

Sampling and Reconstruction (Derivation of Equations)

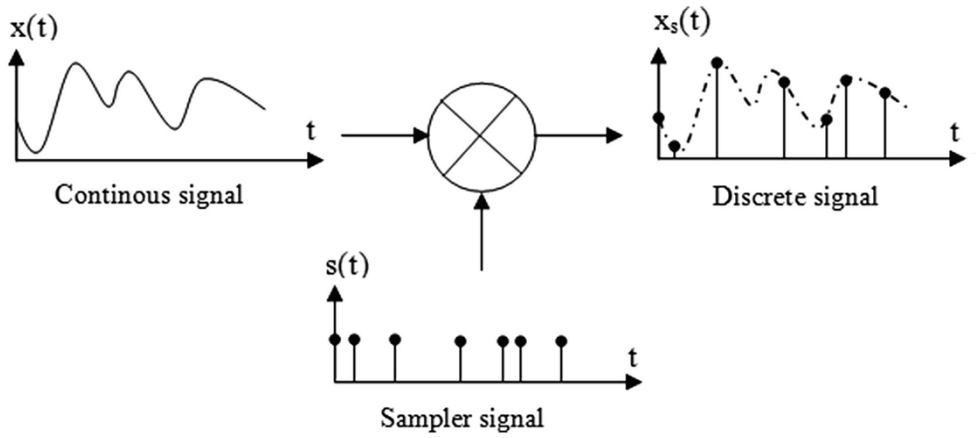
XMUT220 Signals and Systems

Topics

- Sampling of signals.
- Derivation of equations for sampling of signals.
- Reconstruction of signals.
- Derivation of equation for reconstruction of signals.
  - Zero-order hold
  - First-order hold
- Decimation and interpolation.

Sampling of Signal

- In signals and systems, sampling is the conversion of a continuous-time signal to a discrete-time signal.
- A common example of sampling is the conversion of a sound wave to a sequence of samples.
- A continuous signal ($x(t)$) is sampled by a sampler signal ($s(t)$) that results in a discrete signal ($x_p(t)$).



Sampling of Signal

- A sample is a value or set of values at a point in time and/or space.
- A sampler is a subsystem or operation that extracts samples from a continuous signal.
- A theoretical ideal sampler produces samples equivalent to the instantaneous value of the continuous signal at the desired points.
- A continuous signal $x(t)$ can be sampled by an Analogue-to-Digital Converter (A/D) to become a discrete signal composed of a set of time samples.

Sampling of Signals

- The equation that represents sampling of signals:

$$x|m| = x(mt_0) = x(m/F) \qquad (\text{Eq. } 1)$$

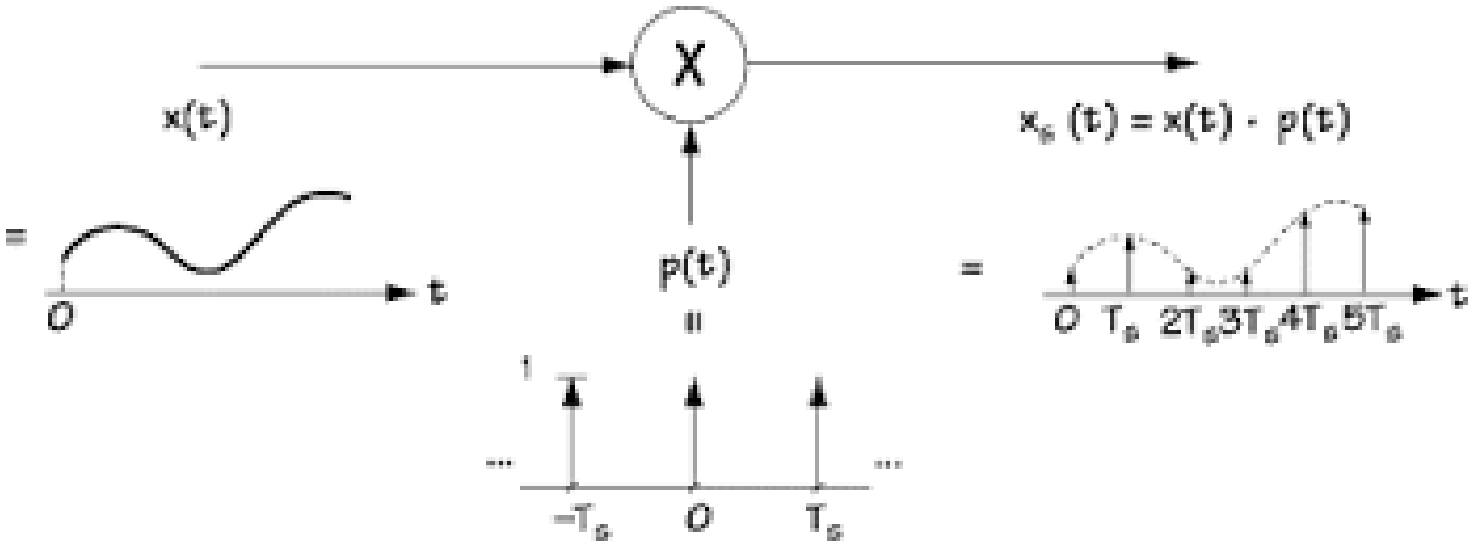
Where:

- $m = \cdots, -1, 0, \ 1, \ldots$
- t_0 is the sampling period, the time interval between two consecutive samples.
- $F = 1/t_0$ is the sampling frequency.

Sampling of Signals

- The sampled signal $x_s(t)$ can be represented as the product of the time signal ($x(t)$) and an impulse train ($p(t)$).

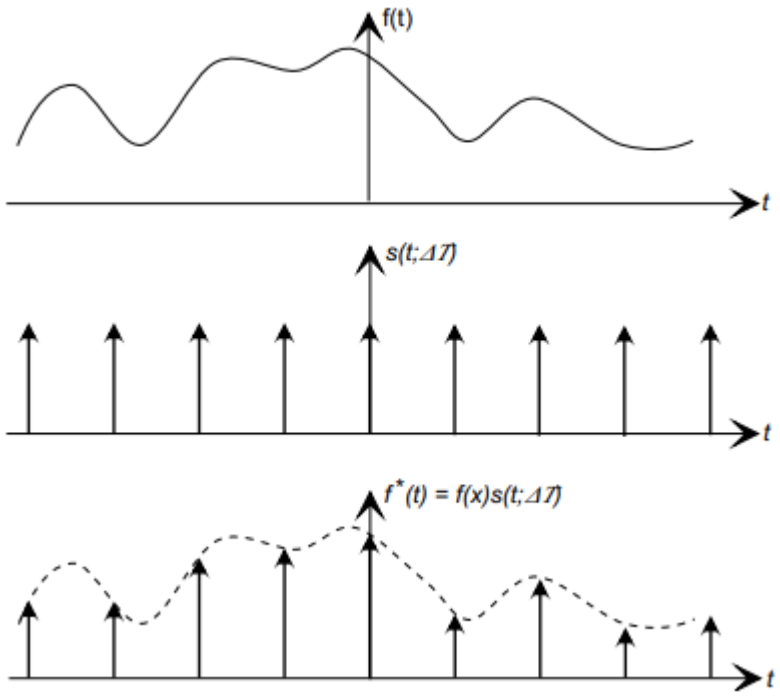
$$x_s(t) = x(t) \cdot p(t)$$



Sampling of Signals

- Diagram shows a sampling by a comb impulse function.
- The function $f(t)$ is multiplied by the sampling function $s(t; \Delta T)$ to produce the sampled waveform:

$$f^*(t) = f(t) \cdot s(t; \Delta T)$$



Sampling of Signals

- The sampling function (also called a comb function):

$$x_s(t) = x(t) \cdot \mathrm{comb}(t) \qquad (\mathrm{Eq.}\,2)$$

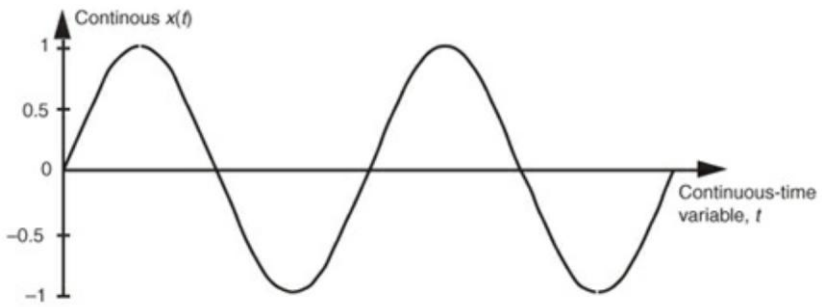
- Knowing comb function:

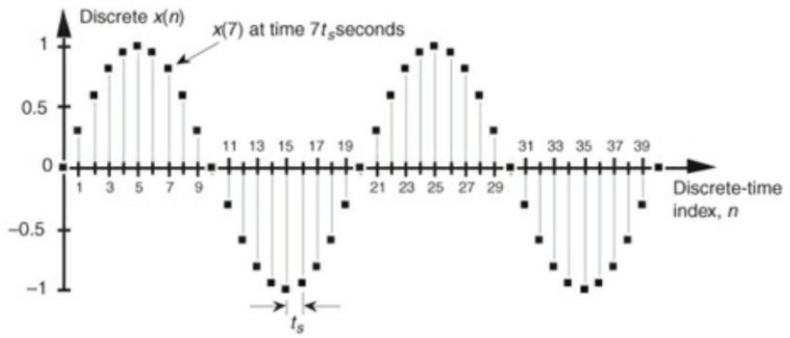
$$\mathrm{comb}(t) = \sum_{m=-\infty}^{\infty} \delta(t - mt_0)$$

- So, applying comb function to the sampling equation:

$$x_s(t) = x(t) \sum_{m=-\infty}^{\infty} \delta(t - mt_0)$$

Sampling of Signals





- Since the original signal can be considered to be consisting of infinite discrete-time samples.

$$x(t) = \sum_{m=-\infty}^{\infty} x[m]$$

- Thus, the equation for sampled signal becomes:

$$x_s(t) = \sum_{m=-\infty}^{\infty} x[m] \cdot \delta(t - mt_0) \qquad (\text{Eq. } 3)$$

Sampling of Signals

- The Fourier transform of the sampled signal can be found as:

$$X_s(j\omega) = \int_{-\infty}^{\infty} x_s(t) \cdot e^{-j\omega t} dt \qquad (\text{Eq. } 4)$$

- Substituting equation (3) into equation (4):

$$X_s(j\omega) = \int_{-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} x[m] \cdot \delta(t - mt_0) \right] e^{-j\omega t} dt$$

- Thus, rearranging the equation, the Fourier transform of the sampled signal becomes:

$$X_s(j\omega) = \sum_{m=-\infty}^{\infty} x[m] \int_{-\infty}^{\infty} \delta(t - mt_0) \cdot e^{-j\omega t} dt$$

Sampling of Signals

- Then, equating the integral part of the equation:

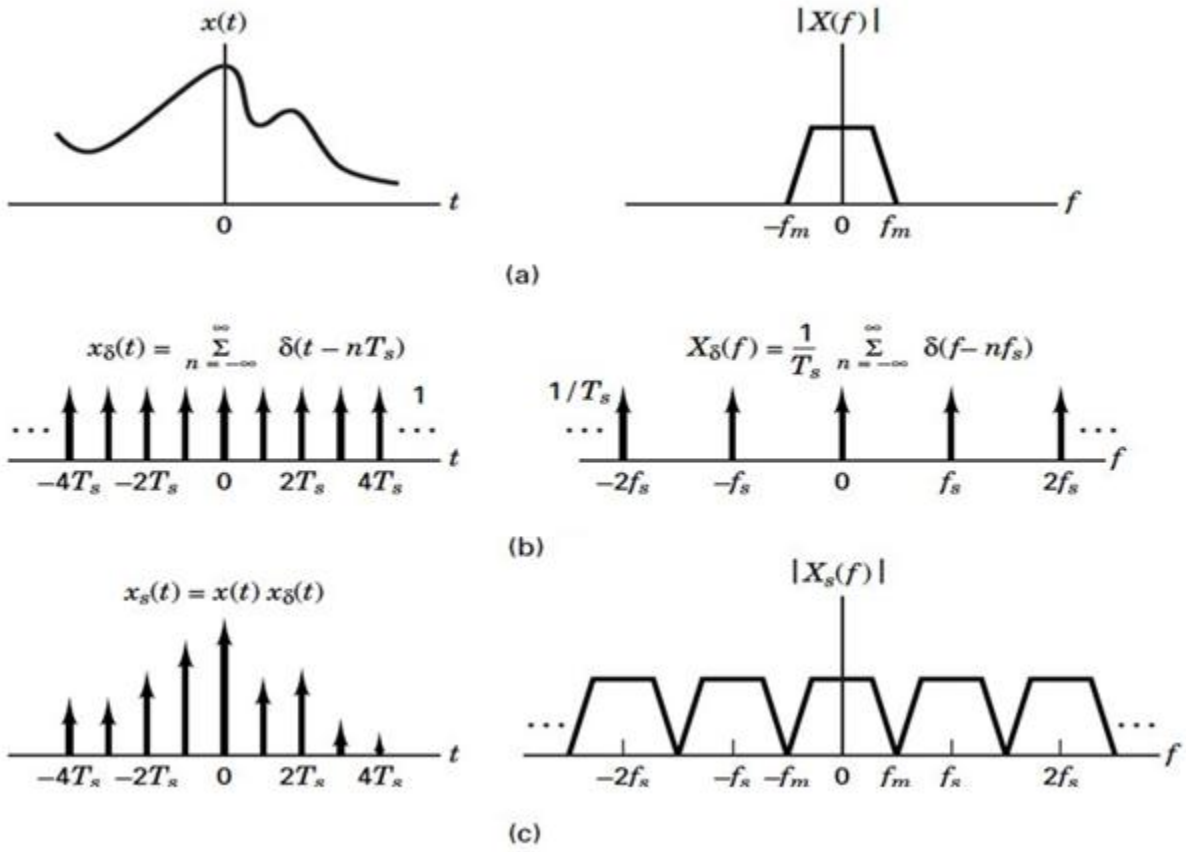
$$X_s(j\omega) = \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-j2\pi f m t_0}$$

- So, we can prove from the derivation process before that:

$$X_s(j\omega) = X_s(f)$$

Sampling of Signals

- Time domain and frequency domain plots of (a) sampled signals, (b) sampling signal and (c) discrete-time sampling samples.



Sampling of Signals

- Note that this frequency spectrum is periodic with period $F = 1/t_0$:

$$X_s(f+F) = \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-j2\pi f m t_0} \cdot e^{-j2\pi F m t_0}$$

- So

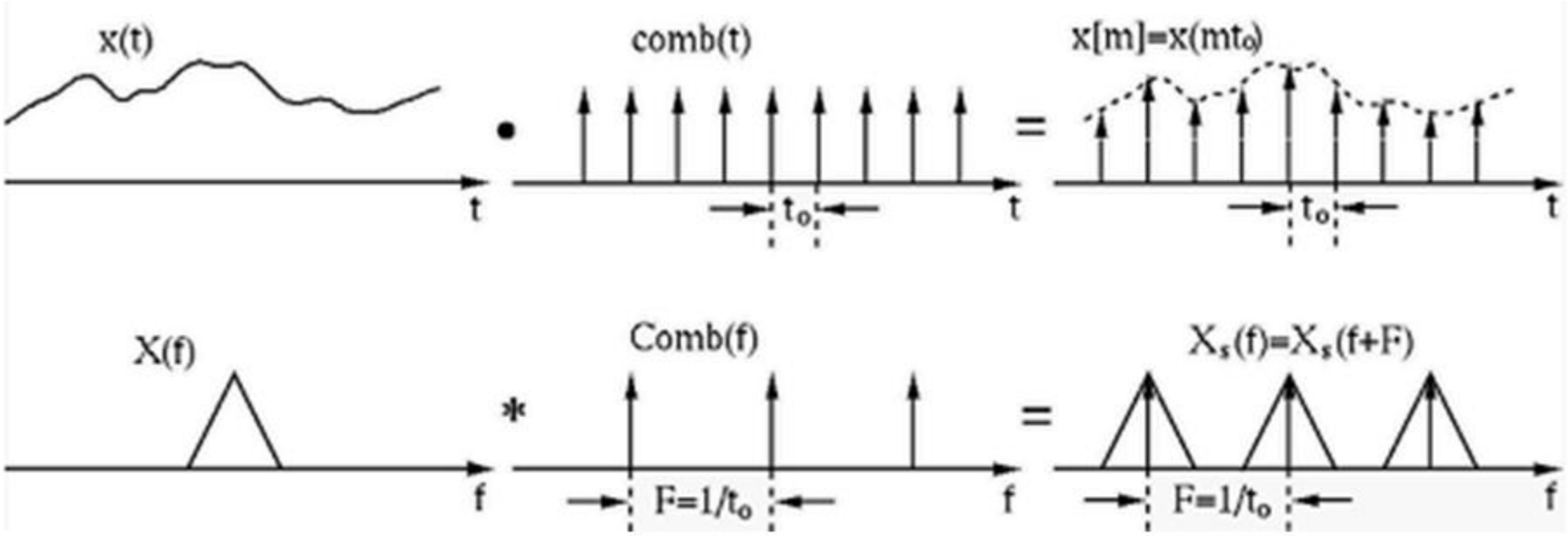
$$X_s(f+F) = \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-j2\pi f m t_0}$$

- Thus, we can prove that:

$$X_s(f+F) = X_s(f)$$

Sampling of Signals

- For sampling process, a multiplication in time domain becomes a convolution in frequency domain.



Signals to be sampled | Sampling Signals | Sampled Signals

Sampling of Signals

- Let $X(f) = \mathcal{F}[x(t)]$, then:

$$\text{comb}(f) = \mathcal{F}[\text{comb}(t)]$$

- Thus

$$\text{comb}(f) = F \sum_{n=-\infty}^{\infty} \delta(f - nF)$$

- Then, the spectrum of the sampled signal is:

$$X_s(f) = X(f) * F \sum_{n=-\infty}^{\infty} \delta(f - nF)$$

Sampling of Signals

- If we assume unit sampling rate $F = 1$ and therefore $t_0 = 1/F = 1$, then:

$$X_s(f) = \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-j2\pi f_m} \qquad (\text{Eq. } 5)$$

- Or

$$X_s(\omega) = \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-j\omega m} \qquad (\text{Eq. } 6)$$

Sampling of Signals

- Discrete-time samples of the discrete signal can be obtained from the spectrum $X_s(\omega)$.
- Applying equation (6) , we multiply $e^{j\omega n}$ on both sides of the above equation and integrate with respect to f over 0 to 1:

$$\int_0^1 X_s(\omega) \cdot e^{j\omega n} df = \sum_{m=-\infty}^{\infty} x[m] \int_0^1 e^{j\omega(m-n)f} df$$

- From standard Fourier transform

$$\int_0^1 X_s(\omega) \cdot e^{j\omega n} df = \sum_{m=-\infty}^{\infty} x[m] \cdot \delta[m-n] \quad \text{(Eq. 7)}$$

$$\int_0^1 X_s(\omega) \cdot e^{j\omega n} df = x[n]$$

Sampling of Signals

- Due to the analogy between the transform coefficients:

$$\int_0^1 e^{-j\omega kt}\, df = \int_0^1 e^{-j2knf}\, df$$

$$\int_0^1 e^{-j2knf}\, df = \int_0^1 e^{-j2knt}\, dt$$

- So, apply Euler rule:

$$\int_0^1 e^{-j2knt}\, dt = \int_0^1 \cos(2\pi kt)\, dt + j\int_0^1 \sin(2\pi kt)\, dt$$

- Integral of these sinusoids of period $1/k$ over 1 is zero, except when $k = 0$ when $\cos(2\pi t) = 1$.

Sampling of Signals

- The previous equation (7) is the inverse Fourier transform.

$$\int_0^1 e^{-j2knt}\, dt = \delta[k]$$

- We rewrite these two equations below:

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

- So

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j2\pi f n}$$

- Thus, we can prove that:

$$X_s(\omega) = X_s(f)$$

Sampling of Signals

- Then,

$$x[n] = \int_0^1 X_s(f) \cdot e^{-j\pi f n} df$$

$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X_s(\omega) \cdot e^{j\omega n} d\omega$$

- This is the discrete-time Fourier transform (DTFT), which is sometimes also expressed as a function of :

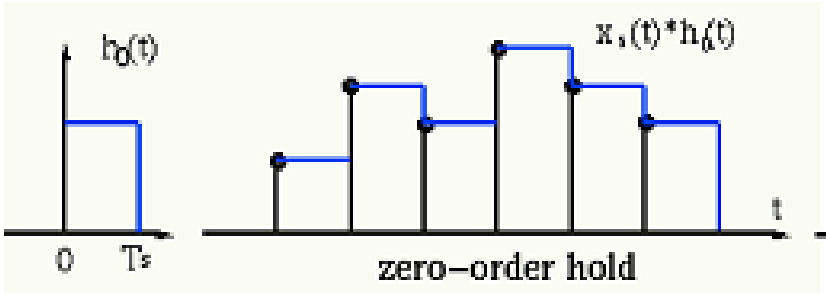
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-jn\omega}$$

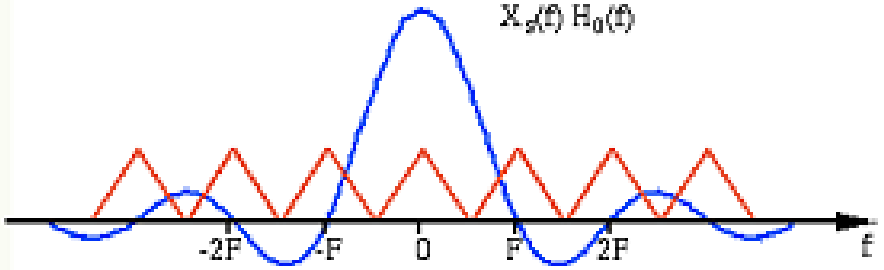
$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega}) \cdot e^{jn\omega} d\omega$$

Reconstruction of Signals

- The reconstruction of the continuous signal from its samples can be realized in either frequency domain or time domain.
- In time domain, the reconstruction is implemented by convolution process with some function $h(t)$ to fill the gaps between the discrete samples.
- Such process corresponds to certain low-pass filtering with $H(f) = F|h(t)$ in frequency domain.
- This is then to attenuate all replicas of the signal spectrum $X(f)$ except the middle one (e.g. low frequency components of the signal).

Zero-Order Hold





- First consider a square impulse function:

$$h(t) = \begin{cases} 1, & [t] < t_0/2 \\ 0, & \text{otherwise} \end{cases}$$

- Its Fourier spectrum is a sinc function:

$$H_0(f) = \mathcal{F}[h(t)] = \frac{\sin(\pi f t_0)}{\pi f}$$

Zero-Order Hold

- The impulse response of a zero-order hold filter is the same as the square impulse, but right-shifted by $t_0/2$:

$$h_0(t) = \begin{cases} 1, & 0 \le t < t_0 \\ 0, & \text{otherwise} \end{cases}$$

- The spectrum of zero-order hold is:

$$H_0(f) = \mathcal{F}[h_0(t)] = \left[\frac{\sin(\pi f t_0)}{\pi f}\right] e^{-j2\pi f t_0/2}$$

Zero-Order Hold

- Note its values at some particular frequencies $f = 0$ and $f = kF = k/t_0$:

$$H_0(0) = t_0 = 1/F \quad \text{and} \quad H_0(kF) = 0 \quad k = \pm 1, \pm 2, \cdots$$

- Based on $h_0(t)$, the original signal $x(t)$ can be approximately reconstructed:

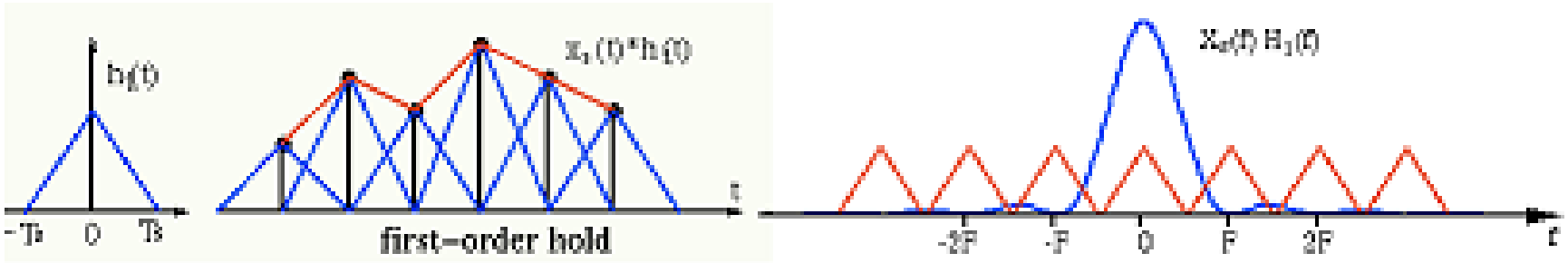
$$\hat{x}_0(t) = h_0(t) * x_s(t)$$

- It is a series of square impulses with their heights modulated by $x[m]$.

Zero-Order Hold

- The convolution process corresponds a filtering process in frequency domain by the sinc function $H_0(f)$.
- This behaves as a poor low-pass filter that attenuates all replicas of the original spectrum $X(f)$.
- This also rescales $X_s(f)$ by a factor of $1/F$ so that its magnitude is back to that of $X(f)$.
- Recall that a factor of F was introduced during the sampling process.

First-Order Hold



- The impulse response of a first-order hold filter is:

$$h_1(t) = \begin{cases} 1 - |t|/t_0, & |t| < t_0 \\ 0, & \text{otherwise} \end{cases}$$

- Note that $h_1(t)$ is related to the square impulse $h(t)$:

$$h_1(t) = \frac{1}{t_0} h(t) * h(t)$$

- Or

$$h_1(t) = Fh(t) * h(t)$$

First-Order Hold

- Correspondingly, in frequency domain, we have:

$$H_1(f) = \mathcal{F}|h_1(t)| = \mathcal{F}[Fh(t) * h(t)] = FH^2(f)$$

- So, the spectrum of first-order hold is:

$$H_1(f) = F\left[\frac{\sin(\pi f t_0)}{(\pi f)^2}\right]^2$$

- Note its values at some particular frequencies (e.g. $f = 0$ and $f = kF = k/t_0$):

$$H_1(0) = \frac{F}{F^2} = \frac{1}{F}$$

Where: $H_0(kF) = 0$ for $k = \pm 1, \pm 2, \ldots$

First-Order Hold

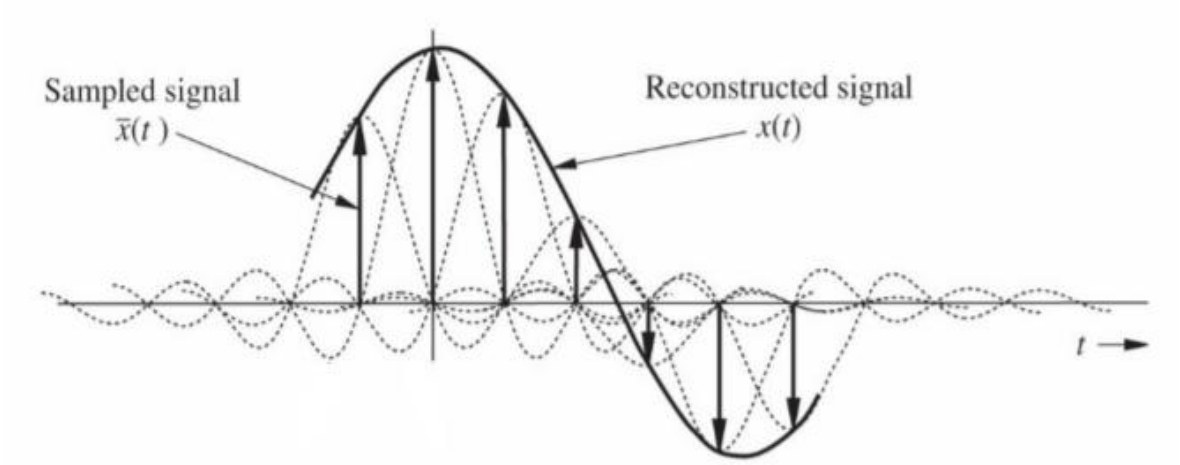
- Based on $h_1(t)$, the original signal $x(t)$ can be approximately reconstructed:

$$\hat{x}(t) = h_1(t) * x_s(t)$$

- This is the linear convolution of the sample train.
- It is a straight line connecting every two consecutive samples.
- This interpolation corresponds a low-pass filtering in frequency domain by with a scaling factor .

Sinc Function

- Reconstruction often uses the sinc function for each sampling impulse.
- Result of reconstruction is relatively more accurate than the zero-order and first-order methods.



Sinc Function

- The sinc interpolation formula is defined as:

$$x(t) = \sum_{n=-\infty}^{+\infty} x_n \cdot \mathrm{sinc}\left[\frac{\pi}{T}(t - nT)\right]$$

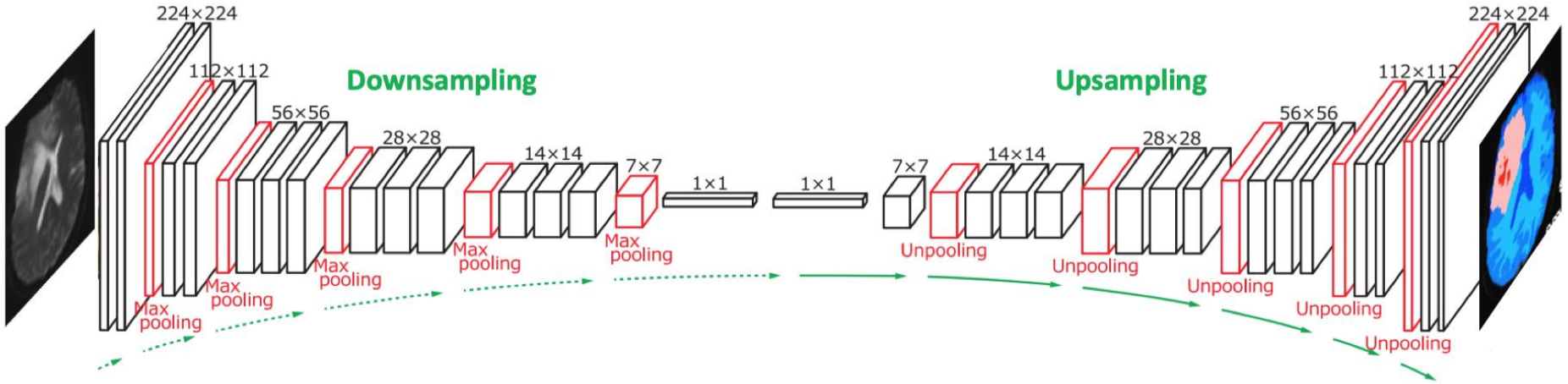
  Where: T is the sampling period used to determine x_n from the original signal, and $x(t)$ is the reconstructed signal.

- The function $\mathrm{sinc}(x)$ is defined by $\mathrm{sinc}(x) = \sin(x)/x$ for $\neq 0$, with $\mathrm{sinc}(0) = 1$.

- The above formula represents a linear convolution between the sequence x_n and scaled and shifted samples of the sinc function.

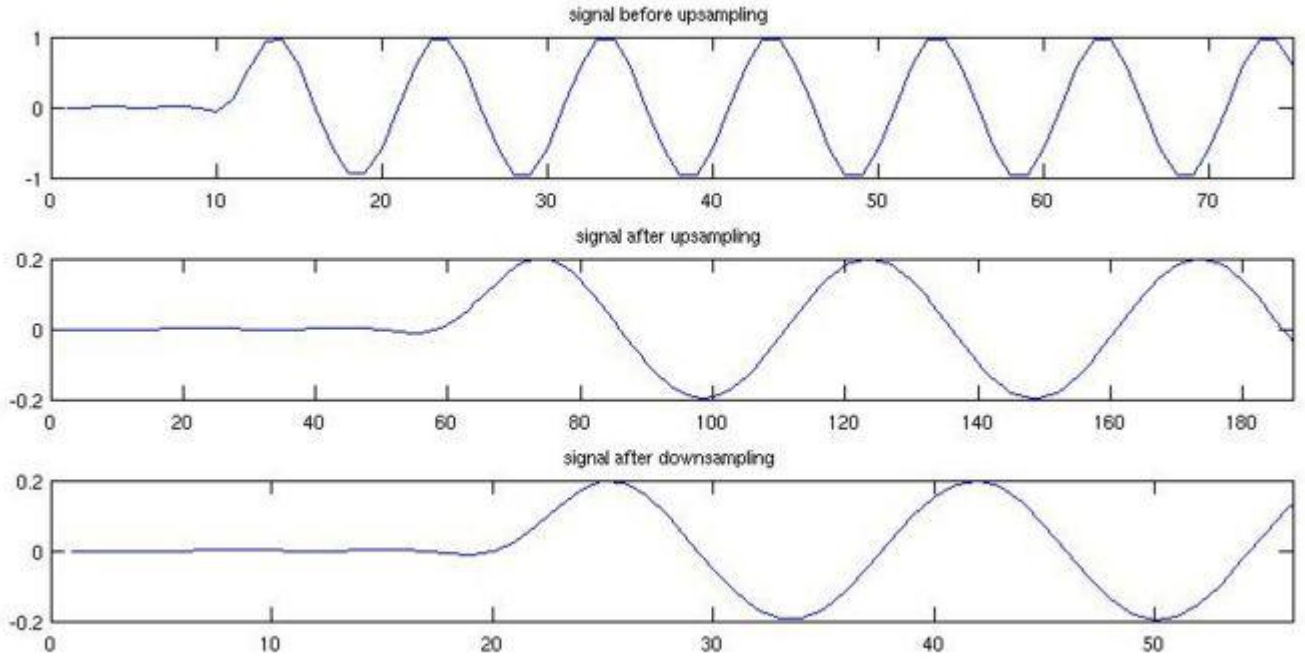
Decimation and Interpolation

- Two topics which are related very closely to discrete-time sampling and reconstruction are the down sampling (decimation) and up sampling (interpolation) methods.
- Very common for data compression application in the digital communication and signal transmissions in the telecommunication.



Decimation and Interpolation

- Down sampling (decimation) is process of reducing the sampling rate. In practice, this usually implies low-pass-filtering a signal, then throwing away some of its samples.
- Up sampling (interpolation) is the process of inserting zero-valued samples between original samples to increase the sampling rate (this is called “zero-stuffing”).



Decimation and Interpolation

Down sampling (decimation):

- Reduce the sampling rate of a discrete-time signal.
- Low sampling rate reduces storage and computation requirements.

Up sampling (interpolation):

- Increase the sampling rate of a discrete-time signal.
- Higher sampling rate preserves fidelity.

Decimation and Interpolation

- Given a discrete-time sequence samples which are resampled at different sampling condition using down sampling or up sampling.
- When sampled sequence is typically replaced by sequence $x_b[n]$, which is simply every Nth value of $x_p[n]$, that is:

$$x_b[n] = x_p[nN]$$

- Or, equivalently

$$x_b[n] = x[nN]$$

- In this case, $x_p[n]$ and $x[n]$ are equal at the integer multiples of N.

Decimation and Interpolation

- To determine the effect in the frequency domain of decimation, we wish to establish the relationship between $X_b(e^{j\omega})$ i.e. the Fourier transform of $X_b[n]$ and $X(e^{j\omega})$.

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_b[k] \cdot e^{-j\omega k}$$

- Knowing that $x_b[n] = x_p[nN]$

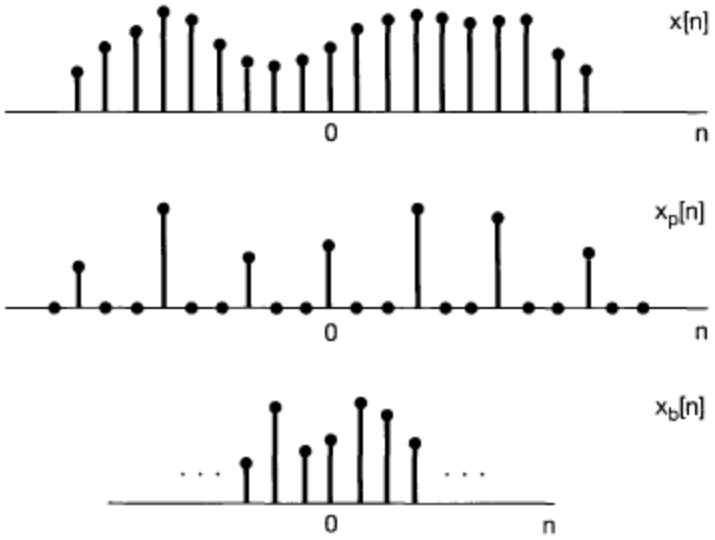
$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_p[kN] \cdot e^{-j\omega k}$$

Decimation and Interpolation

- If we let $n = kN$, or equivalently $k = n/N$, we can write:

$$X_b(e^{j\omega}) = \sum_{n}^{\infty} x_p[n] \cdot e^{-j\omega n/N}$$

Where: n = integer multiple of N



Decimation and Interpolation

- If n is not an integer multiple of N, we can also write:

$$X_b\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} x_p[n] \cdot e^{-j\omega n/N}$$

- Evaluating the right-hand side of the equation above as the Fourier transform of $x_p[n]$

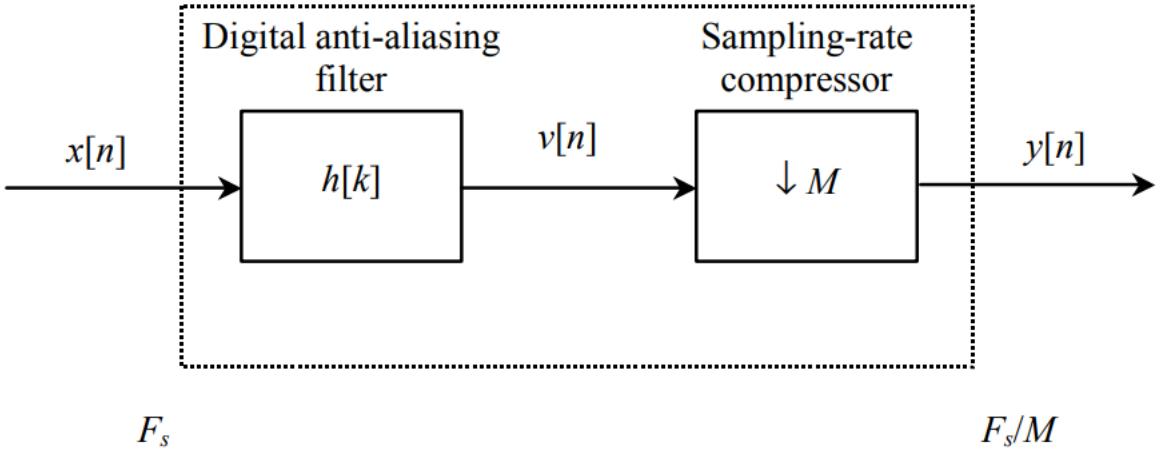
$$\sum_{n=-\infty}^{+\infty} x_p[n] \cdot e^{-j\omega n/N} = X_p\left(e^{j\omega/N}\right)$$

- Thus, we can conclude that the Fourier transform:

$$X_b\left(e^{j\omega}\right) = X_p\left(e^{j\omega/N}\right)$$

Down Sampling (Decimation)

- Down sampling (decimation): its sampling rate is reduced from F_s to F_s/M by discarding $M-1$ samples for every M samples in the original sequence.
- Decimation factor M: M = input rate/output rate

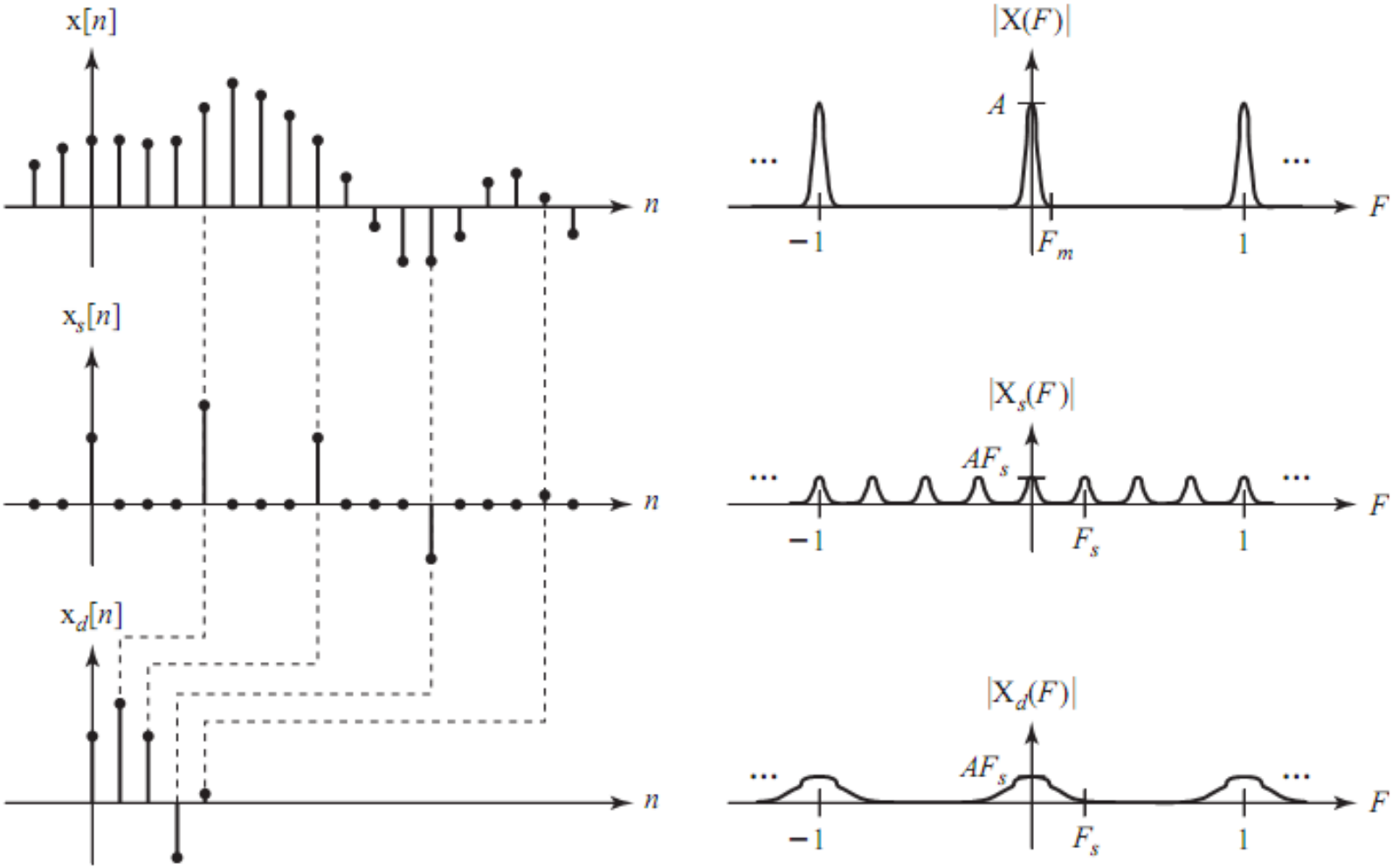


- For M-fold decimation, or down-sampling,

$$y[n] = v[nM] = \sum_{k=-\infty}^{\infty} h[k]\, x[nM-k]$$

Down Sampling (Decimation)

- Down sampling (decimation) of a signal: magnitude (left) and spectral density (right).



Down Sampling (Decimation)

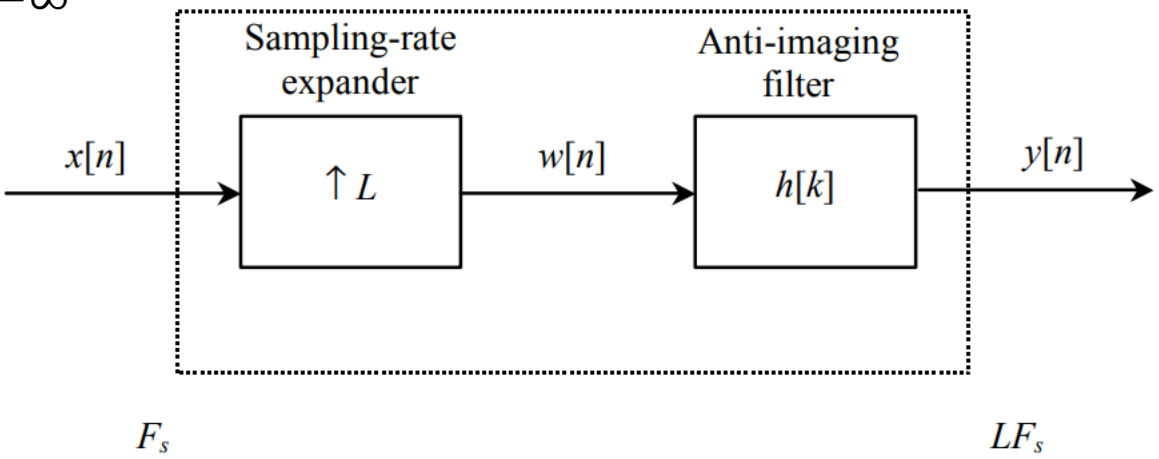
- As shown in the figure above, down sampling (decimation) by a factor of 1/5 times
- Reducing the sampling rate of discrete-time signal by a discrete-time sampling signal by a factor of M = 5.
- Sampled decimated signal ($x_d(t)$) appears at 5 x higher frequency.
- Looking at the decimated frequency spectrum, the frequency spectrum of decimated sampling signal appears as being scaled by a factor of 5.

Up Sampling (Interpolation)

- Up sampling (interpolation) works by inserting $(L-1)$ zero-valued samples for each input sample. The sampling rate therefore increases from F_S to LF_S.
- Interpolation factor L: L = output rate/input rate.

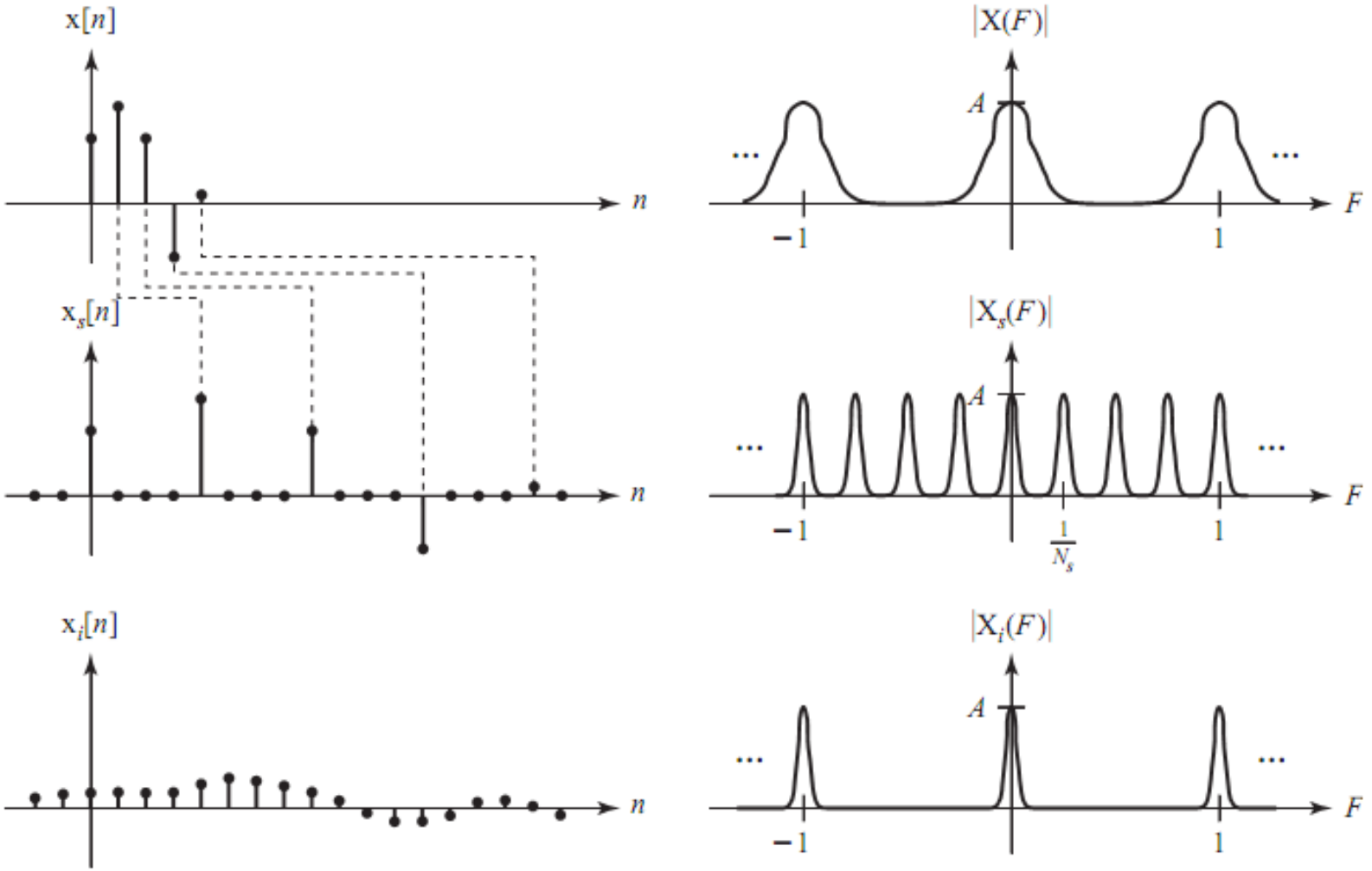
- For L-fold interpolation: Where:

$$y[n] = L \sum_{k=-\infty}^{\infty} h[k]\, w[n-k] \qquad w[n] = \begin{cases} x[n/L], & n/L = \text{int} \\ 0, & n/L = \text{nonint} \end{cases}$$



Up Sampling (Interpolation)

- Up sampling (interpolation) of a signal: magnitude (left) and spectral density (right).

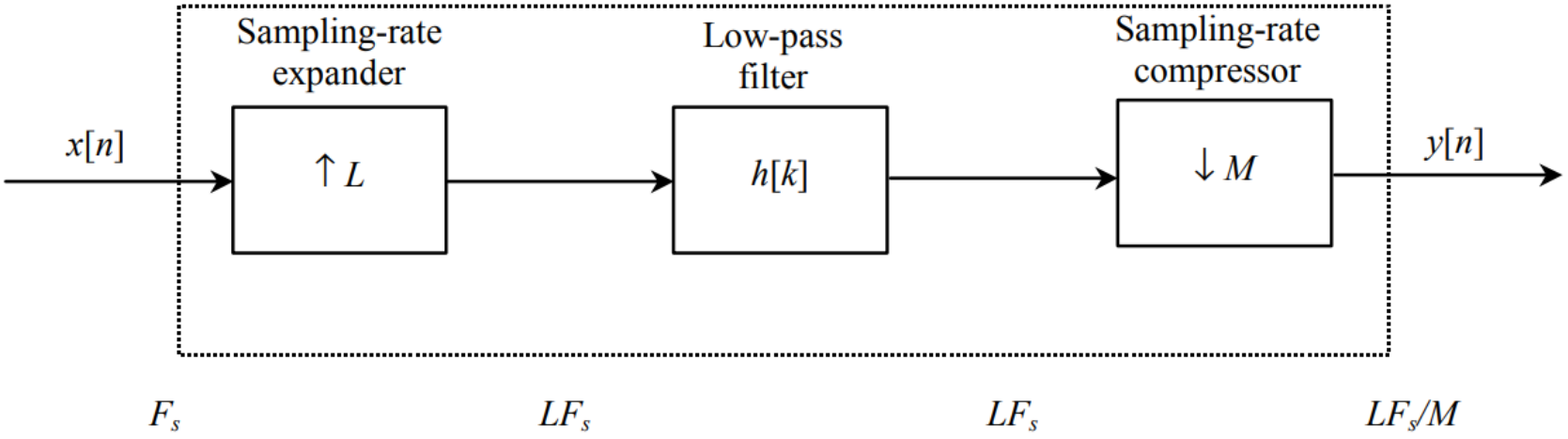


Up Sampling (Interpolation)

- As shown in the figure above, an up sampling (interpolation) by a factor of 5 times.
- Increasing the sampling rate of discrete-time signal by a discrete-time sampling signal by a factor of L = 5.
- Sampled interpolated signal ($x_i(t)$) appears at 5 x lower frequency.
- Looking at the interpolated frequency spectrum, the frequency spectrum of interpolated sampling signal appears as if being scaled by a factor of 1/5.

Sampling Rate Conversion

- Common use of decimation and interpolation is for sampling-rate conversion
- A digital signal $x[n]$ is sampled at an interval T_1, and we wish to obtain a signal $y[n]$ sampled at an interval T_2.
- Providing the ratio T_1/T_2 is a rational number i.e. L/M



Sampling Rate Conversion

- Accomplished by L-fold expansion, followed by low-pass filtering and then M-fold decimation, as in the figure above.
- Interpolation should be performed first and decimation second, to preserve the desired spectral characteristics of $x[n]$.
- Cascading the two in this manner, both of the filters can be combined into one single low-pass filter.
- An example of sampling-rate conversion is for data conversion from a CD format (f_s = 44.1 kHz) is transferred onto a DAT format (f_s = 48 kHz).

Sampling Rate Conversion

- Sampling-rate is increased from 44.1 kHz to 48 kHz - the non-integer factor has to be approximated by a rational number:

$$\frac{L}{M} = \frac{48\ \text{kHz}}{44.1\ \text{kHz}} = \frac{160}{147} = 1.08844$$

- Sampling-rate conversion is achieved by interpolating by L i.e. from 44.1 kHz to [44.1 kHz x 160] = 7056 kHz.

- Then, decimating by M i.e. from 7056 kHz to [7056 kHz/147] = 48 kHz.