

Sampling and Reconstruction (Aliasing)

XMUT220 Signals and Systems

Topics

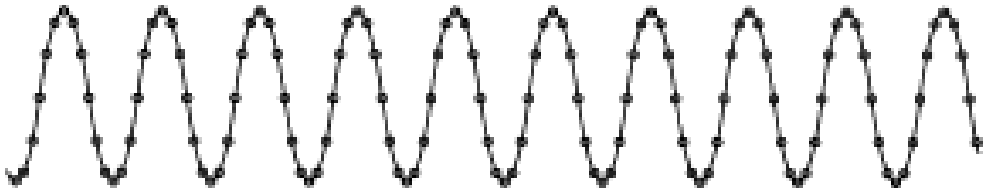
- Fundamental of aliasing.
- Why does aliasing occur?
- Why is aliasing is a problem?
- Preventing aliasing.
- Examples of aliasing.
- Aliasing in the time domain.
- Aliasing in the frequency domain.

Aliasing

- Aliasing:

  “Each of a set of signal frequencies that, when sampled at a given uniform rate, would give the same set of sampled values, and thus might be incorrectly substituted for one another when reconstructing the original signal”.

- Aliasing is when a sinusoid of one frequency takes on the appearance or identity of a different frequency sinusoid.



Adequately Sampled Signal



Aliased Signal Due to Undersampling

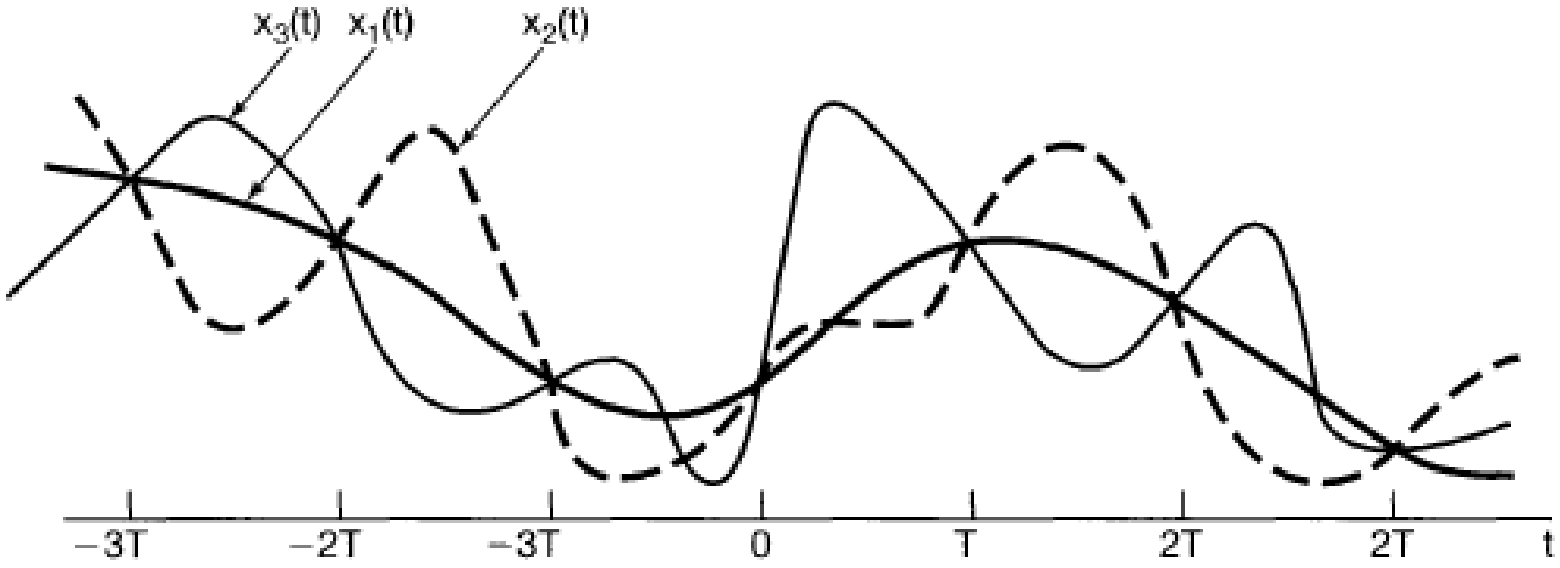
Why Does Aliasing of Signals Occur?

- Similarly, aliasing is always a possibility when sampling a signal, and simple measures should be taken to prevent it from occurring.
- Signals, given the opportunity, will always alias.
- The fundamental reason for aliasing of signals is the fact that discrete-time sinusoids are not unique functions of frequency.
- Sinusoids of frequency ω_1 and $\omega_1 + k2\pi$ are identical for all integers k.
- This is because the sample index n is always integer valued. For example:

$$\cos\left(\frac{\pi}{4}\right) = \cos\left(\frac{9\pi}{4}\right) = \cos\left[\left(\frac{\pi}{4}\right)n + 2\pi n\right] = \cos\left[\left(\frac{\pi}{4}\right)n\right]$$

Why Does Aliasing of Signals Occur?

- In the figure below we illustrate three different continuous-time signals, all of which have identical values at integer multiples of T.
- That is, clearly, an infinite number of signals can generate a given set of samples.



Why Does Aliasing of Signals Occur?

- This property does not apply to continuous-time sinusoids because time is not limited to integers, but takes on a continuum of values.
- Hence, discrete-time sinusoids are only unique over a 2π interval of frequency.
- Typically, we work with the range of discrete-time frequencies $-\pi < \omega \leq \pi$.
- Aliasing of signals can occur when we sample e.g. convert a continuous-time signal to a discrete-time signal.
- Continuous-time sinusoids with distinct frequencies are always unique.

Why Does Aliasing of Signals Occur?

- When sampling converts a continuous-time sinusoid to a discrete-time sinusoid, the discrete-time sinusoid does not inherit the uniqueness of its continuous-time predecessor.
- Consider two continuous-time sinusoids:

$$x_1(t) = \cos\left[\left(\frac{\pi}{2}\right)t\right]$$

and

$$x_2(t) = \cos\left[\left(\frac{9\pi}{2}\right)t\right]$$

- These sinusoids have distinct frequency and thus are distinct.

Why Does Aliasing of Signals Occur?

- However, if we sample them at intervals of $T_s = 0.5$ second, then:

$$x_1[n] = x_1(0.5n) = \cos\left[\left(\frac{\pi}{4}\right)n\right]$$

while

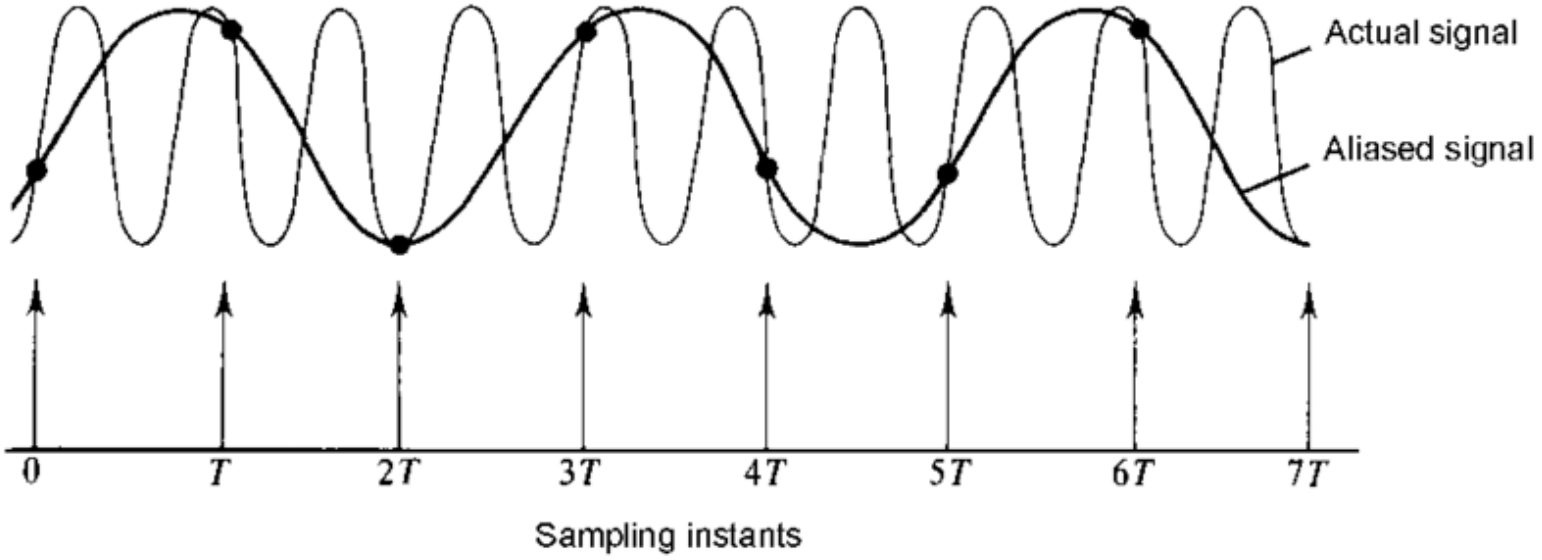
$$x_2[n] = x_2(0.5n) = \cos\left[\left(\frac{9\pi}{4}\right)n\right]$$

- The discrete-time frequency of $x_1[n]$ and $x_2[n]$ differ by 2π and thus $x_1[n] = x_2[n]$.
- We say that the sinusoid with continuous-time frequency $9\pi/2$ rads/sec has "aliased" to the same discrete-time frequency ($\pi/4$) as the sinusoid with continuous-time frequency $\pi/2$ rads/sec.

Why is Aliasing of Signals a Problem?

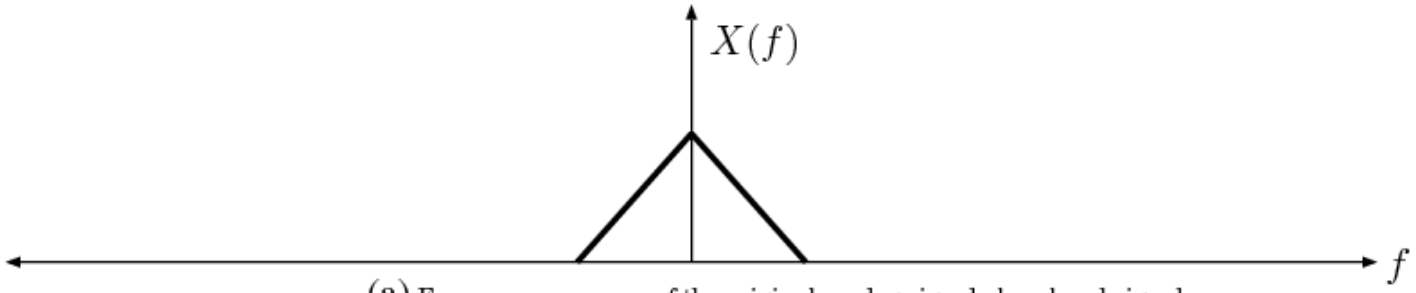
- If two (or more) distinct continuous-time sinusoids contribute to the same discrete-time sinusoid, we have no way of distinguishing the original continuous-time sinusoids.
- There is no way to determine the amplitude and phase to assign to the aliased continuous-time sinusoids if we are trying to analyse the signal or convert back to a continuous-time signal.
- For example, if we observe that the amplitude of the discrete-time sinusoid at a particular frequency is 2, all we know is that the sum of an unknown number of continuous-time sinusoids produces an amplitude of 2.

Why is Aliasing of Signals a Problem?

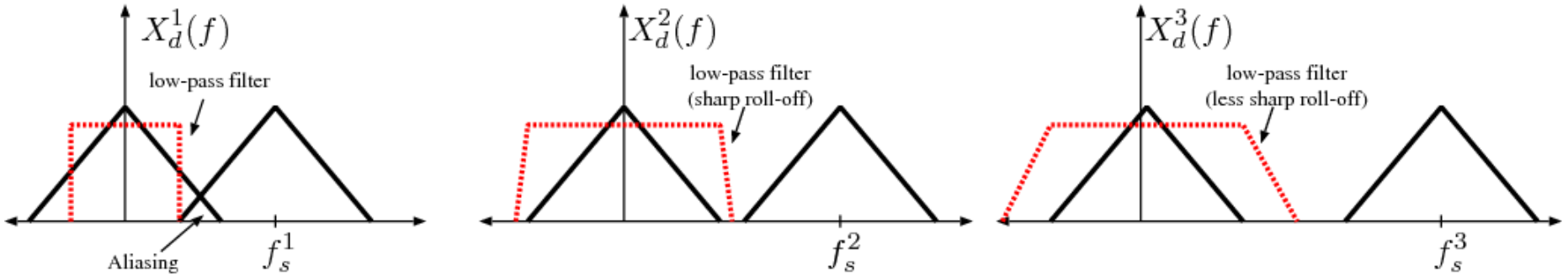


- There are infinite combinations of continuous-time sinusoids that could sum to produce an amplitude of 2, and there is no way to uniquely determine their amplitudes.

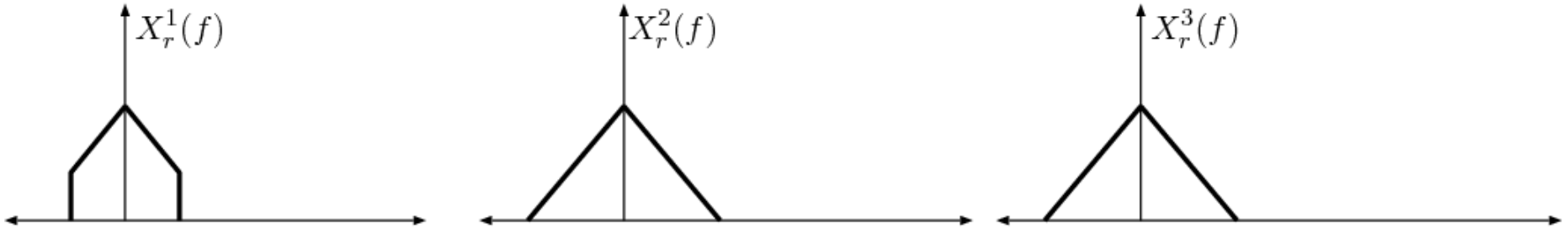
Why is Aliasing of Signals a Problem?



(a) Frequency response of the original analog signal : baseband signal



(b) Frequency responses of the discretized signal of (a) with different sampling frequences



(c) Frequency responses of recovered signal by low-pass filtering

Why is Aliasing of Signals a Problem?

From results of the frequency spectra of continuous-time signal sampling:

- Original signal
- Sampled signal with different sampling frequencies.
- Recovered signal by low-pass filtering.

From frequency spectra of aliased and non-aliased signals:

- Aliased reconstructed signal has different frequency spectrum than the original continuous-time signal.
- Non-aliased reconstructed signal will have the frequency spectrum of original continuous-time signal.

Preventing Aliasing of Signals

- The most common way to prevent aliasing is to limit the range of continuous-time sinusoids, so there is a unique or one-to-one mapping between continuous- and discrete-time frequency.
- If we use the range $-\pi < \omega \leq \pi$ for discrete-time frequency, then the corresponding range of continuous-time frequency is:

$$-f_s/2 < f \leq f_s/2$$

  Where: $f_s = 1/T_s$ is the sampling frequency in Hz.
- This result immediately follows from the fact that:

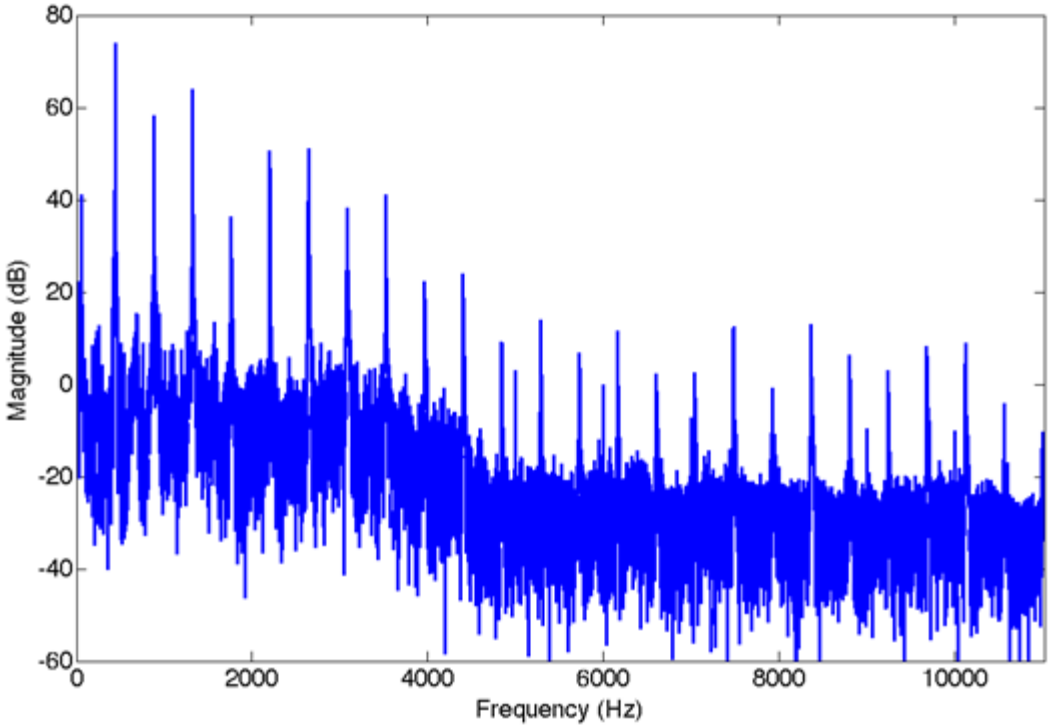
$$\omega = 2\pi f T_s.$$

Preventing Aliasing of Signals

- This leads us to the Nyquist Sampling Theorem, which states that there is a unique correspondence between discrete- and continuous-time sinusoids.
- It experiences no aliasing if the sampling frequency is chosen to exceed twice the highest continuous-time frequency present in the signal.
- In this case, one can uniquely reconstruct the continuous-time signal from its samples.
- In practice, the presence of noise and interference, typically makes it impossible to know in advance the highest frequency present in the continuous time signal.
- A continuous-time low-pass filter is applied to the data before sampling to limit the bandwidth to a desired range. This is called an anti-aliasing filter.

Example of Aliasing

- An aliasing example is a saxophone playing the sound of note 'A'.
- The frequency of note ‘A’ is at 440 Hz.



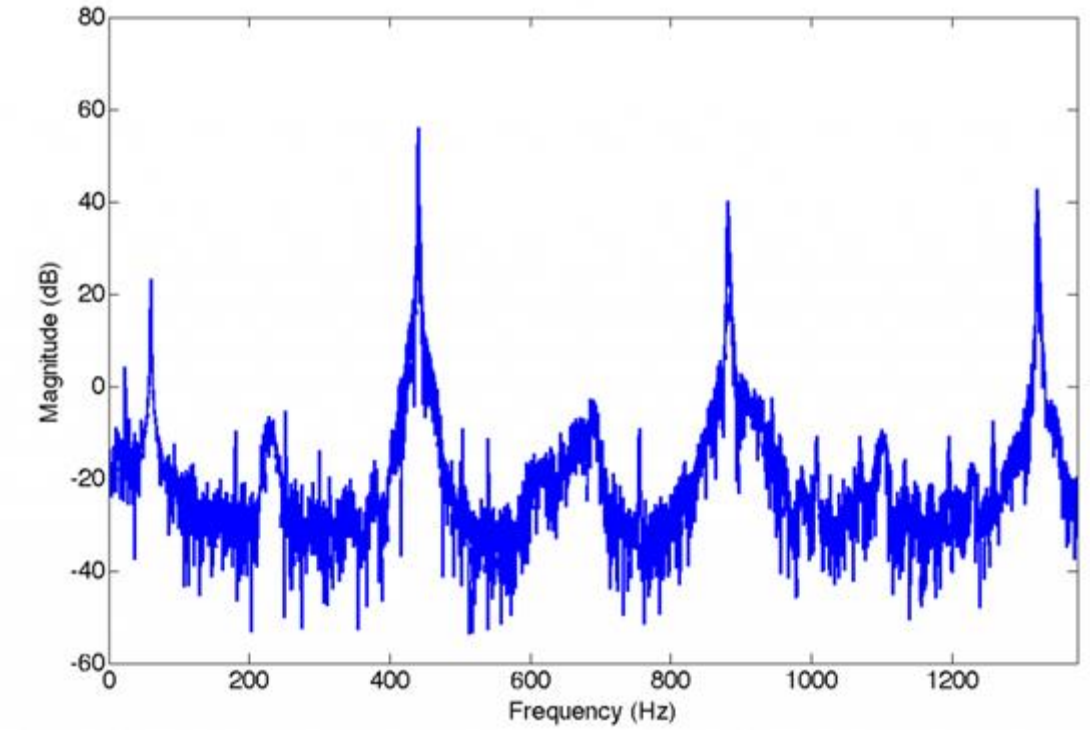
- As shown above, notice on the peaks at periodic intervals in the frequency spectrum plot of the sampled sound signal.

Example of Aliasing

- The spectrum of this note, shown in the previous figure, exhibits spectral peaks at harmonics, i.e. integer multiples of 440 Hz.
- Notice the peaks at frequencies: 440 Hz, 880 Hz, 1,320 Hz, etc.
- The amplitude of the harmonics is quite small above 4 kHz.
- The spectral peak near 0 Hz occurs at 60 Hz and is due to power line noise contaminating the recording.
- Then, we sample this signal at 2,756 Hz with and without an antialiasing filter.

Example of Aliasing

- We sample this signal at 2,756 Hz with an anti aliasing filter.
- Shown in the figure below is the frequency spectrum of the sound of the ‘A’ note.



Example of Aliasing

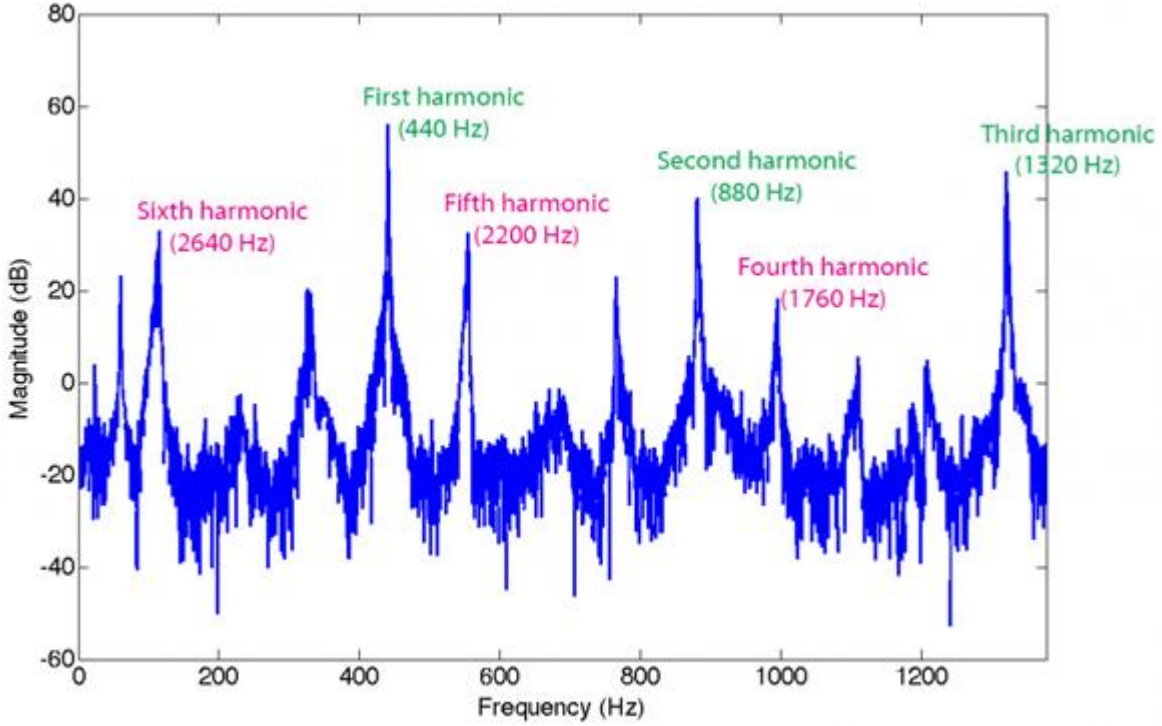
- At this sampling frequency the maximum continuous-time sinusoid frequency that can be represented without aliasing is:

$$F(\text{sampling}) = F(\text{max})/2 = 2{,}756/2 = 1{,}378 \text{ Hz}$$

- Notice the peaks at those harmonics in the spectrum at:
  - 440 Hz (1st harmonic),
  - 880 Hz (2nd harmonic), and
  - 1,320 Hz (3rd harmonic).
- Thus, the antialiasing filter limits the signal to the first three harmonics.

Example of Aliasing

- The following figures depict the spectrum of the sampled sound without the antialiasing filter applied before sampling.



- The aliased fourth, fifth, and sixth harmonics are labelled in figure above.

Example of Aliasing

- Furthermore, you can also find the frequency spectrum plot the spectral peaks at 324 Hz and 764 Hz which are aliases of the seventh and eighth harmonics.
- Note that in general a sinusoid of frequency f_0 aliases to $kf_s - f_0$ and $f_0 - mf_s$ where k, m are integers.
- A wind instrument like a saxophone creates a very tonal sound because the only significant contributors are harmonics of the fundamental.
- Aliasing introduces energy at frequencies which are not harmonically related to the fundamental, and introduces a muddy character to the sound, or a buzzing aspect.

Aliasing Example in Time Domain

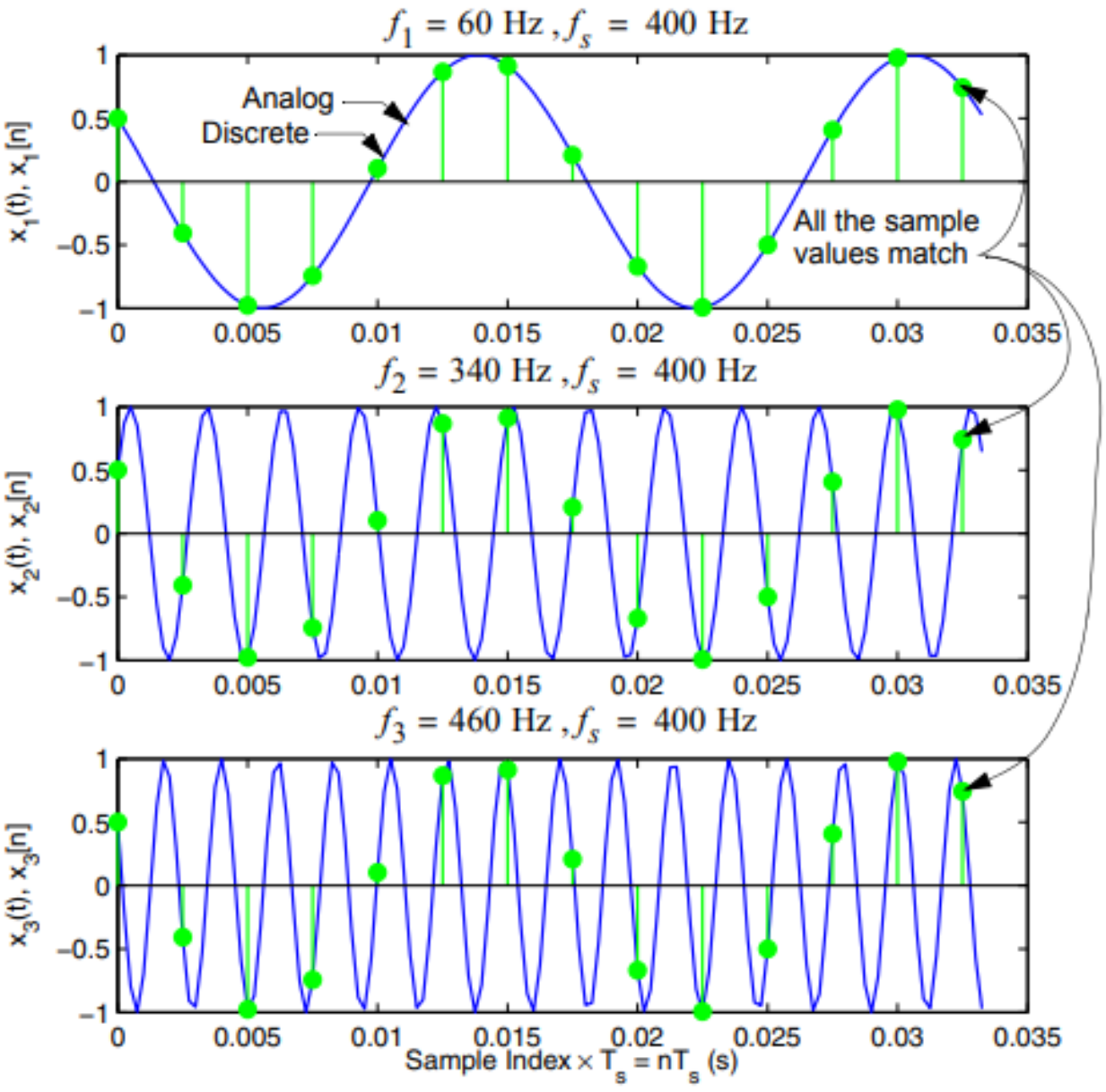
- We are sampling sinusoid signal inputs with the following frequencies: 60 Hz, 340 Hz, or 460 Hz with sampling frequency, f_s = 400 Hz
- The analogue signals are:
  - $x_1(t) = \cos(2\pi 60t + \pi/3)$
  - $x_2(t) = \cos(2\pi 340t - \pi/3)$
  - $x_3(t) = \cos(2\pi 460t + \pi/3)$
- We sample $x_i(t)$, $i = 1, 2, 3$ at sampling rate f_s = 400 Hz.

Aliasing Example in Time Domain

- This example further illustrates the signals sampling and aliasing/folding in the time domain.
- Notice the characteristics of the signals in time-domain plots:
  - For $x_1(t) = \cos(2\pi 60t + \pi/3)$, the sampled signal looks like a discrete signal with frequency of 60 Hz.
  - For $x_2(t) = \cos(2\pi 340t - \pi/3)$, the sampled signal looks like a discrete signal with frequency of 60 Hz.
  - For $x_3(t) = \cos(2\pi 460t + \pi/3)$, the sampled signal looks like a discrete signal with frequency of 60 Hz.
- Aliasing phenomenon is happening in $x_2(t)$ and $x_3(t)$ signals.

Aliasing Example in Time Domain

- Time-domain plots of signal samplings:
  - $f_1 = 60$ Hz
  - $f_1 = 340$ Hz
  - $f_1 = 460$ Hz
- Note: Sampling frequency is 400 Hz



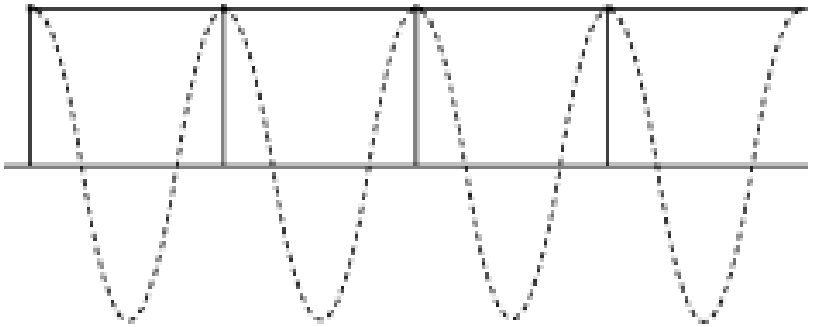
Aliasing Example in Time Domain

We illustrate the time domain plots of the case study shows a continuous time signal with frequency = 1 kHz to be sampled with a sampling signal.

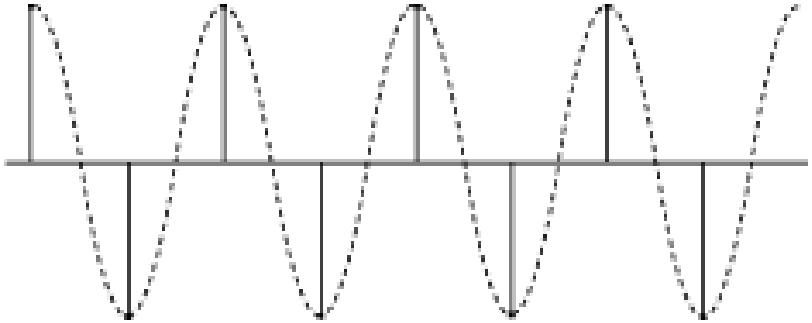
Then, the sampling rate of the sampling signal is increased from:

a. The rate of 1 sample per cycle.
b. The rate of 2 samples per cycle.
c. The rate of 7/4 samples per cycle.
d. The rate of 10 samples per cycle.

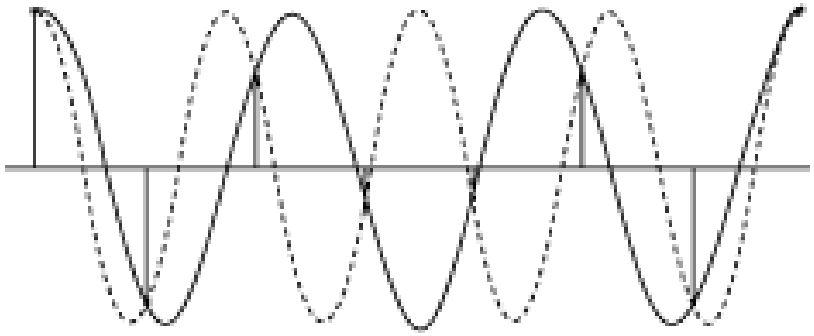
Aliasing Example in Time Domain



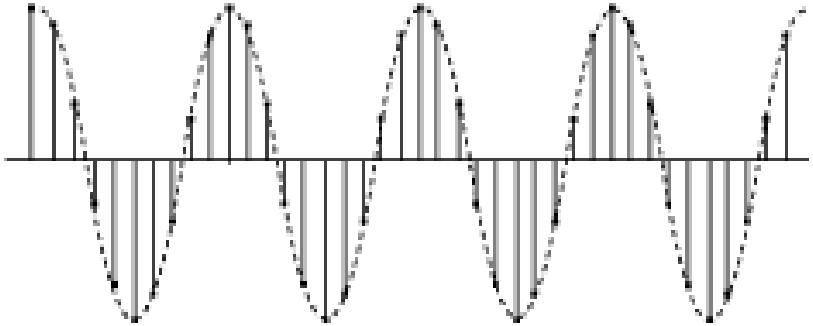
A. 1 sample/1 cycle



C. 2 samples/cycle



B. 7 samples/4 cycles



D. 10 samples/cycle

Aliasing Example in Time Domain

- When it is 1 sample/cycle, the sampled signal appears in (a) as a flat line signal at a specific amplitude.
- When it is 7/4 samples/cycle, the sampled signal in (b) appears as a triangular signal that has been shifted (or modulated) by a specific phase angle.
- When it is 2 samples/cycle, the sampled signal in (c) appears as a triangular signal with a constant magnitude.
- When it is 10 samples/cycle, the sampled signal in (d) appears as a sinusoidal signal with the given specified frequency.

Aliasing Example in Frequency Domain

- Given an aliasing case study showing a sinusoidal signal is being sampled at $f = 6$ samples/second, while its frequency f is increased from 1 Hz to 12 Hz.

$$x(t) = \cos(2\pi f t + \emptyset)$$

- We will evaluate the sampled signals in the time domain and frequency domain for the following sampling frequency cases:

(a) 1 Hz (b) 2 Hz (c) 3 Hz (d) 4 Hz

(e) 5 Hz (f) 6 Hz (g) 7 Hz (h) 8 Hz

(i) 9-12 Hz

Aliasing Example in Frequency Domain

- Consider the continuous signal:

$$x(t) = \cos(2\pi f t + \emptyset)$$

- Applying Euler theorem formulae to represent the given signal as above

$$x(t) = \frac{1}{2}\left[e^{j(2\pi f t+\emptyset)} + e^{-j(2\pi f t+\emptyset)}\right]$$

- Then,

$$x(t) = c_1 e^{j2\pi f t} + c_{-1} e^{-j2\pi f t}$$

Where: $c_1 = \frac{1}{2} e^{j\emptyset}$ and $c_{-1} = \frac{1}{2} e^{-j\emptyset}$

- These c_1 and c_2 coefficients are the two non-zero coefficients for the frequency components $e^{j2\pi f t}$ and $e^{-j2\pi f t}$.

Aliasing Example in Frequency Domain

- When this signal is sampled at a rate of $F = 1/t_0$, it becomes a discrete signal:

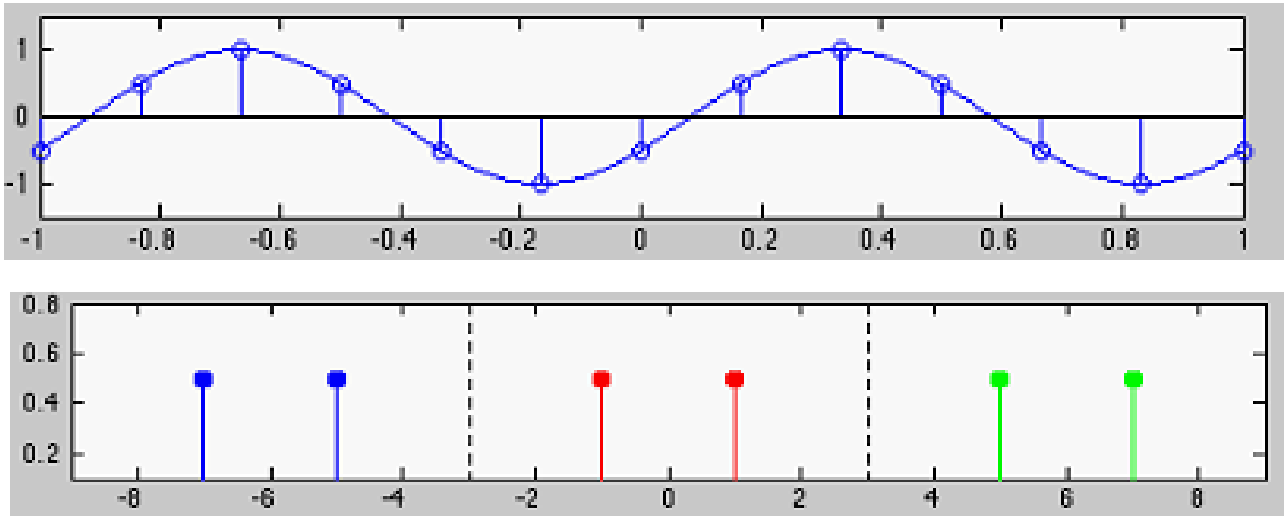
$$x|m| = \cos(2\pi f_m t_0 + \emptyset)$$
$$= \cos(2\pi f_m / F + \emptyset)$$

- This discrete signal can be represented as follow:

$$x|m| = \left(\frac{e^{j\emptyset}}{2}\right) e^{j2\pi f_m/F} + \left(\frac{e^{-j\emptyset}}{2}\right) e^{-j2\pi f_m/F}$$
$$= c_1 e^{j2\pi f_m/F} + c_{-1} e^{-j2\pi f_m/F}$$

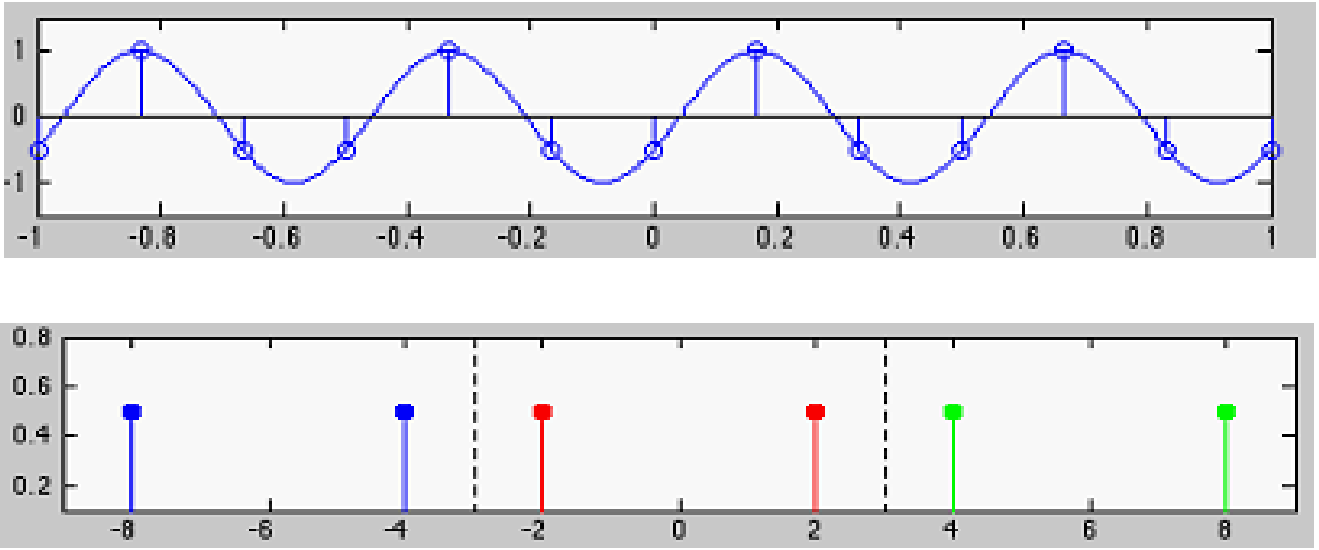
Aliasing in Frequency Domain ($f = 1$)

- Notice the discrete sampled signal and reconstructed signal in the time domain plot and its frequency spectra in the frequency domain plot.
- When $f = 1$, in this case $f < F/2 = 3$, the two non-zero frequency components are present.
- The c_1 and c_{-1} are inside the middle period $-F/2 < f < F/2$ of the spectrum.



Aliasing in Frequency Domain ($f = 2$)

- When $f = 2$, in this case $f < F/2 = 3$, the two non-zero frequency components moved away from the y-axis.
- The c_1 and c_{-1} are still inside the middle period, although now they have moved further outward.

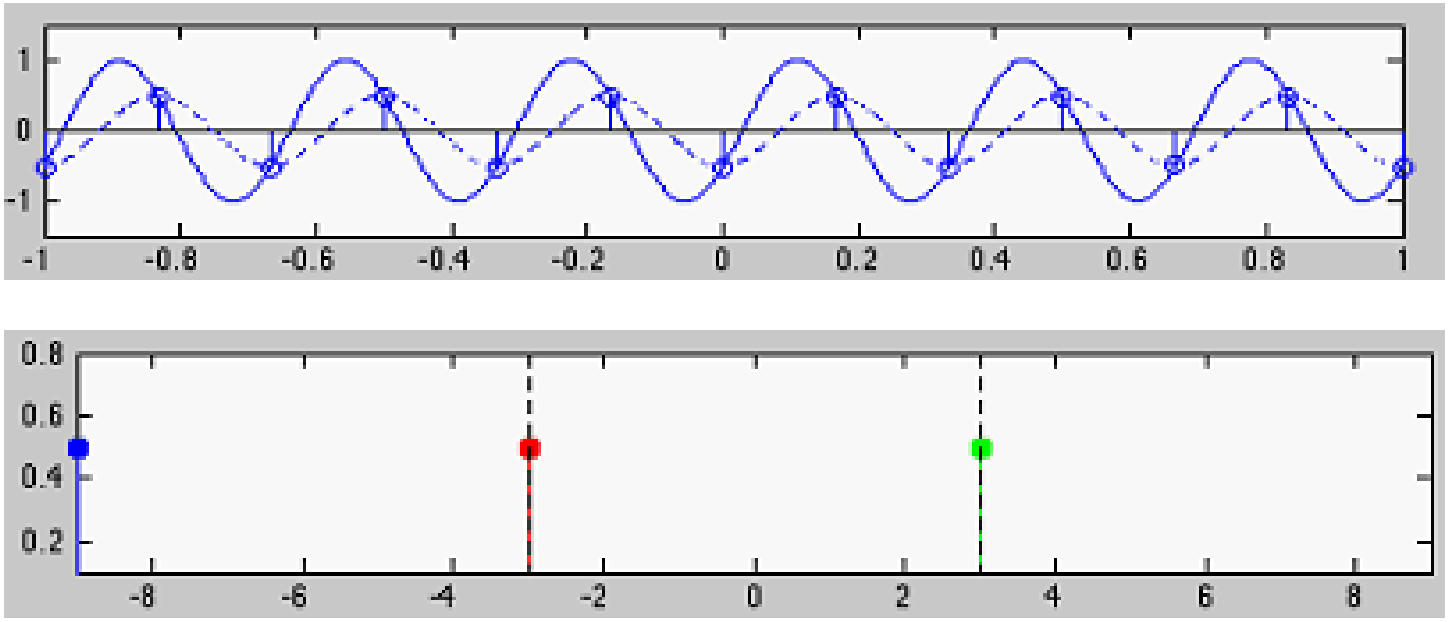


Aliasing in Frequency Domain ($f = 3$)

- When $f = 3$, in this case $f = F/3$, the two non-zero frequency components move to be on top of the sampling boundaries.
- Depending on the relative phase difference between the signal and the sampling process, the signal is distorted to different amount.
- In the worst case, when the two samples in a signal period happen to be at the zero-crossings, same as a zero signal $x(t) = 0$.

Aliasing in Frequency Domain ($f = 3$)

- Note that for f = 3, the spectra of the impulses of the signal are superimposed on the sampling boundary lines e.g. the signal is marginally aliased.
- Notice the dash line (-) in the time domain plot which is the reconstructed signal.



Aliasing in Frequency Domain ($f = 4$)

- When $f = 4$, in this case $f > F/2 = 3$ the signal is:

$$x(t) = \cos(2\pi 4t + \emptyset)$$

- In frequency domain, the coefficients for $f = 4$ and $f = -4$ are:

$$c_1 = e^{j\emptyset}/2$$

and

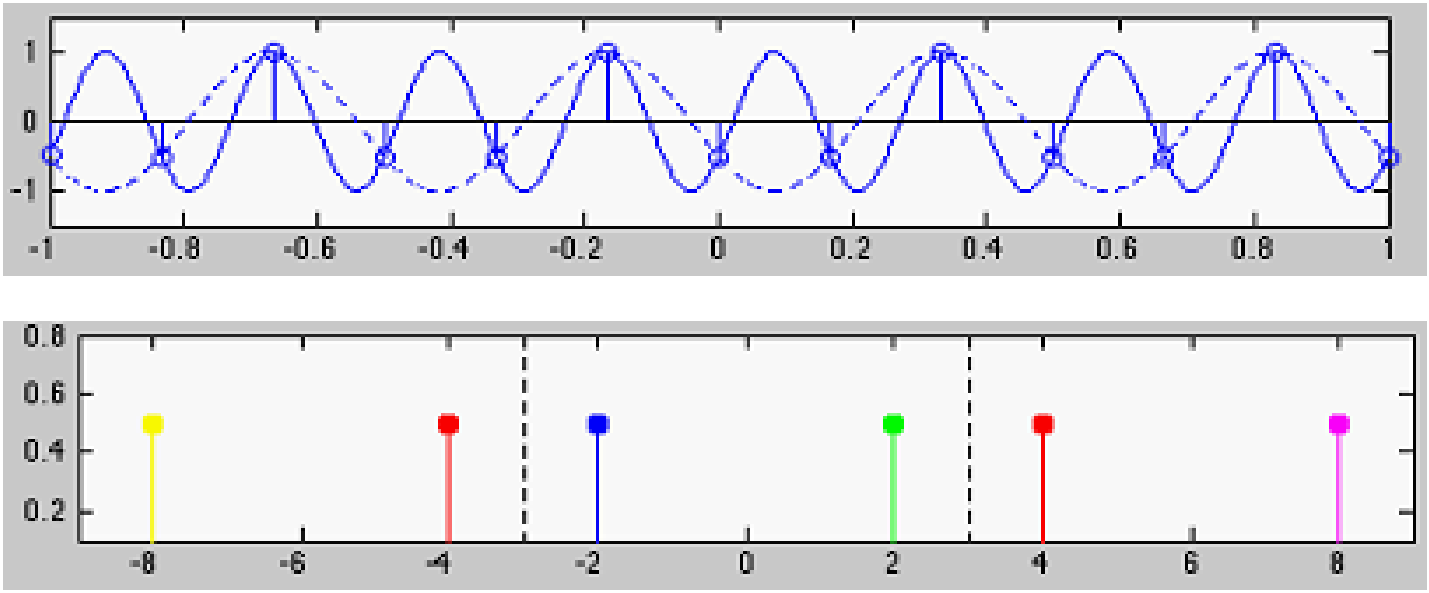
$$c_{-1} = e^{-j\emptyset}/2$$

- These respectively move out of the middle period.
- While their replicas in the two neighbouring periods move inside the middle period.

Aliasing in Frequency Domain ($f = 4$)

Specifically:

- c_1 on the left for the aliased frequency 4 Hz moves in from the left to become coefficient for $-F = 4 - 6 = -2$, and
- c_{-1} for frequency -4 Hz moves in from the right to become coefficient for $-4 + F = 2$.



Aliasing in Frequency Domain ($f = 4$)

- The reconstructed signal based on these coefficients inside the middle period becomes:

$$x'[\mathrm{m}] = c_{-1}\, e^{j2\pi f_m/F} + c_1 e^{-j2\pi f_m/F}$$

$$= \left(\frac{e^{-j\emptyset}}{2}\right) e^{j2\pi f_m/F} + \left(\frac{e^{j\emptyset}}{2}\right) e^{-j2\pi f_m/F}$$

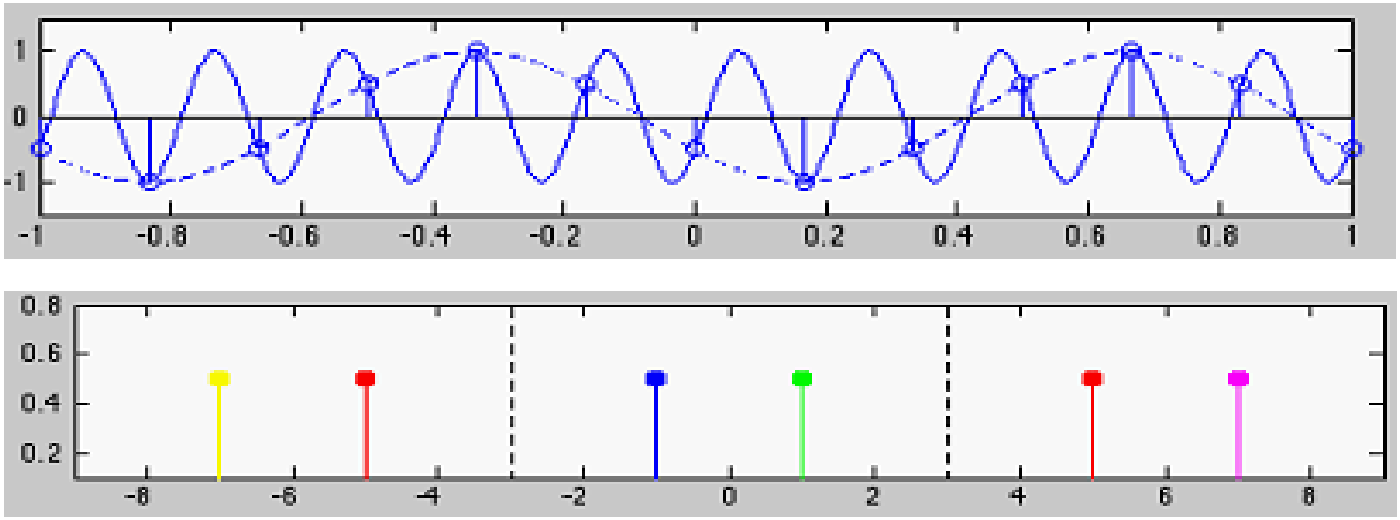
$$= \cos(-2\pi f_m t_0 - \emptyset)$$

$$= \cos(-(2\pi f_m t_0 + \emptyset))$$

- This phenomenon is called aliasing or folding.

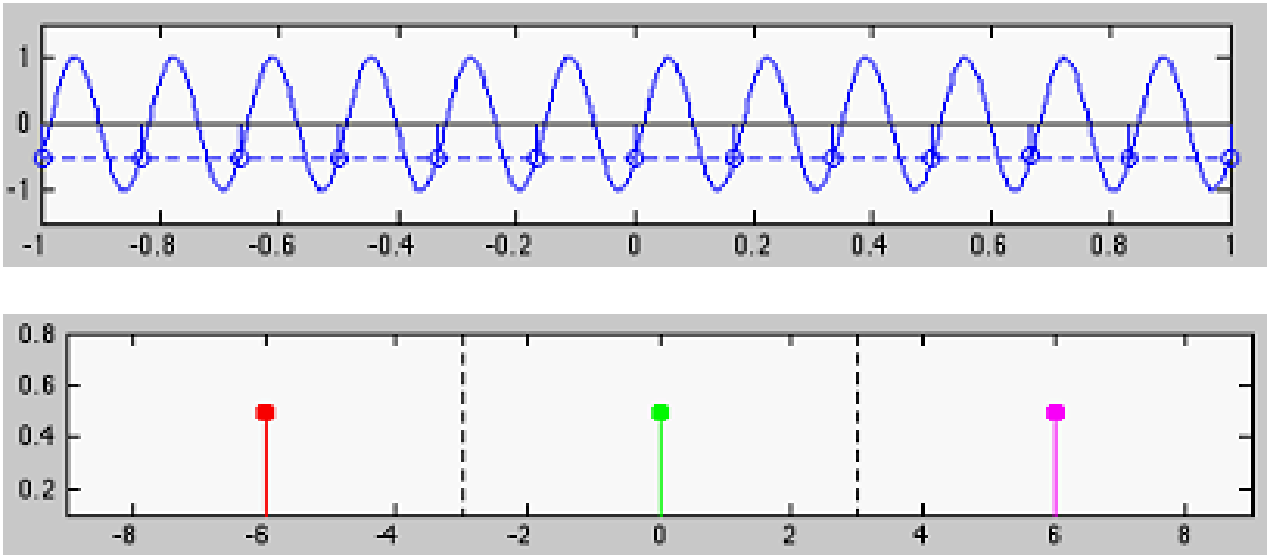
Aliasing in Frequency Domain ($f = 5$)

- When $f = 5$, in this case $f > F/2 = 3$, e.g. similar folding as the $f = 4$ case occurs.
- This time, the two frequency components moving further away from the centre.
- But, replicas moving further towards the centre.



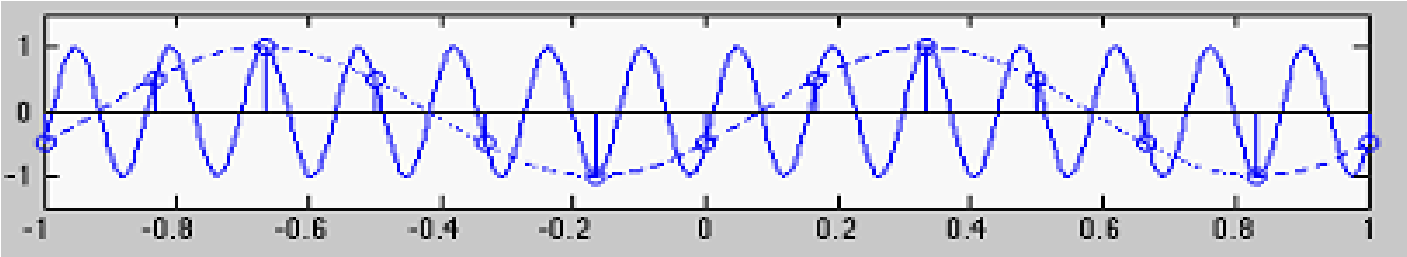
Aliasing in Frequency Domain ($f = 6$)

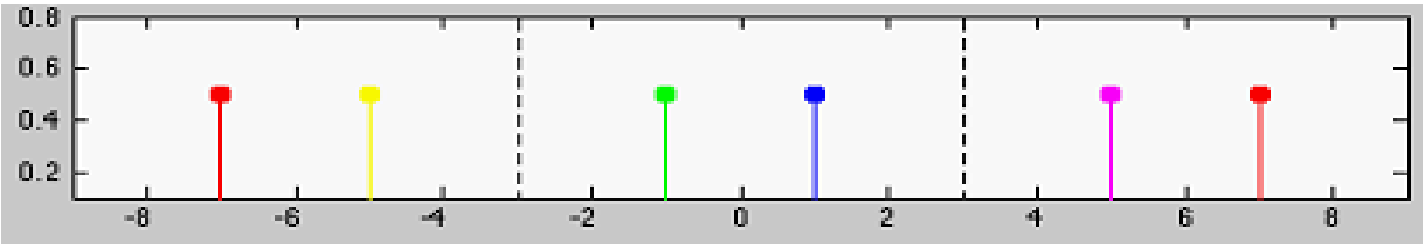
- When $f = 6$, in this case $f = F$, e.g. one sample is taken per period.
- The aliased frequency happens to be zero.
- The reconstructed signal is of course a constant.



Aliasing in Frequency Domain ($f = 7$)

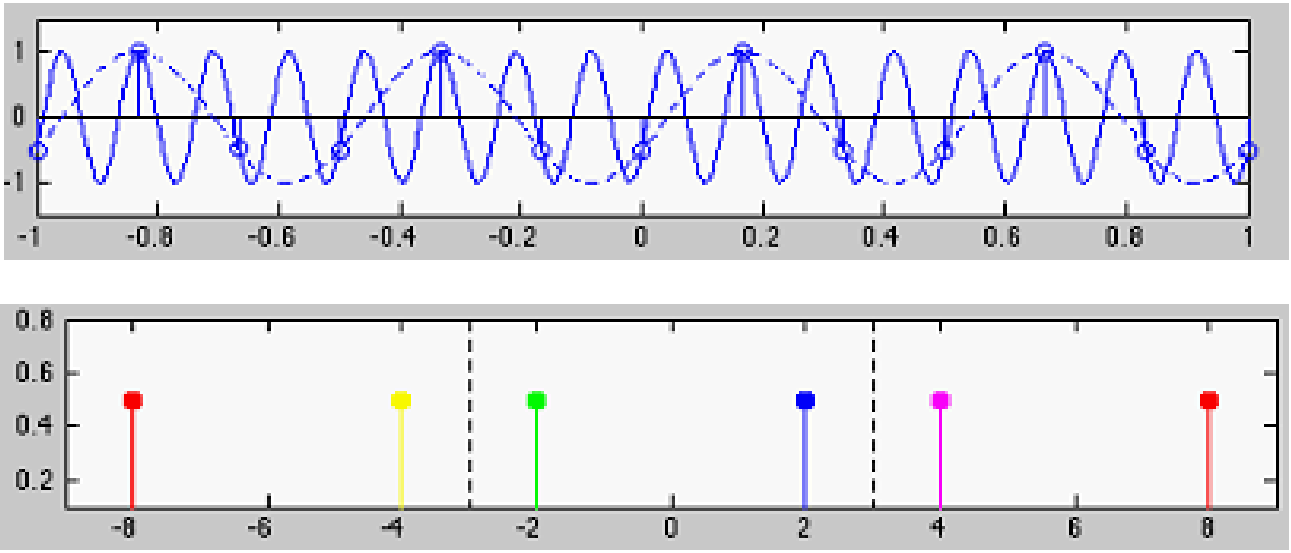
- When $f = 7$, aliasing occurs.
- Similar to the cases when $f = 4, 5$.
- Except now:
  - c_1 is for aliased frequency $7 - F = 7 - 6 = 1$.
  - c_{-1} is for aliased frequency $-7 + 6 = -1$.





Aliasing in Frequency Domain ($f = 8$)

- When $f = 8$, aliasing occurs similar to the cases when $f = 5$.
- Except now:
  - c_1 is for aliased frequency $8 - F = 8 - 6 = 2$.
  - c_{-1} is for aliased frequency $-8 + F = -8 + 6 = -2$.



Aliasing in Frequency Domain ($f = 9 - 12$)

- In the remaining four cases for f = 9, 10, 11, and 12, the two coefficients inside the middle period are identical to those in the previous four cases when f = 3, 4, 5, and 6 respectively.
- Except now they are replicated coefficients moved from not the immediate neighbouring periods but, two periods further out.
- Consequently the reconstructed signals based on the middle period are also identical to those in the previous cases.

Aliasing in Frequency Domain ($f = 9 - 12$)

- Frequency spectra of the remaining four cases of f = 9, 10, 11, and 12

