

z-Transform (Introduction)

XMUT220 Signals and Systems

Topics

- Derivation of z-Transform Equation.
- Region of Convergence (ROC).
- Inverse z-Transform.
- Properties of z-Transform.
- Relationship with Laplace transform.

Derivation of Equation

- The z-transform of a sequence $x[n]$ is:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

- An operator $\mathbb{Z}\{\cdot\}$ that transforms a sequence to a function:

$$\mathbb{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n} = X(z)$$

- In both cases, z is a continuous complex variable:

$$z = Ae^{j\phi} = A(\cos\phi + j\sin\phi)$$

Derivation of Equation

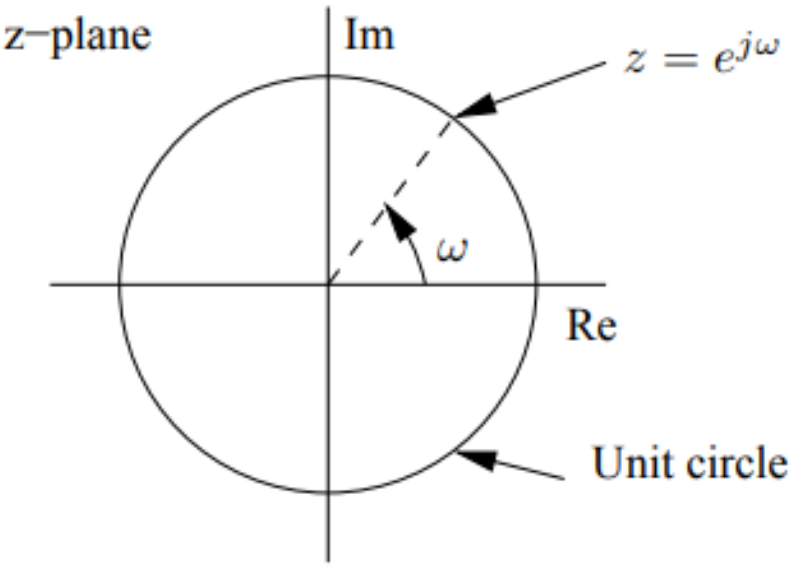
- Fourier transform from the z-transform by making the substitution $z = e^{j\omega}$.
- This corresponds to restricting $|z| = 1$ (e.g. a circle).
- Also, with $z = re^{j\omega}$:

$$X\left(re^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] \cdot (re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} (x[n]r^{-n}) \cdot e^{j\omega}$$

- z-transform is the Fourier transform of the sequence $x[n]r^{-n}$.

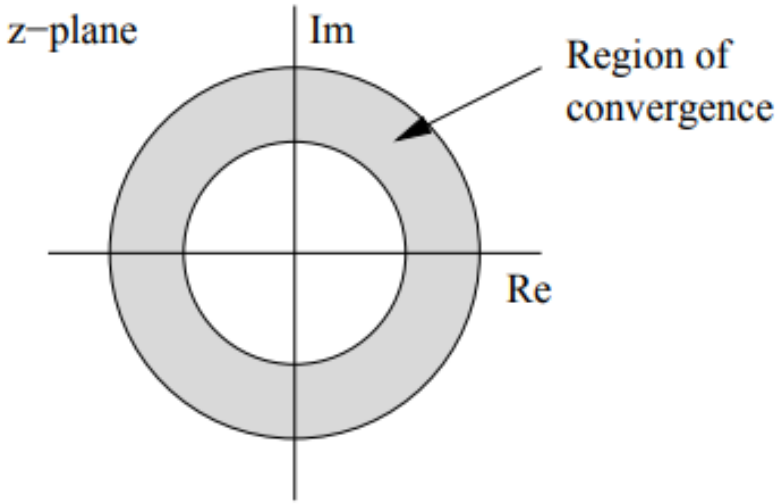
Derivation of Equation

- For $r = 1$, this becomes the Fourier transform of $x[n]$.
- The Fourier transform therefore corresponds to the z-transform evaluated on a unit circle.
- The inherent periodicity in frequency of the Fourier transform is captured naturally under this interpretation.



Derivation of Equation

- The Fourier transform does not converge for all sequences i.e. the infinite sum may not always be finite.
- Similarly, the z-transform does not converge for all sequences or for all values of z.
- The set of values of z for which the z-transform converges is called the region of convergence (ROC).



Derivation of Equation

- The Fourier transform of $x[n]$ exists if the sum of the following equation converges.

$$\sum_{n=-\infty}^{\infty} |x[n]|$$

- However, the z-transform of $x[n]$ is just the Fourier transform of the sequence $x[n]r^{-n}$.
- The z-transform therefore exists (or converges) if:

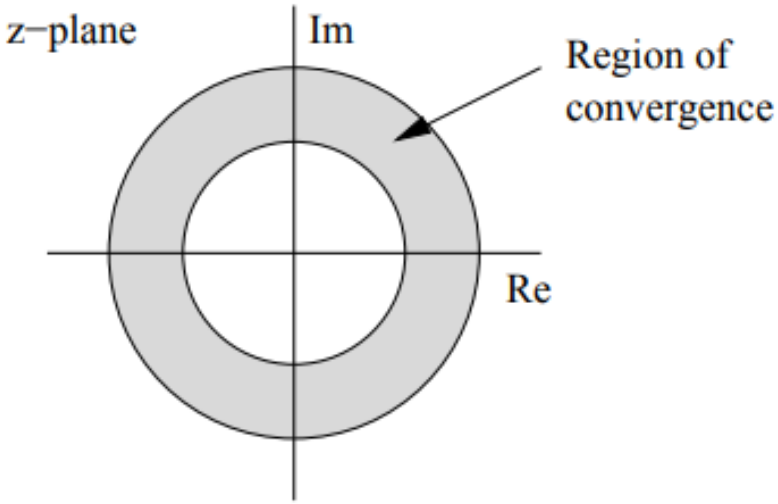
$$X(z) = \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

Derivation of Equation

- This leads to the condition for the existence of the z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} |x[n]||z^{-n}| < \infty$$

- The ROC therefore consists of a ring in the z-plane.
- In specific cases, the inner radius of this ring may include the origin, and the outer radius may extend to infinity.



- If the ROC includes the unit circle $|z| = 1$, then the Fourier transform will converge.

Derivation of Equation

- Most useful z-transforms can be expressed in the form:

$$X(z) = \frac{P(z)}{Q(z)}$$

  Where: $P(z)$ and $Q(z)$ are polynomials in z.

- The values of z for which $P(z) = 0$ are called the zeros of $X(z)$, and the values with $Q(z) = 0$ are called the poles.
- The zeros and poles completely specify $X(z)$ to within a multiplicative constant.

Derivation of Equation

- Consider the following signal.

$$x[n] = a^n u[n]$$

- This has the z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=-\infty}^{\infty} (az^{-1})^n$$

- Convergence requires that:

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$$

which is only the case if $|az^{-1}| < 1$, or equivalently $|z| > |a|$.

Derivation of Equation

- In the ROC, the series converges to a value since it is just a geometric series:

$$x[n] = a + ar + ar^2 + ar^3 + \cdots + \mathrm{a}r^{n-1}$$

$$= \sum_{k=0}^{n-1} a\, r^k = a\left(\frac{1-r^n}{1-r}\right)$$

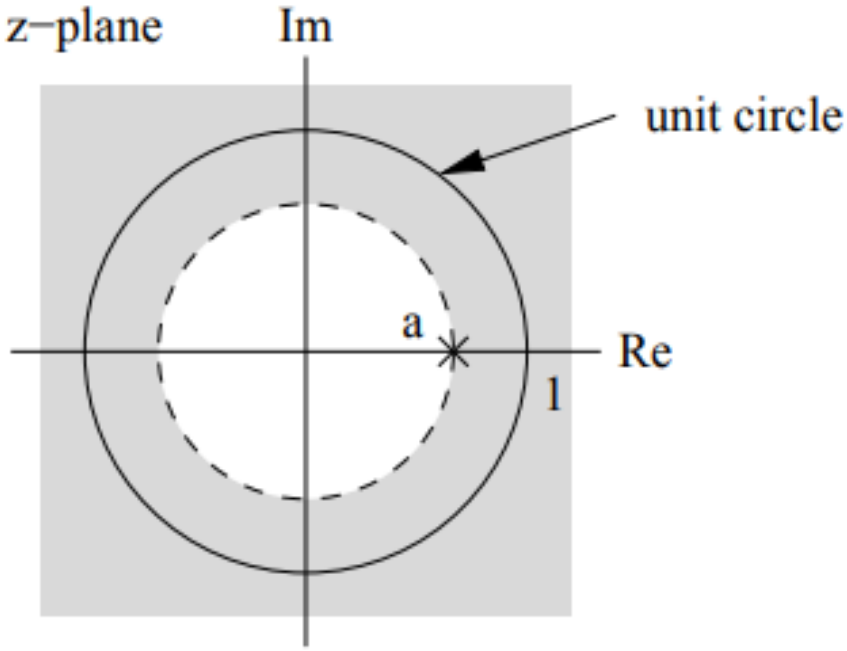
- As a result

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$

- The z-transform has a region of convergence for any finite value of a.

Derivation of Equation

- The Fourier transform of $x[n]$ only exists if the ROC includes the unit circle, which requires that $|a| < 1$.
- On the other hand, if $|a| > 1$ then the ROC does not include the unit circle, and the Fourier transform does not exist.



- This is consistent with the fact that for these values of a the sequence $a^n u[n]$ is exponentially growing, and the sum therefore does not converge.

Derivation of Equation

- Now consider the sequence: $x[n] = -a^n u[-n-1]$.
- This sequence is left-sided because it is nonzero only for $n \leq -1$.
- The z-transform is:

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n[-n-1]z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

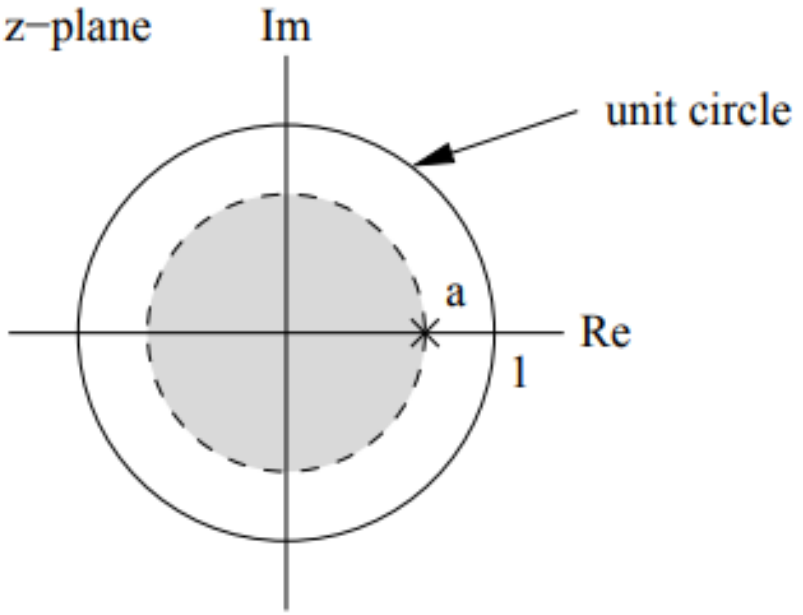
$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

- For $|a^{-1}z| < 1$, or $|z| < |a|$, the series converges to:

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|$$

Derivation of Equation

- Note that the expression for the z-transform (and the pole zero plot) is exactly the same as for the right-handed exponential sequence — only the region of convergence is different.



- Specifying the ROC is therefore critical when dealing with the z-transform.

Derivation of Equation

- The signal $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$ is the sum of two real exponentials.
- The z-transform is:

$$X(z) = \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \right\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} + \left(-\frac{1}{3}\right)^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n$$

Derivation of Equation

- From the example for the right-handed exponential sequence, the first term in this sum converges for $|z| > 1/2$, and the second for $|z| > 1/3$.
- The combined transform $X(z)$ therefore converges in the intersection of these region.
- Namely when $|z| > 1/2$. In this case:

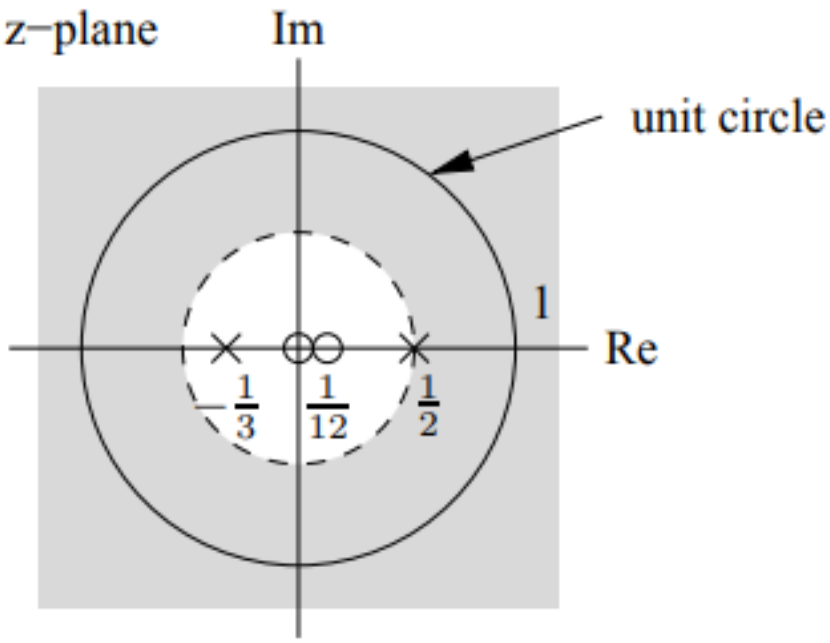
$$X(z) = \frac{1}{1 - 1/2\, z^{-1}} + \frac{1}{1 + 1/3\, z^{-1}} = \frac{2z(z - 1/12)}{(z - 1/2)(z + 1/3)}$$

Derivation of Equation

- The pole-zero plot and region of convergence of the signal are:

- The poles of the signal are:
  - Pole 1: -1/3.
  - Pole 2: 1/2.

- The zeros of the signal are:
  - Zero 1: 0.
  - Zero 2: 1/12.



Derivation of Equation

- Given a signal:

$$f(x) = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

- It has z-transform:

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}}\left(\frac{z^N - a^N}{z-a}\right)$$

- Since there are only a finite number of non-zero terms, the sum always converges when az^{-1} is finite.

Derivation of Equation

- There are no restrictions on a (e.g. $|a| < \infty$), and the ROC is the entire z-plane with the exception of the origin $z = 0$ (where the terms in the sum are infinite).
- The N roots of the numerator polynomial are at:

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 0, 1, \cdots, N-1$$

since these values satisfy the equation $z^N = a^N$.

- The zero at $k = 0$ cancels the pole at $z = a$, so there are no poles except at the origin, and the zeros are at:

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 1, \cdots, N-1$$

Property of Region of Convergence

The properties of the ROC depend on the nature of the signal. Assuming that the signal has a finite amplitude and that the z-transform is a rational function:

- The ROC is a ring or disk in the z-plane, centred on the origin ($0 \leq r_R < |z| < r_L \leq \infty$).
- The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform includes the unit circle.
- The ROC cannot contain any poles.
- If $x[n]$ is finite duration (i.e. zero, except on finite interval $-\infty < N_1 \leq n \leq N_2 < \infty$), then the ROC is the entire z-plane, except perhaps at $z = 0$ or $z = \infty$.

Property of Region of Convergence

- If $x[n]$ is a right-sided sequence then the ROC extends outward from the outermost finite pole to infinity.
- If $x[n]$ is left-sided then the ROC extends inward from the innermost non-zero pole to $z = 0$.
- A two-sided sequence (neither left nor right-sided) has a ROC consisting of a ring in the z-plane, bounded on the interior and exterior by a pole (and not containing any poles).
- The ROC is a connected region.

Inverse z-Transform

Inspection method

- If one is familiar with (or has a table of) common z-transform pairs, the inverse can be found by inspection.
- For example, one can invert the z-transform:

$$X(z) = \left(\frac{1}{1 - \frac{1}{2} z^{-1}} \right), \qquad |z| > \frac{1}{2}$$

- Using the z-transform pair:

$$a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}, \qquad \text{for } |z| > |a|$$

Inverse z-Transform

- By inspection we recognise that:

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

- Also, if $X(z)$ is a sum of terms, then one may be able to do a term-by-term inversion by inspection, yielding $x[n]$ as a sum of terms.

Inverse z-Transform

Partial Fraction Expansion

- For any rational function, we can obtain a partial fraction expansion and identify the z-transform of each term.
- Assume that $X(z)$ is expressed as a ratio of polynomials in z^{-1}:

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Inverse z-Transform

- It is always possible to factor $X(z)$ as:

$$X(z) = \frac{b_0 \prod_{k=1}^{M}(1 - c_k z^{-1})}{a_0 \prod_{k=1}^{N}(1 - d_k z^{-1})}$$

Where: the c_k's and d_k's are the non-zero zeros and poles of $X(z)$.

Inverse z-Transform

- If the order of equation $M < N$ and the poles are all first order, then $X(z)$ can be expressed as:

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

- In this case, the coefficients A_k are given by:

$$A_k = (1 - d_k z^{-1}) X(z)\big|_{z = d_k}$$

Inverse z-Transform

- If the order of equation $M \geq N$ and the poles are all first order, then an expansion of the form:

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

- can be used, and the B_r's be obtained by long division of the numerator by the denominator.
- The A_k's can be obtained using the same equation as for $M < N$.

Inverse z-Transform

- The most general form for the partial fraction expansion, which can also deal with multiple-order poles, is:

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1,k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}$$

- Ways of finding the C_m's can be found in most standard signal and system textbooks.

Inverse z-Transform

- The terms $B_r z^{-r}$ correspond to shifted and scaled impulse sequences, and invert to terms of the form $B_r \delta[n-r]$.
- The fractional terms:

$$\frac{A_k}{1 - d_k z^{-1}}$$

  correspond to exponential sequences.

- For these terms, the ROC properties must be used to decide whether the sequences are left-sided or right-sided.

Inverse z-Transform

Inverse by Partial Fractions

- Consider the sequence $x[n]$ with z-transform:

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}$$

- Since $M = N = 2$ this can be expressed as:

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

Inverse z-Transform

- The value B_0 can be found by long division:

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}$$

- The coefficients A_1 and A_2 can be found using:

$$A_k = (1 - d_k z^{-1})X(z)\big|_{z = d_k}$$

- So

$$A_1 = \left.\frac{1 + 2z^{-1} + z^{-2}}{1 - z^{-1}}\right|_{z^{-1} = 2} = \frac{1 + 4 + 4}{1 - 2} = -9$$

Inverse z-Transform

- And then

$$A_2 = \left.\frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{1}{2}z^{-1}}\right|_{z^{-1}=1} = \frac{1+2+1}{\frac{1}{2}} = 8$$

- Thus

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{(1 - z^{-1})}$$

- Using the fact that the ROC is $|z| > 1$, the terms can be inverted one at a time by inspection to give:

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

Inverse z-Transform

Power Series Expansion

- If the z-transform is given as a power series in the form:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-1}$$

- Or

$$X(z) = \cdots + x[-2]z^{-1} + x[-1]\,z^{-1} + x[0] + x[1]\,z^{-1} + x[2]\,z^{-2} + \cdots$$

- then any value in the sequence can be found by identifying the coefficient of the appropriate power of z^{-1}.

Inverse z-Transform

Example: finite-length sequence

- The z-transform:

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1})$$

- This can be multiplied out to give:

$$X(z) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

Inverse z-Transform

- By inspection, the corresponding sequence is therefore:

$$X[n] = \begin{cases} 1 & n = -2 \\ -\frac{1}{2} & n = -1 \\ -1 & n = 0 \\ \frac{1}{2} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

- or

$$x[n] = 1\delta[n+2] - \frac{1}{2}\delta[n+1] - 1\delta[n] + \frac{1}{2}\delta[n-1]$$

Inverse z-Transform

Example: power series expansion

- Consider the z-transform:

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|$$

- Using the power series expansion for $\log(1 + x)$, with $|x| < 1$, gives:

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

- The corresponding sequence is therefore:

$$x[n] = \begin{cases} (-1)^{n+1}\, a^n/n & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

Inverse z-Transform

Example: power series expansion by long division

- Consider the transform:

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$

- Since the ROC is the exterior of a circle, the sequence is right-sided.
- We therefore divide it to get a power series in powers of z^{-1}:

$$1 + az^{-1} + a^2z^{-2} + \cdots$$

Inverse z-Transform

- Divided by $1 - z^{-1}$, or

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \cdots$$

- As a result:

$$x[n] = a^n u[n]$$

Inverse z-Transform

Example: power series expansion for left-sided sequence.

- Consider instead the z-transform:

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| < |a|$$

- Because of the ROC, the sequence is now a left-sided one.
- Thus, we divide to obtain a series in powers of z:

$$-a^{-1}z - a^{-2}z^2 - \cdots$$

- Divided by $-a + z$, thus:

$$z[n] = -a^n u[-n-1]$$

Table of z-Transform

No	Sequence	Transform	ROC
1	$\delta[n]$	1	All z
2	$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$-u[n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4	$\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5	$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6	$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7	$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8	$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9	$\begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
10	$\sin(\omega_0 n)u[n]$	$\frac{\sin\omega_0\, z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11	$\cos(\omega_0 n)u[n]$	$\frac{1-\cos\omega_0\, z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
12	$[a^n \sin\omega_0 n]u[n]$	$\frac{(a\sin\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
13	$[a^n \cos\omega_0 n]u[n]$	$\frac{1-(a\cos\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Properties of z-Transform

We look into the following properties of Z-Transform:

- Linearity
- Time shifting
- Convolution
- Exponential Multiplication
- Differentiation
- Conjugation
- Time Reversal
- Initial and Final Value Theorem

Property of z-Transform

- If $X(z)$ denotes the z-transform of a sequence $x[n]$ and the ROC of $X(z)$ is indicated by R_x, then this relationship is indicated as:

$$x[n] \leftrightarrow X(z), \qquad \mathrm{ROC} = R_x$$

- Furthermore, with regard to nomenclature, we have two sequences such that:

$$x[n] \leftrightarrow X_1(z), \qquad \mathrm{ROC} = R_{x1}$$

and

$$x[n] \leftrightarrow X_2(z), \qquad \mathrm{ROC} = R_{x2}$$

Property of z-Transform

Linearity

- The z-transform of a linear combination of signals then it will be the same as the linear combination of the z-transforms of each of the individual signals.
- The linearity property is as follows:

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(z) + aX_2(z)$$

- ROC contains $R_{x1} \bigcap R_{x2}$.

Property of z-Transform

Example: multiplied sequence with constants.

Let a sequence given as: $w[n] = \alpha x[n] + \beta y[n]$. Then

$$W(z) = \sum_{n=-\infty}^{\infty} [\alpha x[n] + \beta y[n]] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \alpha x[n]\, z^{-n} + \sum_{n=-\infty}^{\infty} \beta y[n]\, z^{-n}$$

$$= \alpha \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} + \beta \sum_{n=-\infty}^{\infty} y[n]\, z^{-n}$$

$$= \alpha X(z) + \beta X(z)$$

Property of z-Transform

Time Shifting

- The time-shifting property is as follows:

$$x[n-n_0] \leftrightarrow z^{-n_0}X(z), \qquad \text{ROC} = R_x$$

- The ROC may change by adding or deleting $z = 0$ or $z = \infty$.

- This is shown as follows:

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n} = \sum_{n=-\infty}^{\infty} x[m]z^{-(m+n_0)}$$

$$= z^{-n_0} \sum_{n=-\infty}^{\infty} x[m]z^{-n} = z^{-n_0}X(z)$$

Property of z-Transform

Example: shifted exponential sequence

- Consider the z-transform:

$$X(z) = \frac{1}{z - \left(\frac{1}{4}\right)}, \qquad |z| > \frac{1}{4}$$

- From the ROC, this is a right-sided sequence.
- Rewriting the equation given above:

$$X(z) = \frac{z^{-1}}{1 - \left(\frac{1}{4}\right) z^{-1}} = z^{-1}\left[\frac{1}{1 - \left(\frac{1}{4}\right) z^{-1}}\right], \qquad |z| > \frac{1}{4}$$

Property of z-Transform

- The term in brackets corresponds to an exponential sequence $(1/4)^n u[n]$.
- The factor z^{-1} shifts this sequence one sample to the right.
- The inverse z-transform is therefore:

$$x[n] = \left(\frac{1}{4}\right)^{n-1} u[n-1]$$

- Note that this result could also have been easily obtained using a partial fraction expansion.

Property of z-Transform

Convolution

- This property states that:

$$x_1[n] * x_2[n] \leftrightarrow X_1[n] \cdot X_2[n], \qquad \text{ROC } R_{x1} \bigcap R_{x1}$$

Property of z-Transform

Example: evaluating a convolution using the z-transform.

- The z-transforms of the signals $x_1 = a^n u[n]$ and $x_2 = u[n]$ are:

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$

- and

$$X_2(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}, \qquad |z| > 1$$

Property of z-Transform

- For $|a| < 1$, the z-transform of the convolution $y[n] = x_1[n] * x_2[n]$ is:

$$Y(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})}$$

$$= \frac{z^2}{(z - a)(z - 1)}, \qquad |z| > 1$$

Property of z-Transform

- Using a partial fraction expansion:

$$Y(z) = \frac{1}{1-a}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right), \qquad |z| > 1$$

- So

$$y[n] = \frac{1}{1-a}\left[u[n] - a^{n+1}u[n]\right]$$

Property of z-Transform

Exponential Multiplication

- This property is:

$$(Z_0)^n x[n] \leftrightarrow X(z/z_0), \qquad \text{ROC} = |z_0|R_x$$

  where the notation $|z_0|R_x$ indicates that the ROC is scaled by $|z_0|$ (i.e. inner and outer radii of the ROC scaled by $|z_0|$).

- All pole-zero locations are similarly scaled by a factor z_0.
- If $X(z)$ had a pole at $z = z_1$, then $X(z/z_0)$ will have a pole at $z = z_0 z_1$.

Property of z-Transform

- If z_0 is positive and real, this operation can be interpreted as a shrinking or expanding of the z-plane — poles and zeros change along radial lines in the z-plane.
- If z_0 is complex with unit magnitude ($z_0 = e^{j\omega_0}$), then the scaling operation corresponds to a rotation in the z-plane by an angle ω_0.
- That is, the poles and zeros rotate along circles centred on the origin.

Property of z-Transform

- This can be interpreted as a shift in the frequency domain, associated with modulation in the time domain by $e^{j\omega_n}$.
- If the Fourier transform exists, this becomes:

$$e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$$

Property of z-Transform

Example: exponential multiplication

- The z-transform pair:

$$u[n] \leftrightarrow \frac{1}{1-z^{-1}}, \qquad |z| > 1$$

- can be used to determine $x[n] = r^n \cos(\omega_0 n) u[n]$.

Property of z-Transform

- Since $\cos(\omega_0 n) = 1/2\, e^{j\omega_0 n} + 1/2\, e^{-j\omega_0 n}$, the signal can be rewritten as:

$$x[n] = \frac{1}{2}(re^{j\omega_0})^n u[n] + \frac{1}{2}(re^{-j\omega_0})^n u[n]$$

- So,

$$X(z) = \frac{1/2}{1 - re^{j\omega_0} z^{-1}} + \frac{1/2}{1 - re^{-j\omega_0} z^{-1}}, \qquad |z| > r$$

$$= \frac{1 - r\cos\omega_0 z^{-1}}{1 - 2re^{j\omega_0} z^{-1} + r^2 z^{-2}}, \qquad |z| < r$$

Property of z-Transform

Differentiation

- The differentiation property states that:

$$nx[n] \leftrightarrow -z\left(\frac{dX(z)}{dz}\right), \qquad \text{ROC} = R_x$$

- This can be seen as follows, since:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-1}$$

Property of z-Transform

- We have:

$$-z\left(\frac{dX(z)}{dz}\right) = -z\sum_{n=-\infty}^{\infty}(-n)x[n]z^{-n-1}$$

$$= \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = \mathbb{Z}\{nx[n]\}$$

Property of z-Transform

Example: second order pole

- The z-transform of the sequence:

$$x[n] = na^n u[n]$$

- This can be found using:

$$a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}, \qquad |z| > 0$$

- To be:

$$X(z) = -z\left[\frac{d}{dz}\left(\frac{1}{1 - az^{-1}}\right)\right] = \frac{az^{-1}}{(1 - az^{-1})^2}, \qquad |z| > 0$$

Property of z-Transform

Conjugation

- This property is described as:

$$x^*[n] \leftrightarrow X^*(z^*), \qquad \mathrm{ROC} = R_x$$

- Proof of conjugate values in z-transform:

$$Y(z) = \sum_{n=-\infty}^{\infty} x^*[n] z^{-n} = \left(\sum_{n=-\infty}^{\infty} x[n] (z^*)^{-n} \right)^* = X^*(z^*)$$

- Since ROC depends only on magnitude $|z|$, it does not change.

Property of z-Transform

Time Reversal

- Here, time reversal property is:

$$x^*[-n] \leftrightarrow X^*(1/z^*), \qquad \text{ROC} = \frac{1}{R_x}$$

- The notation $1/R_x$ means that the ROC is inverted, so if R_x is the set of values such that $r_R < |z| < r_L$.
- Then the ROC is the set of values of z such that:

$$1/r_L < |z| < 1/r_R$$

Property of z-Transform

Example: time-reversed exponential sequence

- The signal below is a time-reversed version of a sequence $a^n u[n]$.

$$x[n] = a^{-n} u[-n]$$

- The z-transform is therefore:

$$X(z) = \frac{1}{1 - az} = \frac{-a^{-1} z^{-1}}{1 - a^{-1} z^{-1}}, \qquad |z| < |a^{-1}|$$

Property of z-Transform

Initial and Final Value Theorem

Initial value theorem

- Initial value and final value theorems of z-transform are defined for causal signal
- If $x[n]$ is zero for $n < 0$, then:

$$x[0] = \lim_{z \to \infty} X(z)$$

- This is used to find the initial value of the signal without taking inverse z-transform.

Property of z-Transform

Final value theorem

- For a causal signal $x[n]$, the final value theorem states that:

$$x[\infty] = \lim_{z \to 1} [z - 1]\, X(z)$$

- This is used to find the final value of the signal without taking inverse z-transform.

Table of z-Transform

No	Property	Sequence	Transform	ROC
1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2.
2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the region.
3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X\left(e^{-j\omega_0}z\right)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e. the set of points $\{\|a\|z\}$for z in R.
4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R(i.e. R^{-1} is equal to the set of point z^{-1}, where z is in R).
5	Time expansion	$x_{(k)}[n]\begin{cases} x[r], n = rk \\ 0, n \neq rk \end{cases}$ for some integer r	$X\left(z^k\right)$	$R^{1/k}$(i.e. the set of point $z^{1/k}$, where z is in R).
6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
7	Convolution	$x_1[n] * x_2[n]$	$X_1(z) \cdot X_2(z)$	At least the intersection of R_1 and R_2.
8	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$.
9	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\left(\frac{1}{1 - z^{-1}}\right)X(z)$	At least the intersection of R and $\|z\| > 1$.
10	Differentiation in the z-domain	$nx[n]$	$-z\left[\frac{dX(z)}{dz}\right]$	R
11	Initial value theorem	If $x[n] = 0$ for $x < 0$, then $x[0] = \lim_{z\to\infty} X(z)$		

Relationship with Laplace Transform

- Continuous-time systems and signals are usually described by the Laplace transform.
- Letting $z = e^{sT}$, where s is the complex Laplace variable:

$$s = d + j\omega$$

- We have:

$$z = e^{(d+j\omega)T} = e^{dT}e^{j\omega T}$$

- Therefore, magnitude:

$$|z| = e^{dT}$$

Relationship with Laplace Transform

- And phase shift:

$$\measuredangle z = \omega T = 2\pi(f/f_s) = 2\pi(\omega/\omega_s)$$

Where: ω_s is the sampling frequency.

As ω varies from $-\infty$ to ∞ , the s-plane is mapped to the z-plane:

- The $j\omega$ axis in the s-plane is mapped to the unit circle in the z-plane.
- The left-hand s-plane is mapped to the inside of the unit circle.
- The right-hand s-plane maps to the outside of the unit circle.