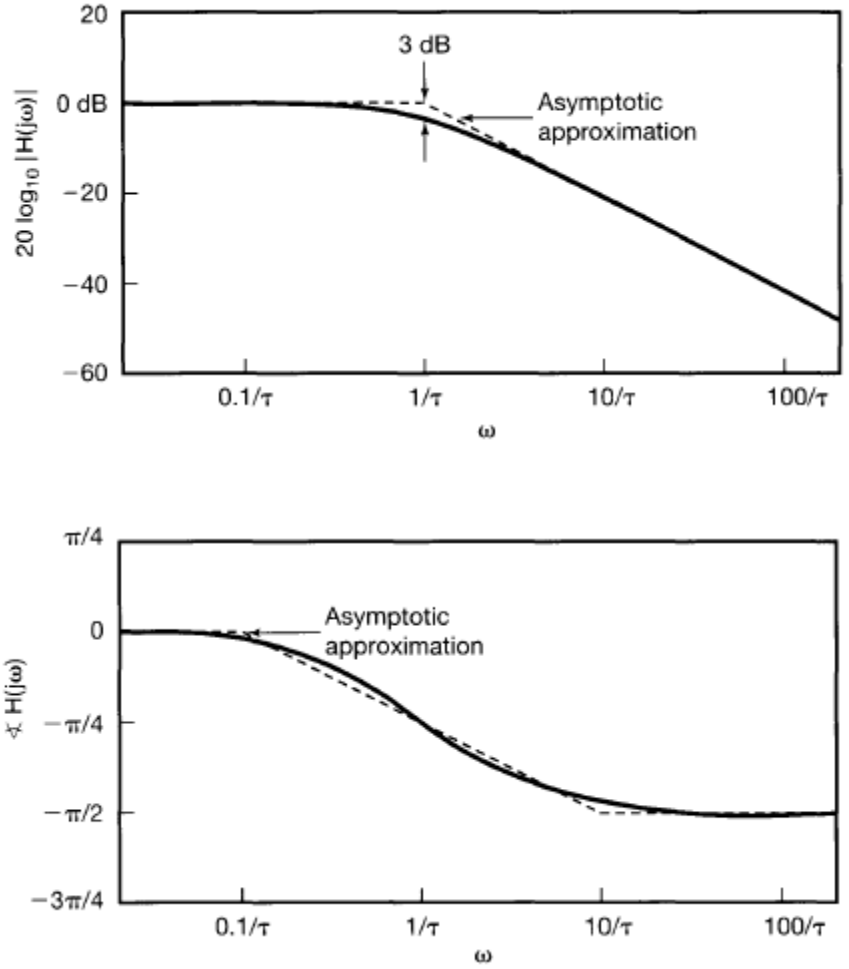
$$\measuredangle H(\omega) = -\tan^{-1}(\omega\tau)$$

- The approximation is at 0, when $\omega \leq 0.1/\tau$.

$$\measuredangle H(\omega) = -\tan^{-1}(\omega\tau) = 0,$$

- It is at $-\pi/2$, when $\omega \gg 10/\tau$

$$\measuredangle H(\omega) = -\tan^{-1}(\omega\tau) = -\frac{\pi}{2}$$



Frequency response of 1st order system (magnitude plot (left) and phase plot (right))

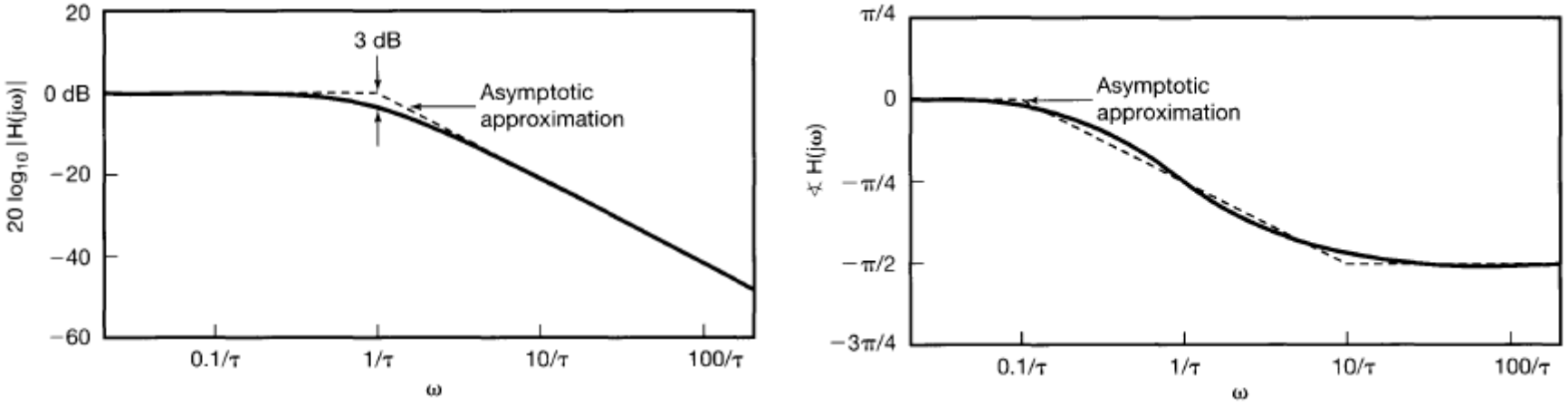
Phase Response of 1st Order System

- So, when $0.1/\tau \leq \omega \leq 10/\tau$.

$$\measuredangle H(\omega) = -\tan^{-1}(\omega\tau) = -\left(\frac{\pi}{4}\right)[\log(\omega\tau) + 1]$$

- This decreases linearly (from 0 to $-\pi/2$) as a function of $\log(\omega)$ in the range:

$$\frac{0.1}{\tau} \leq \omega \leq \frac{10}{\tau}$$



Frequency response of 1st order system (magnitude plot (left) and phase plot (right))

Phase Response of 1st Order System

- The range from one decade below the break frequency to one decade above the break frequency.
- Correct asymptotic value of $\measuredangle H(\omega)$ is 0 for $\omega \leq 1/\tau$.
- Correct asymptotic value of $\measuredangle H(\omega)$ is $-\pi/2$ for $\omega \gg 1/\tau$.
- Furthermore, the approximation agrees with the actual value of $\measuredangle H(\omega)$ at the break frequency $\omega = 1/\tau$, at which point:

$$\measuredangle H\left(j\frac{1}{\tau}\right) = -\frac{\pi}{4}$$

- This asymptotic approximation is also plotted in the graph, if desired, we can modify the straight-line approximation to obtain a more accurate sketch of $\measuredangle H(\omega)$.

Summary of Continuous 1st Order System

- In the frequency response of the first-order system, we can again see the inverse relationship between time and frequency.
- In time domain, as we make τ smaller, we speed up the time response of the system [i.e., $h(t)$ becomes more compressed toward the origin, and the rise time of the step response is reduced].
- In frequency domain, we simultaneously make the break frequency large [i.e., $H(\omega)$ becomes broader, since $|H(\omega)| \cong 1$ for a larger range of frequencies].
- This can also be seen by multiplying the impulse response by τ and observing the relationship between $\tau h(t)$ and $H(\omega)$
- Thus, $\tau h(t)$ is a function of t/τ and $H(\omega)$ is a function of $\omega\tau$, and from this we see that changing τ is essentially equivalent to a scaling in time and frequency.

Discrete 1st Order System

- Given a discrete first-order causal LTI system described by the difference equation:

$$y[n] - ay[n-1] = x[n]$$

with $|a| < 1$.

- The frequency response of this system is:

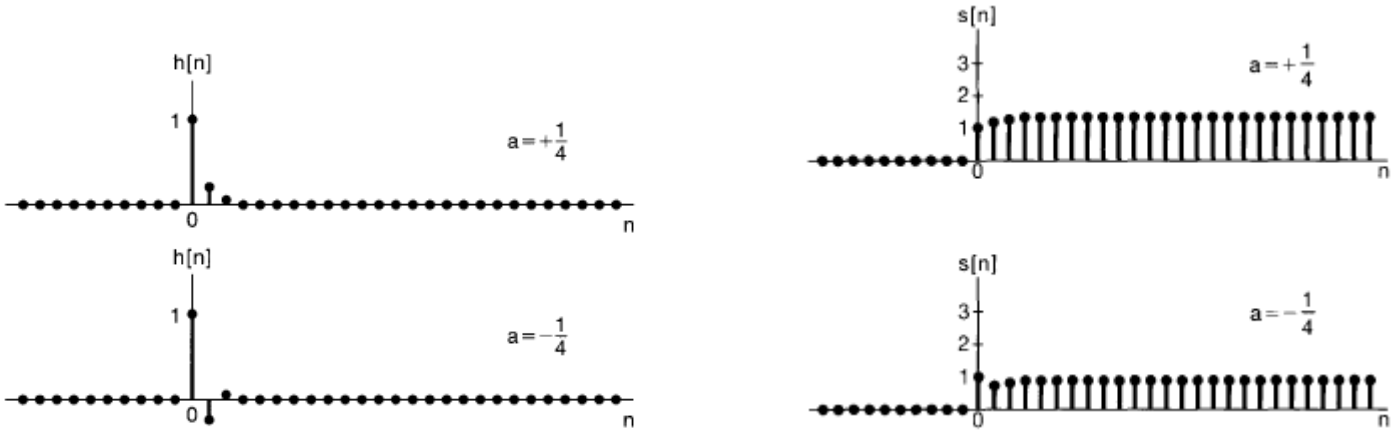
$$H\left(e^{j\omega}\right) = \frac{1}{1 - ae^{-j\omega}}$$

Transient Response of Discrete 1st Order System

- The impulse response of the system is:

$$h[n] = a^n u[n]$$

The impulse response of the system is illustrated in the graph (left) for several values of a.



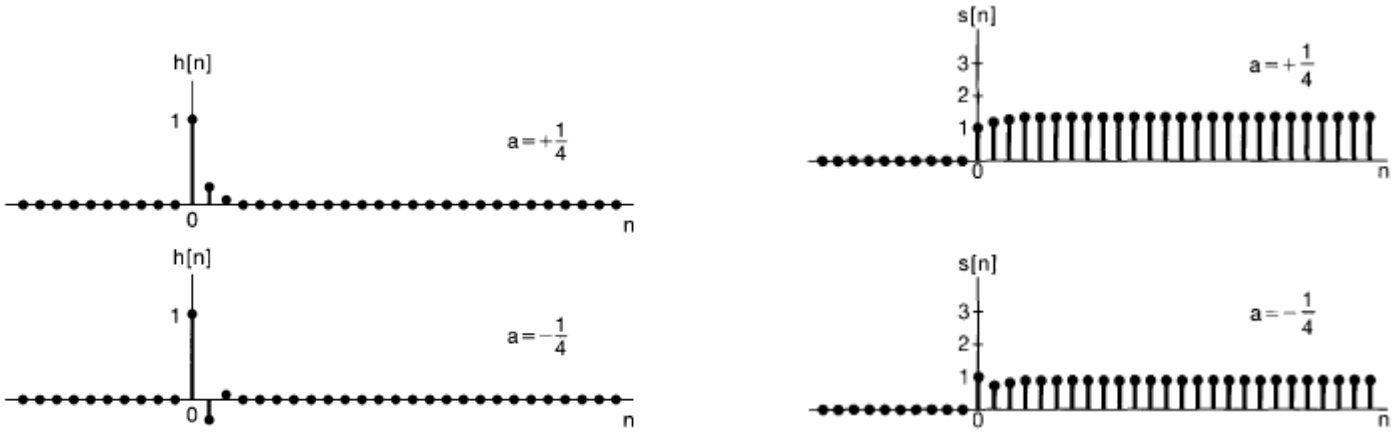
Transient response of discrete 1st order system with impulse (left) and step inputs (right)

Transient Response of Discrete 1st Order System

- The step response of the system is:

$$s[n] = h[n] * u[n] = \left[\frac{1 - a^{n+1}}{1 - a}\right] u[n]$$

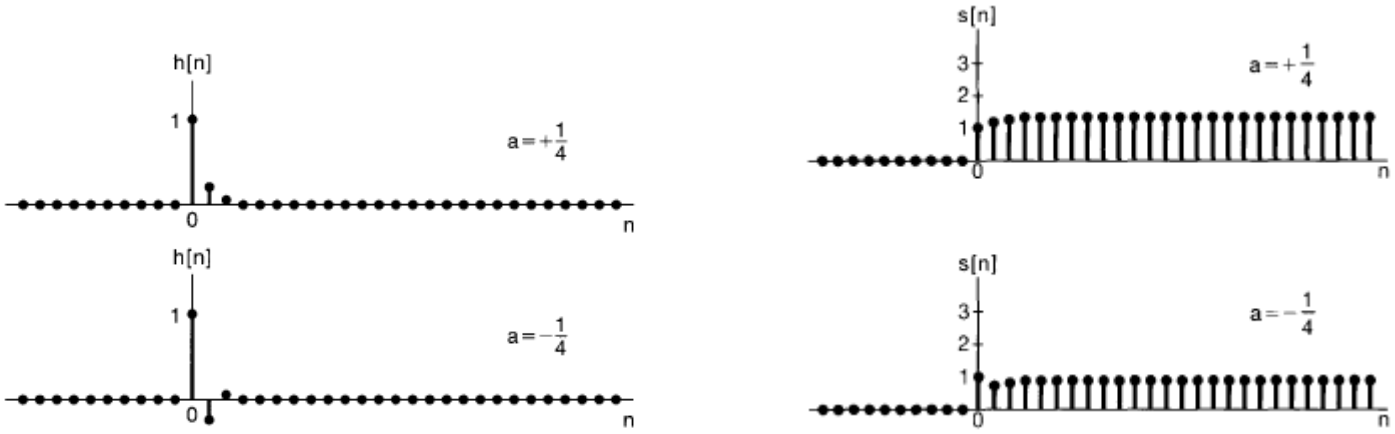
The step response is illustrated in the graph (right).



Transient response of discrete 1st order system with impulse (left) and step inputs (right)

Transient Response of Discrete 1st Order System

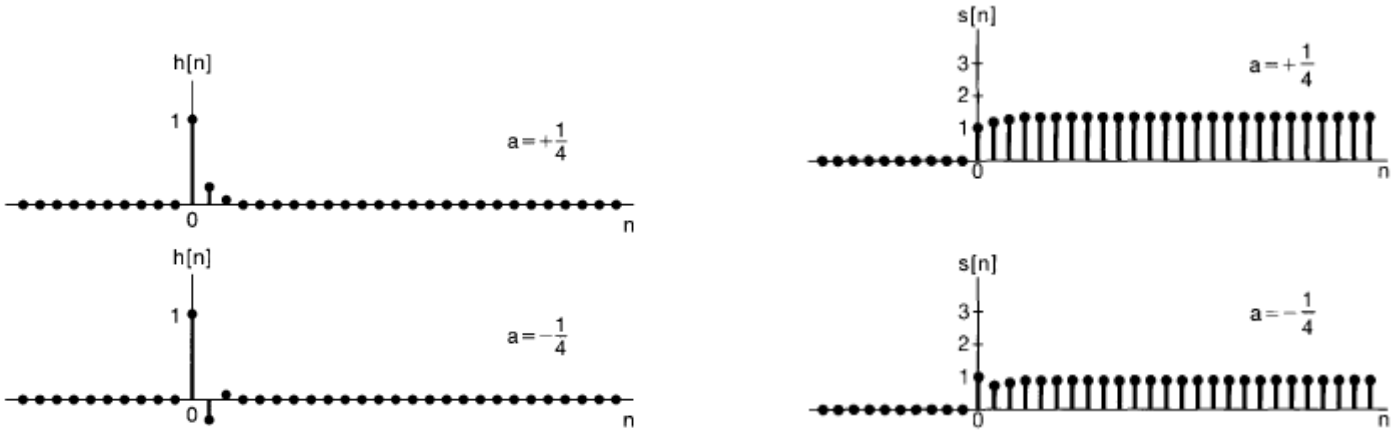
- The magnitude of the parameter a plays a role similar to that of the time constant τ in the continuous-time first-order system.
- Specifically, $|a|$ determines the rate at which the first-order system responds.



Transient response of discrete 1st order system with impulse (left) and step inputs (right)

Transient Response of Discrete 1st Order System

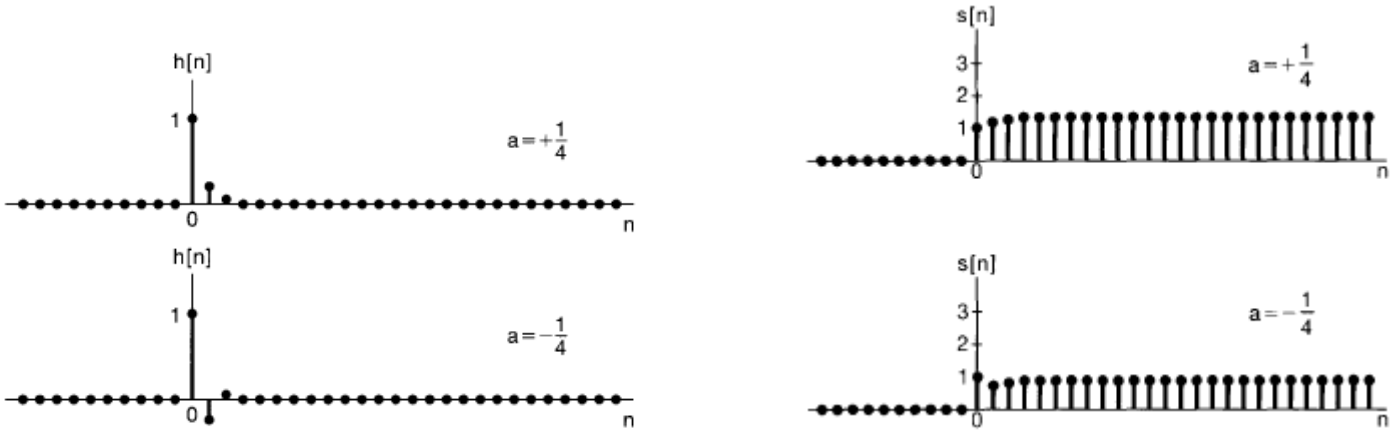
- From the graphs, we see that $h[n]$ and $s[n]$ converge to their final value at the rate at which $|a|^n$ converges to zero.
- The impulse response decays sharply and the step response settles quickly for $|a|$ small.



Transient response of discrete 1st order system with impulse (left) and step inputs (right)

Transient Response of Discrete 1st Order System

- For $|a|$ nearer to 1, these responses are slower.
- Note that unlike its continuous-time counterpart, the discrete first-order system can display oscillatory behavior.
- This occurs when $a < 0$, in which case the step response exhibits both overshoot of its final value and ringing.



Transient response of discrete 1st order system with impulse (left) and step inputs (right)

Frequency Response of Discrete 1st Order System

- The magnitude and phase of the frequency response of the discrete first-order system are, respectively,

$$\left|H\left(e^{j\omega}\right)\right| = \frac{1}{\sqrt{1 - a^2 - 2a\ \cos\omega}}$$

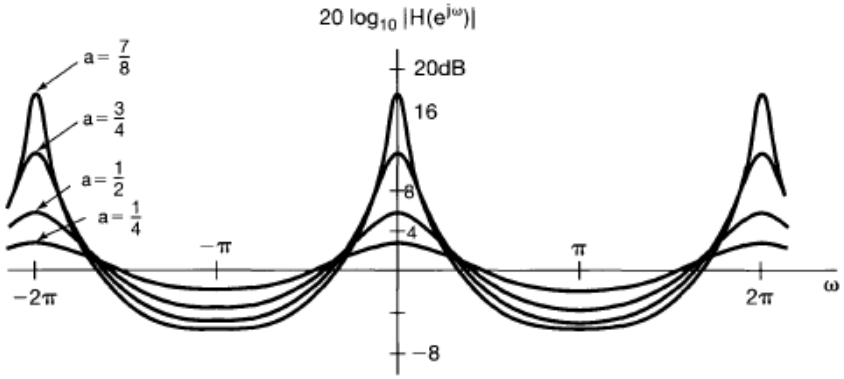
- and

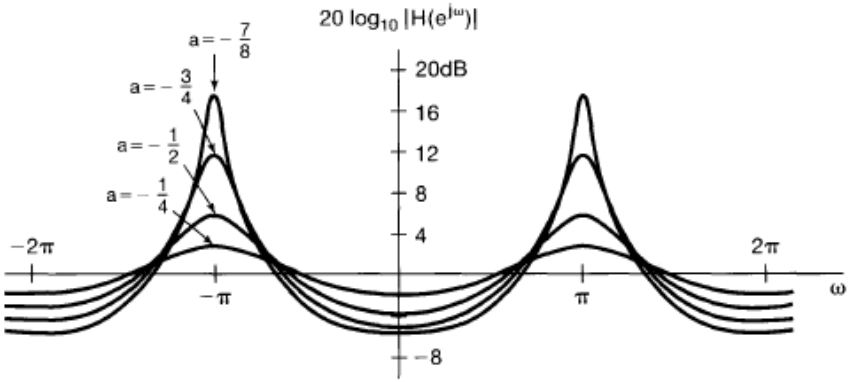
$$\measuredangle H\left(e^{j\omega}\right) = -\tan^{-1}\left[\frac{a\sin\omega}{1 - a\cos\omega}\right]$$

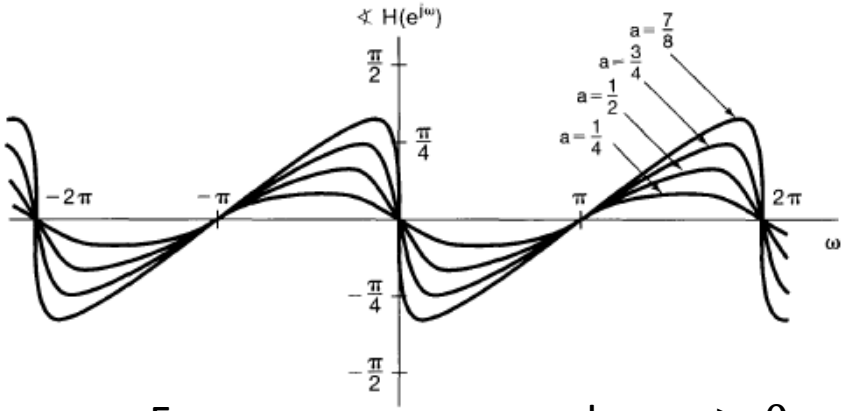
- The magnitude plot of the frequency response of the discrete system for several values of $a > 0$ is left figures on the top and the phase plot is on the bottom.
- The magnitude and phase graphs are also illustrated for $a < 0$ on the right figures.

Frequency Response of Discrete 1st Order System

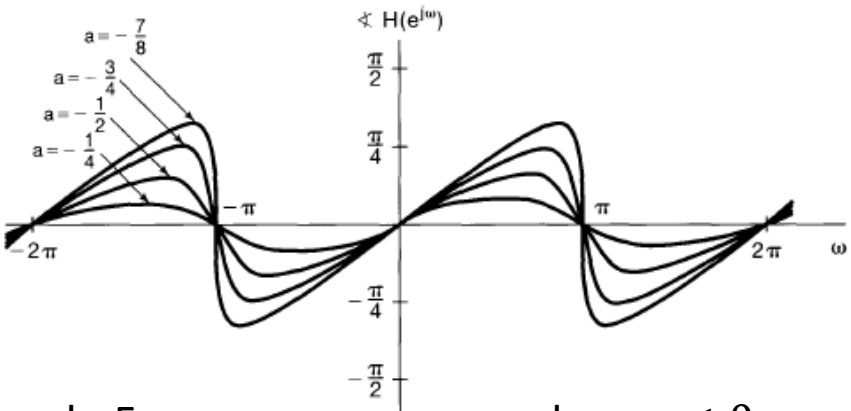
- Frequency response of discrete 1st order system $a > 0$ (left) and $a < 0$ (right):







a. Frequency response when $a > 0$



b. Frequency response when $a < 0$

Frequency Response of Discrete 1st Order System

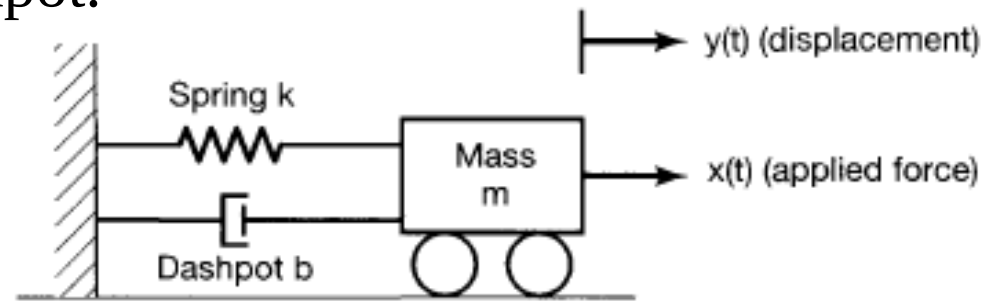
- For $a > 0$, the system attenuates high frequencies [i.e., $H(e^{j\omega})$ is smaller for ω near $\pm\pi$ than it is for ω near 0].
- For $a < 0$, the system amplifies high frequencies and attenuates low frequencies.
- For $|a|$ small, the maximum and minimum values, $1/(1+a)$ and $1/(1-a)$, of $H(e^{j\omega})$ are close together in value, and the graph of $H(e^{j\omega})$ is relatively flat.
- For $|a|$ near 1, these quantities differ significantly-> $H(e^{j\omega})$ is more sharply peaked (e.g. in filter application providing filtering and amplification that is more selective over a narrow band of frequencies).

Continuous 2nd Order System

- The linear constant-coefficient differential equation for a second-order system is:

$$\frac{d^2y(t)}{dt^2} + (2\zeta\omega_n)\frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

- Equations of this type arise in many physical systems, including electrical systems, such as *RLC* circuits and mechanical systems, such as the composed of a spring, a mass, and a viscous damper/dashpot.



Mechanical system consisted of mass-spring-damper components

Continuous 2nd Order System

- In the figure, the input is the applied force $x(t)$ and the output is the displacement of the mass $y(t)$ from some equilibrium position at which the spring exerts no restoring force.
- The equation of motion for this system is (applying the Newton 2nd law, e.g. $\sum F = ma$):

$$m\left(\frac{d^2y(t)}{dt^2}\right) = x(t) - (k)y(t) - b\left(\frac{dy(t)}{dt}\right)$$

Or

$$\frac{d^2y(t)}{dt^2} + \left(\frac{b}{m}\right)\frac{dy(t)}{dt} + \left(\frac{k}{m}\right)y(t) = \left(\frac{1}{m}\right)x(t)$$

Continuous 2nd Order System

- The frequency response for the second-order system of the standardized second order system equation is:

$$H(\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

- Comparing the equation of the system to the standardized second order system equation, we see that if we identify:

$$\omega_n = \sqrt{k/m} \quad \text{and} \quad \zeta = \frac{b}{2\sqrt{km}}$$

- Then the equation of motion for the system of the given figure reduces to standardized second order system equation (except for a scale factor of k on $x(t)$).

Continuous 2nd Order System

- The denominator of $H(\omega)$ can be factored to yield:

$$H(\omega) = \frac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)}$$

Where:

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \quad \text{and} \quad c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

- For $\zeta \neq 1$, c_1 and c_2 are unequal, and using a partial-fraction expansion of the form:

$$H(\omega) = \frac{M}{j\omega - c_1} - \frac{M}{j\omega - c_2}$$

where: $M = \omega_n/2\sqrt{\zeta^2 - 1}$

Impulse Response of Continuous 2nd Order System

The impulse response for the system is (given that $M = \omega_n/2\sqrt{\zeta^2 - 1}$):

$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$$

- If $\zeta = 1$, then $c_1 = c_2 = -\omega_n$, and

$$H(\omega) = \frac{\omega_n^2}{(j\omega + \omega_n)^2}$$

From the table of Fourier transform, we find that in this case the impulse response is:

$$h(t) = \omega_n^2 t e^{-\omega_n t} u(t)$$

So that $h(t)/\omega_n$ is a function of $\omega_n t$.

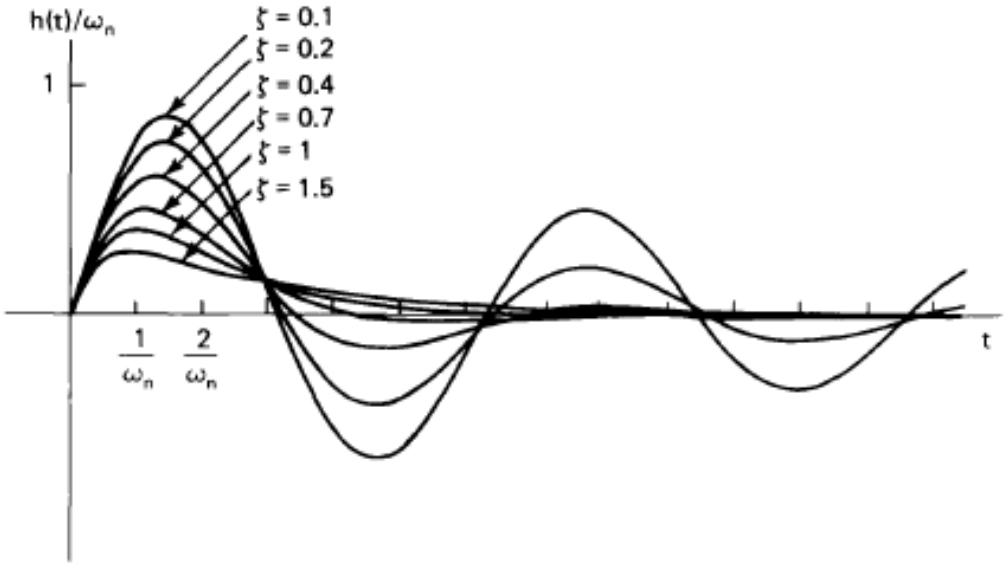
Impulse Response of Continuous 2nd Order System

- Furthermore, the partial fractioned equation can be rewritten as:

$$H(\omega) = \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1}$$

from which we see that the frequency response is a function of ω/ω_n.

- Thus, changing ω_n is essentially identical to a time and frequency scaling.



Magnitude frequency response (various ζ)

Damping in Continuous 2nd Order System

- The parameter ζ is the damping ratio and ω_n is the (undamped) natural frequency.
- These terminologies are important when we take a more detailed look at the impulse response and the step response of a second order system.
- For $0 < \zeta < 1$, c_1 and c_2 are complex, and we can rewrite the impulse response equation in the form:

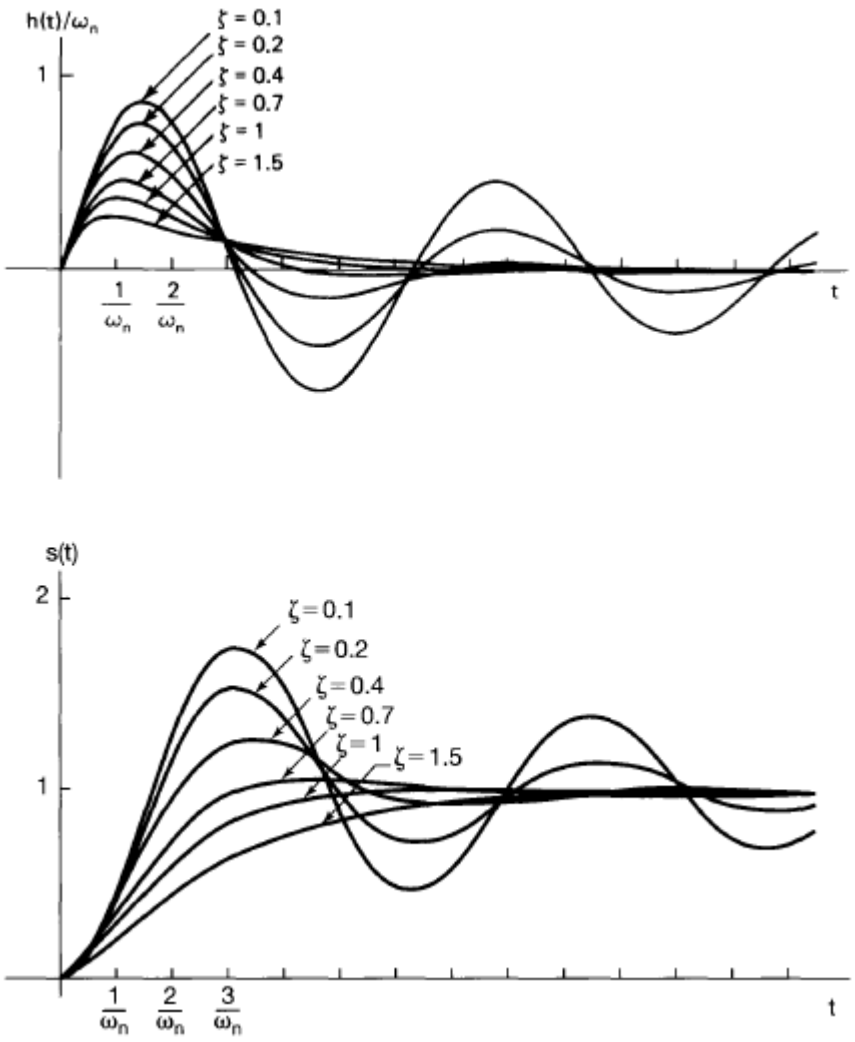
$$h(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{2j\sqrt{1-\zeta^2}}\left[e^{j\left(\omega_n\sqrt{1-\zeta^2}\right)t} - e^{-j\left(\omega_n\sqrt{1-\zeta^2}\right)t}\right]u(t)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\left(\omega_n\sqrt{1-\zeta^2}\right)t\right]u(t)$$

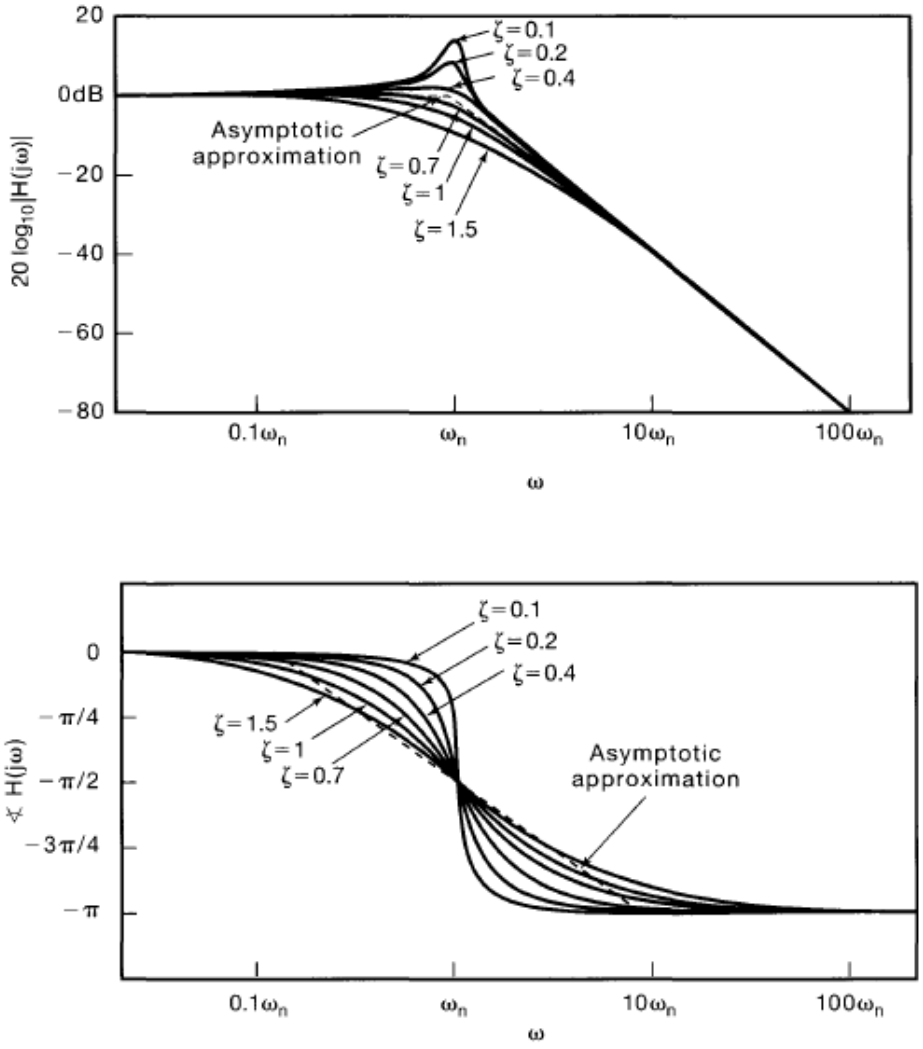
Damping in Continuous 2nd Order System

- So, for $0 < \zeta < 1$, the second-order system has an impulse response that has damped oscillatory behavior, and in this case the system is referred to as being underdamped.
- If $\zeta > 1$, both c_1 and c_2 are real and negative, and the impulse response is the difference between two decaying exponentials. In this case, the system is overdamped.
- The case of $\zeta = 1$, when $c_1 = c_2$, the response is the critically damped case.
- The impulse responses (multiplied by $1/\omega_n$) for second-order systems with different values of ζ are plotted in the figure (top).

Damping in Continuous 2nd Order System



Transient response of 2nd order system with impulse input (top) and step input (bottom) with various damping ratios (ζ)



Frequency response of 2nd order system with various damping ratios (ζ)

Step Response of Continuous 2nd Order System

- For $\zeta \neq 1$ the step response of a second-order system can be calculated from: $h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$.
- Knowing that $M = \omega_n / 2\sqrt{\zeta^2 - 1}$), this yields the expression:

$$s(t) = h(t) * u(t) = \left\{1 + M\left[\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right]\right\} u(t)$$

Where:

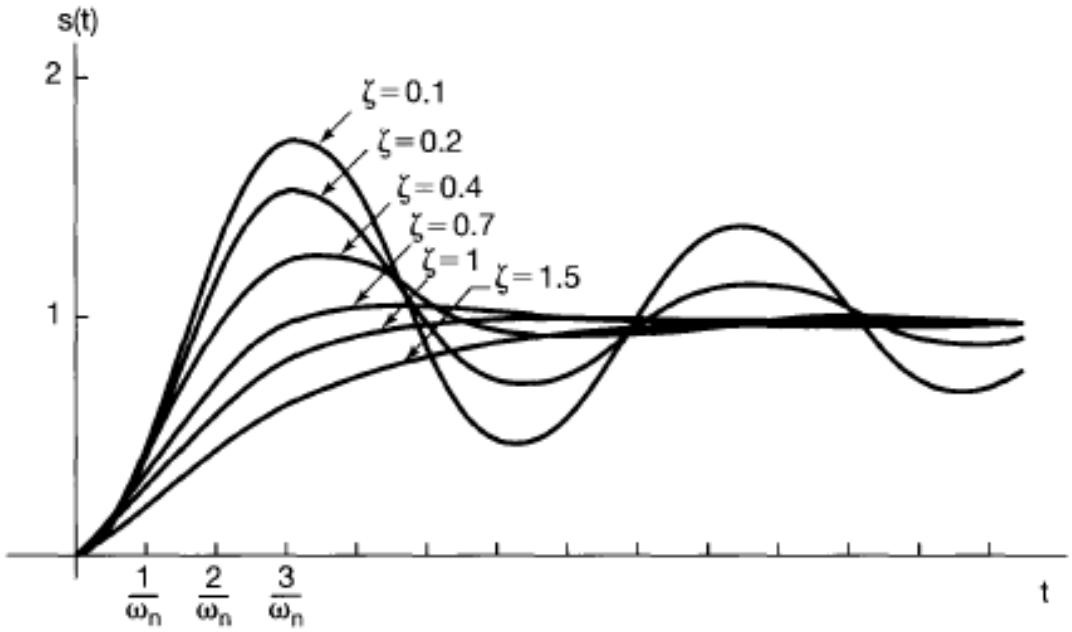
$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$ and $c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$

- For $\zeta = 1$, we can use $h(t) = \omega_n^2 t e^{-\omega_n t} u(t)$ and the step response is:

$$s(t) = [1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}]u(t)$$

Step Response of Continuous 2nd Order System

- The step response of a second-order system is plotted in the given figure for several values of ζ.
- In the underdamped case, the step response exhibits both overshoot.
- The step response exceeds its final value) and ringing (i.e. oscillatory behavior).



Magnitude frequency response (various ζ)

Step Response of Continuous 2nd Order System

- For $\zeta = 1$, the step response has the fastest response (i.e., the shortest rise time) that is possible without overshoot and thus has the shortest settling time.
- As $\zeta \gg 1$, the response becomes slower. This can be seen from impulse response equations.
- As ζ increases, c_1 becomes smaller in magnitude, while c_2 increases in magnitude.
- Therefore, although the time constant $(1/|c_2|)$ associated with $e^{c_2 t}$ decreases, the time constant $(1/|c_1|)$ associated with $e^{c_1 t}$ increases.

Step Response of Continuous 2nd Order System

- Consequently, the term involving $e^{c_1 t}$ in step response equation takes a longer time to decay to zero.
- Thus, it is the time constant associated with $e^{c_1 t}$ that determines the settling time of the step response.
- As a result, the step response takes longer to settle for large values of ζ.
- In terms of the spring-damper-mass system example, as we increase the magnitude of the damping coefficient b beyond the critical value at which ζ in standardized equation of second order system equals 1, the motion of the mass becomes increasingly sluggish.

Step Response of Continuous 2nd Order System

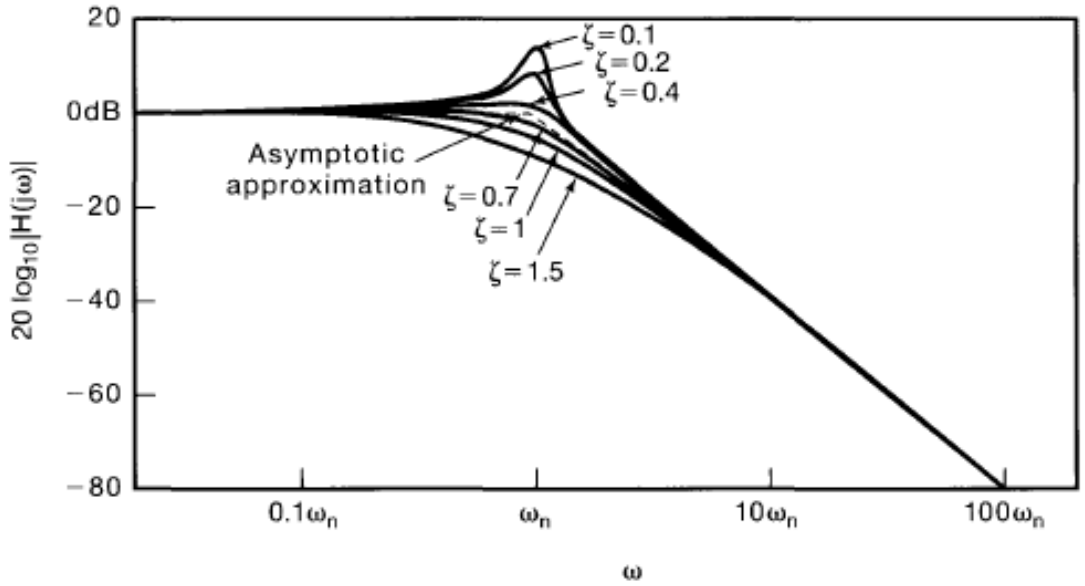
- We found that the value of ω_n essentially controls the time scale of the responses $h(t)$ and $s(t)$.
- For example, in the underdamped case, the larger ω_n is, the more compressed is the impulse response as a function of t, and the higher is the frequency of the oscillations or ringing in both $h(t)$ and $s(t)$.
- From equations for impulse and step responses of second order system, we see that the frequency of the oscillations in $h(t)$ and $s(t)$ is $\omega_n\sqrt{1-\zeta^2}$ which does increase with increasing ω_n.

Step Response of Continuous of 2nd Order System

- This frequency depends explicitly on the damping ratio and does not equal (and is in fact smaller than) ω_n, except in the undamped case, $\zeta = 0$.
- It is for this reason that the parameter ω_n is traditionally referred to as the undamped natural frequency.
- For the spring-dashpot example, we therefore conclude that:
  - The rate of oscillation of the mass equals ω_n when no damper/dash pot is present.
  - The oscillation frequency decreases when we include the damper/dashpot.

Magnitude of FR in Continuous 2nd Order System

- In the given figure, we have depicted the magnitude plot of the frequency response given in the standardized second order system equation for several values of ζ.



- As in the first-order case, the logarithmic frequency scale leads to linear high- and low-frequency asymptotes for the log magnitude.

Magnitude of FR in Continuous 2nd Order System

- From the standardized second order system equation, the magnitude of the frequency response of the 2nd order system:

$$20\log|H(\omega)| = 20\log\frac{1}{\sqrt{\left[1+\left(\frac{j\omega}{\omega_n}\right)^2\right]^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}}$$

$$20\log|H(\omega)| = -10\log\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\}$$

- From this expression, it follows that:

$$20\log|H(\omega)| \cong \begin{cases} 0, & \text{for } \omega \ll \omega_n \\ -40\log\omega + 40\log\omega_n, & \text{for } \omega \gg \omega_n \end{cases}$$

Magnitude of FR in Continuous 2nd Order System

- The low-frequency asymptote of the log magnitude is the 0-dB line.
- The high-frequency asymptote has a slope of -40 dB per decade i.e. $H(\omega)$ decreases by 40 dB for every increase in ω of a factor of 10.
- The two straight-line asymptotes meet at the point $\omega = \omega_n$.
- For this reason, ω_n is referred to as the break frequency of the second-order system.
- This approximation is also plotted (as a dashed line) in given figure.

Phase of FR in Continuous 2nd Order System

- A straight-line approximation to phase of the frequency response $\measuredangle H(\omega)$, is obtained from the standardized second order system equation:

$$\measuredangle H(\omega) = \frac{\frac{2\zeta\omega}{\omega_n}}{1+\left(\frac{j\omega}{\omega_n}\right)^2} = -\tan^{-1}\left(\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right)$$

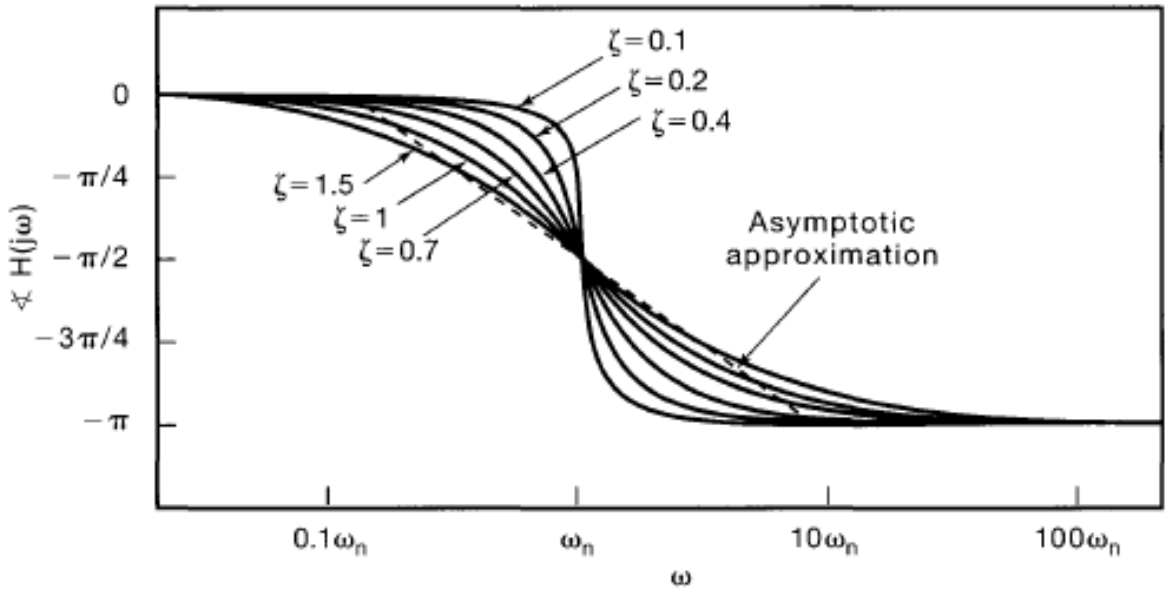
- The approximation is:

$$\measuredangle H(\omega)\begin{cases} 0, & \omega \le 0.1\omega_n \\ -\frac{\pi}{2}\left[\log(\omega/\omega_n)+1\right], & 0.1\omega_n \le \omega \le 10\omega_n \\ -\pi, & \omega \ge 10\omega_n \end{cases}$$

which is also plotted in the given figure.

Phase of FR in Continuous 2nd Order System

- In the given figure, we have depicted the phase plot of the frequency response given in the standardized second order system equation for several values of ζ.



- As in the first-order case, the logarithmic frequency scale leads to linear high- and low-frequency asymptotes for the phase shift of the system.

Phase of FR in Continuous 2nd Order System

- The approximation and the actual value again are equal at the break frequency $\omega = 0.1\omega_n$, where $\measuredangle H(\omega_n) = -\pi/2$
- The asymptotic approximations of magnitude and phase equations that we have obtained for a second-order system do not depend on ζ.
- While the actual plots of $|H(\omega)|$ and $\measuredangle H(\omega)$ certainly do, and thus, to obtain an accurate sketch, especially near the break frequency $\omega = \omega_n$, we must take this into account by modifying the approximations to conform more closely to the actual plots.

Peak in FR of Continuous 2nd Order System

- The actual log magnitude has a peak around $\omega = \omega_n$.
- The discrepancy is most pronounced for small values of ζ.
- Straightforward calculations using approximation of magnitude equation show that, for $\zeta < \sqrt{2}/2 \cong 0.707$, $|H(\omega)|$ has a maximum value at:

$$\omega_{max} = \omega_n \sqrt{1 - \zeta^2}$$

- The value at this maximum point is:

$$|H(\omega_{max})| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

Peak in FR of Continuous 2nd Order System

- For $\zeta > 0.707$, however, $H(\omega)$ decreases monotonically as ω increases from zero.
- The fact that $H(\omega)$ can have a peak is extremely important in the design of frequency-selective filters and amplifiers.
- In some applications, one may want to design such a circuit so that it has a sharp peak in the magnitude of its frequency response at some specified frequency
- Thereby providing large frequency-selective amplification for sinusoids at frequencies within a narrow band.

Peak in FR of Continuous 2nd Order System

- The quality Q of such a circuit is defined to be a measure of the sharpness of the peak.
- For a second-order circuit described by an equation of the form of transfer function of the given system equation, the quality is usually taken to be:

$$Q = \frac{1}{2\zeta}$$

- From the given figure and the above equation, we see that this definition has the proper behavior e.g. the less damping there is in the system, the sharper is the peak in $|H(\omega)|$.

Discrete 2nd Order System

- For the second-order causal LTI system described by:

$$y[n] - 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n]$$

with $0 < r < 1$ and $0 \leq \theta \leq \pi$.

- The frequency response for this system is:

$$H\left(e^{j\omega}\right) = \frac{1}{1 - 2r\cos\theta e^{-j\omega} + r^2 e^{-j2\omega}}$$

- The denominator of $H\left(e^{j\omega}\right)$ can be factored to obtain (note: Euler law of $\cos\theta = \left(e^{j\theta} + e^{-j\theta}\right)/2$:

$$H\left(e^{j\omega}\right) = \frac{1}{[1 - (re^{j\theta})e^{-j\omega}][1 - (re^{-j\theta})e^{-j\omega}]}$$

Discrete 2nd Order System

- For $\theta \neq 0$ or π, the two factors in the denominator of $H(e^{j\omega})$ are different.
- By applying a partial-fraction expansion yields:

$$H(e^{j\omega}) = \frac{A}{1-(re^{j\theta})e^{-j\omega}} + \frac{B}{1-(re^{-j\theta})e^{-j\omega}}$$

Where:

$$A = \frac{e^{j\theta}}{2j\sin\theta}, \qquad B = \frac{e^{-j\theta}}{2j\sin\theta}$$

Impulse Response in Discrete 2nd Order System

- In this case, the impulse response of the system is:

$$h[n] = \left[A\left(re^{j\theta}\right)^n + B\left(re^{-j\theta}\right)^n\right]u[n]$$

Where:

$$A = \frac{e^{j\theta}}{2j\sin\theta}, \qquad B = \frac{e^{-j\theta}}{2j\sin\theta}$$

- Knowing that Euler law of $\sin\theta = \left(e^{j\theta} - e^{-j\theta}\right)/2j$:

$$h[n] = r^n\left\{\frac{\sin[(n+1)\theta]}{\sin\theta}\right\}u[n]$$

Impulse Response in Discrete 2nd Order System

- For $\theta = 0$ or π, the two factors in the denominator of transfer function equation of the system are the same.
- When $\theta = 0$,

$$H(e^{j\omega}) = \frac{1}{(1 - re^{-j\omega})^2}$$

$$h[n] = (n+1)r^n u[n]$$

- When $\theta = \pi$,

$$H(e^{j\omega}) = \frac{1}{(1 + re^{-j\omega})^2}$$

$$h[n] = (n+1)(-r)^n u[n]$$

Impulse Response in Discrete 2nd Order System

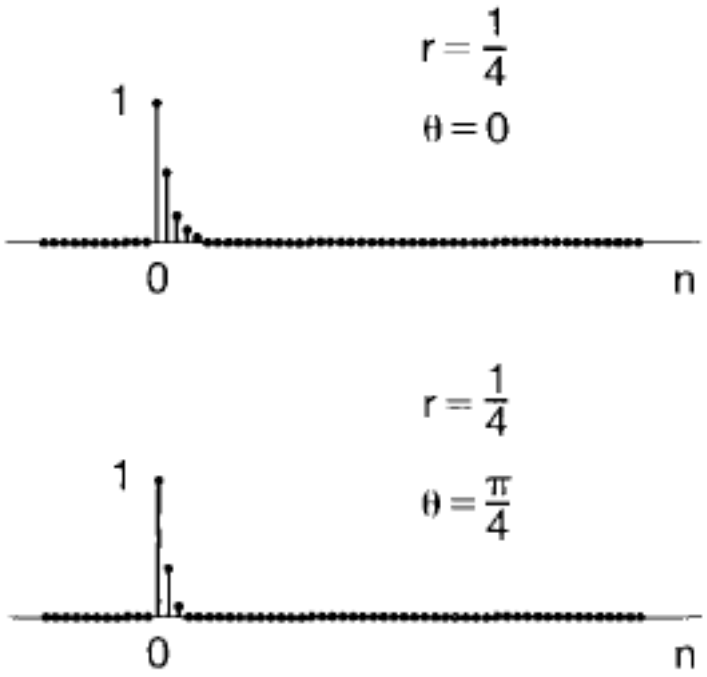
- The rate of decay of $h[n]$ is controlled by r i.e., the closer r is to 1, the slower is the decay in $h[n]$.
- The value of θ determines the frequency of oscillation.
- With $\theta = 0$ there is no oscillation in $h[n]$, while for $\theta = \pi$ the oscillations are rapid.
- The impulse responses for second-order systems are plotted in the given figure for a range of values of r and θ.

Impulse Response in Discrete 2nd Order System

- Transient response of discrete 2nd order system with impulse input.

  Note: each pair of plots for r = 1/4 and $\theta = \pi/4$.

- Parameter θ determines how fast the response to settle down.
- System with smaller θ takes longer time to settle down.
- System with bigger θ has faster time to settle down.



Step Response in Discrete 2nd Order System

- For $\theta \neq 0$ or π,

$$s[n] = h[n] * u[n] = \left[\left(\frac{1-\left(re^{j\theta}\right)^{n+1}}{1-re^{j\theta}}\right)+\left(\frac{1-\left(re^{-j\theta}\right)^{n+1}}{1-re^{-j\theta}}\right)\right]u[n]$$

- For $\theta = 0$,

$$s[n] = \left[\frac{1}{(r-1)^2}-\frac{r}{(r-1)^2}(r)^n+\frac{r}{r-1}(n+1)(r)^n\right]u[n]$$

- For $\theta = \pi$,

$$s[n] = \left[\frac{1}{(r+1)^2}+\frac{r}{(r+1)^2}(-r)^n+\frac{r}{r+1}(n+1)(-r)^n\right]u[n]$$

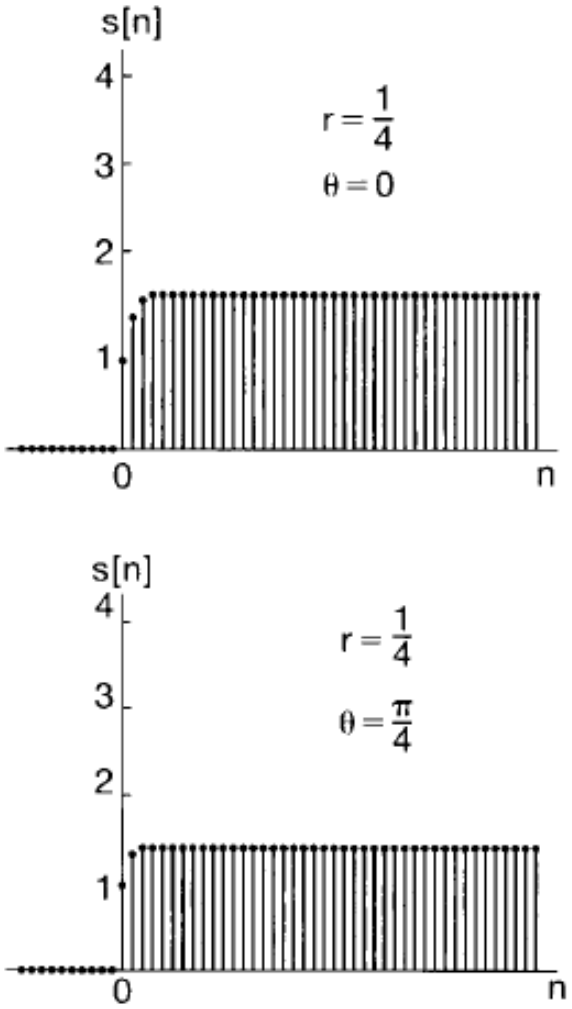
- The step response is plotted in the figure, again for a range of values of r and θ.

Step Response in Discrete 2nd Order System

- Transient response of discrete 2nd order system step inputs.

  Note: each pair of plots for r = 1/4 and $\theta = 0$ or $\pi/4$.

- System with $\theta = 0$ takes longer time to settle down than the system with $\theta = \pi/4$.



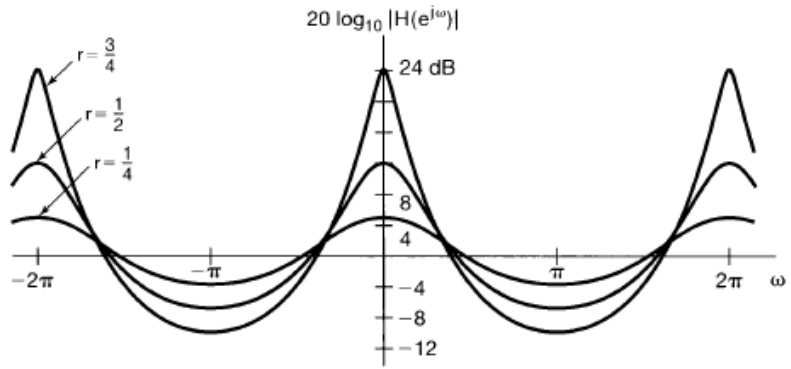
Frequency Response of Discrete 2nd Order System

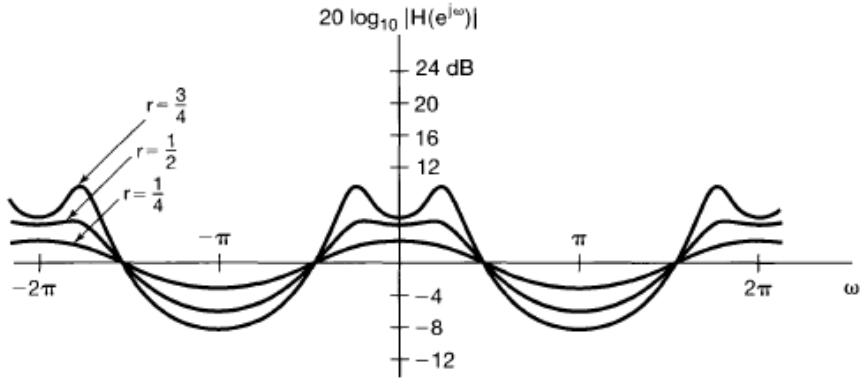
- The transfer function of the discrete second-order system given is the opposite of the underdamped second-order system in continuous time.
- For a special case of $\theta = 0$, it is the critically damped case.
- For any value of $\theta \neq 0$:
  - The impulse response has a damped oscillatory behavior.
  - The step response exhibits ringing and overshoot.
- The frequency response of this system is depicted in the figure for a number of values of r and θ.
- From the figure, we see that a band of frequencies is amplified and r determines how sharply peaked the frequency response is within this band.

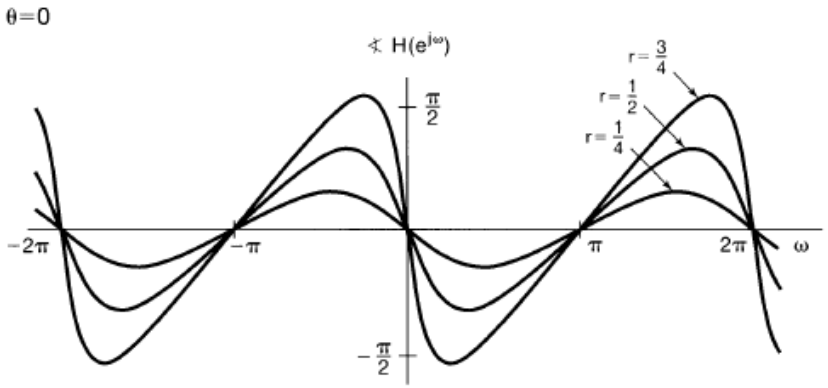
Frequency Response of Discrete 2nd Order System

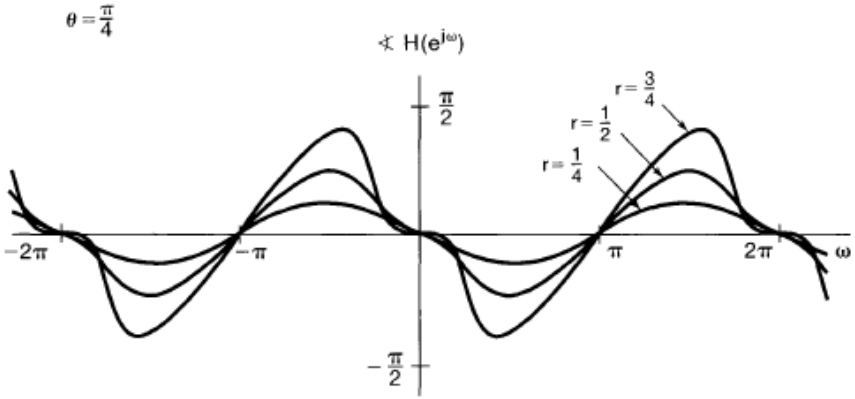
- Frequency response of discrete 2nd order system.

  Note: $\theta = 0$ (left) and $\theta = \pi/4$ (right)









Frequency Response of Discrete 2nd Order System

- The discrete second-order system described in frequency response equation has factors with complex coefficients (unless $\theta = 0$ or π).
- So, for the discrete second-order systems having factors with real coefficients:

$$H(e^{j\omega}) = \frac{1}{(1 - d_1 e^{-j\omega})(1 - d_2 e^{-j\omega})}$$

Where: d_1 and d_2 are both real numbers with $|d_1|, |d_2| < 1$.

- Actually, the equation above is the frequency response for the difference equation:

$$y[n] - (d_1 + d_2)y[n-1] + d_1 d_2 y[n-2] = x[n]$$

Frequency Response of Discrete 2nd Order System

- In this case, applying partial fraction:

$$H(e^{j\omega}) = \frac{A}{1 - d_1 e^{-j\omega}} + \frac{B}{1 - d_2 e^{-j\omega}}$$

- where

$$A = \frac{d_1}{d_1 - d_2}, \qquad B = \frac{d_2}{d_2 - d_1}$$

- Impulse response (e.g. the sum of two decaying real exponentials):

$$h[n] = [A d_1^n + B d_2^n] u[n]$$

- So, the step response:

$$s[n] = \left[A \left(\frac{1 - d_1^{n+1}}{1 - d_1} \right) + B \left(\frac{1 - d_2^{n+1}}{1 - d_2} \right) \right] u[n]$$

Frequency Response of Discrete 2nd Order System

- The frequency response of the discrete second order system corresponds to the cascade of two first-order systems -> Most of its properties deduced from the first-order case.
- The log-magnitude and phase plots for the frequency response of the discrete second order system can be obtained by adding together the plots for each of the two first-order terms.
- Like for first-order systems, the response of the system is fast if $|d_1|$ and $|d_2|$ are small, but the system has a long settling time if either of these magnitudes is near 1.
- Furthermore, if d_1 and d_2 are negative, the response is oscillatory.
- The case when both d_1 and d_2 are positive the impulse and step responses settling without oscillation.

Stability of Discrete Systems

- In the discrete systems, we have restricted attention to those first- and second-order systems that are stable and for which the frequency response can be defined.
- From the equation of the discrete first-order system, it is unstable for $|a| \geq 1$.

$$h[n] = a^n u[n]$$

- From the frequency response of the discrete second-order system, it is unstable if $r \geq 1$.

$$y[n] - 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n]$$

- For the discrete second-order system, it is unstable if either $|d_1|$ or $|d_2|$ exceeds 1.

$$y[n] - (d_1 + d_2)y[n-1] + d_1 d_2 y[n-2] = x[n]$$