LECTURE BLOCK 1A: SIGNALS

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Aim of this lecture

Introduce the concepts of a signal (and system), and their various forms and properties:

- Types of signals,
- Signal transformations,
- Odd/even, periodic signals,
- Signal power and energy.

Reading: Chapter 1 *Signals and Systems*, Oppenheim, Willsky, Nawab 2nd edition (Pearson)

Signals

Signals come in many forms:

- Continuous,
- Discrete,
- Analog,
- Digital,
- Periodic,
- Non-periodic
- Odd or even-symmetry
- Non-symmetry
- … etc

Signals

Signals with special waveforms:

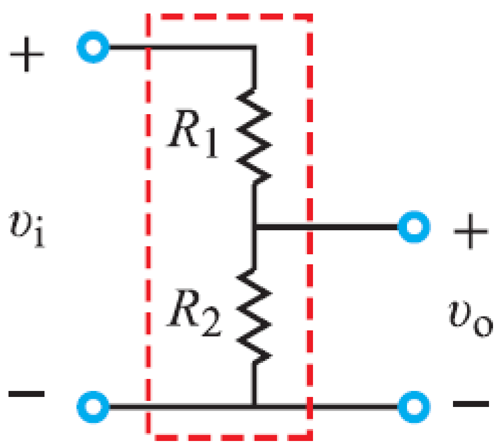
- Unit step,
- Impulses,
- Exponentials
- Sinusoids,
- … etc.

We are also interested with the transformations commonly associated with signals.

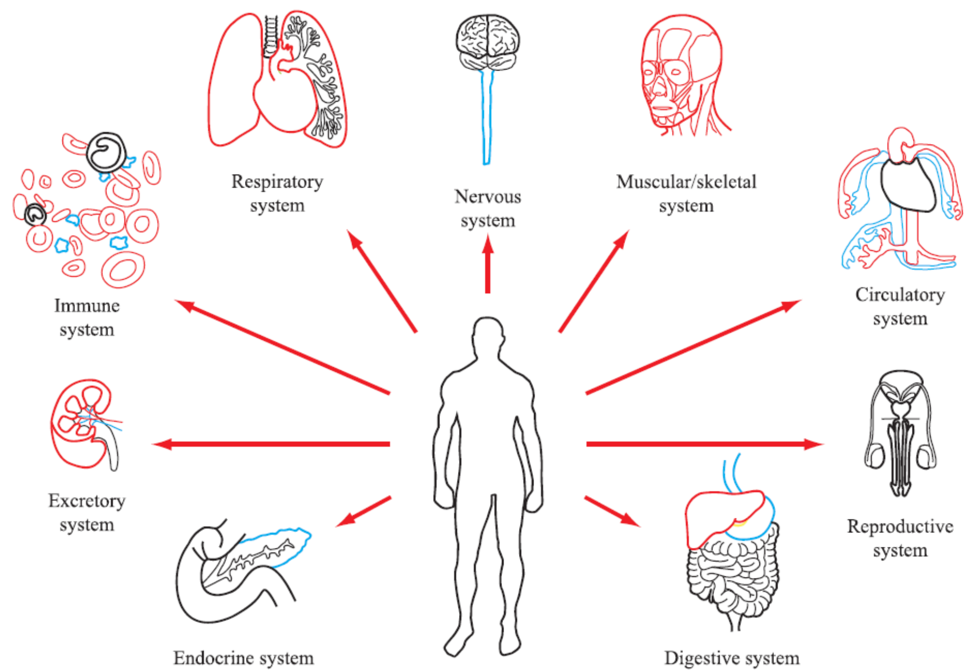
This prepares us to deal with how signals interact with systems.

Overview

A system transforms input signals (excitations)
into output signals (responses)
to perform a certain operation.

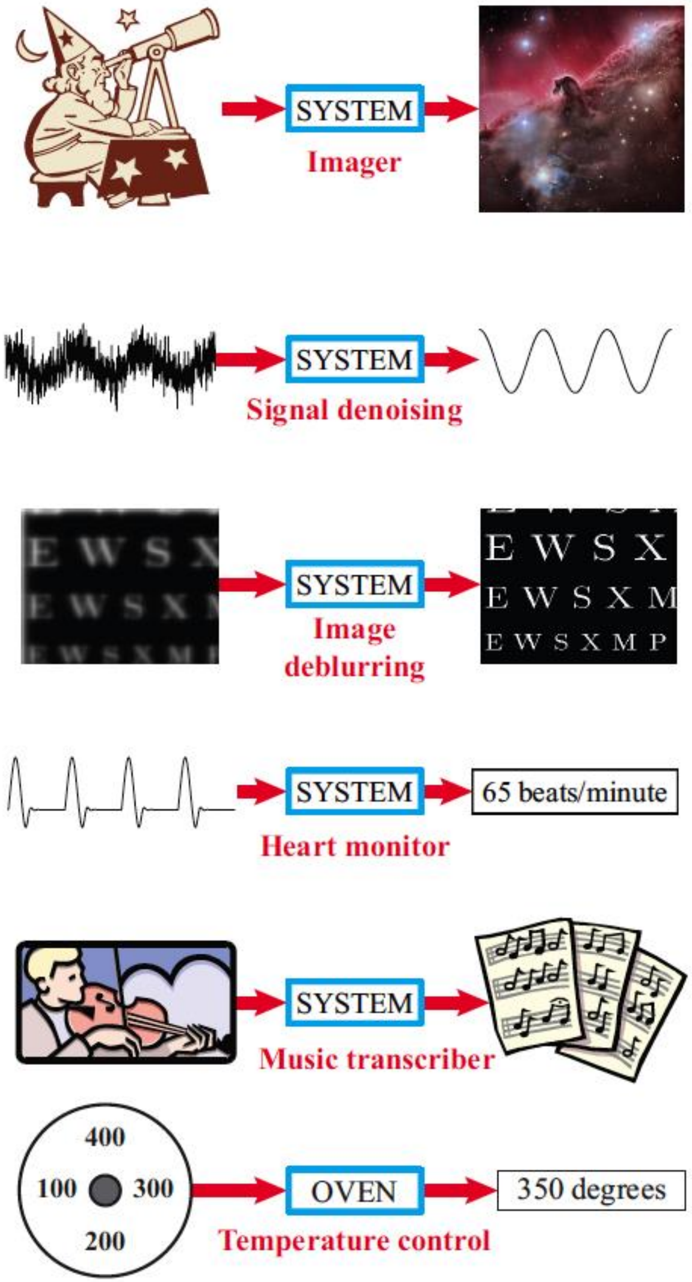


A voltage divider is a system that scales down the input voltage,
$v_o = [R_2/(R_1+R_2)]v_i$



Examples of Systems

- By modelling signals and systems mathematically, we can use the system model to predict the output resulting from a specified input.
- We can also design systems to perform operations of interest.
- Now, we look at the mathematics of signals;
- Subsequently, we will look at systems.



Types of Signals

- Continuous (pic a) vs Discrete (pic b)
- It is a physical quantity being measured.

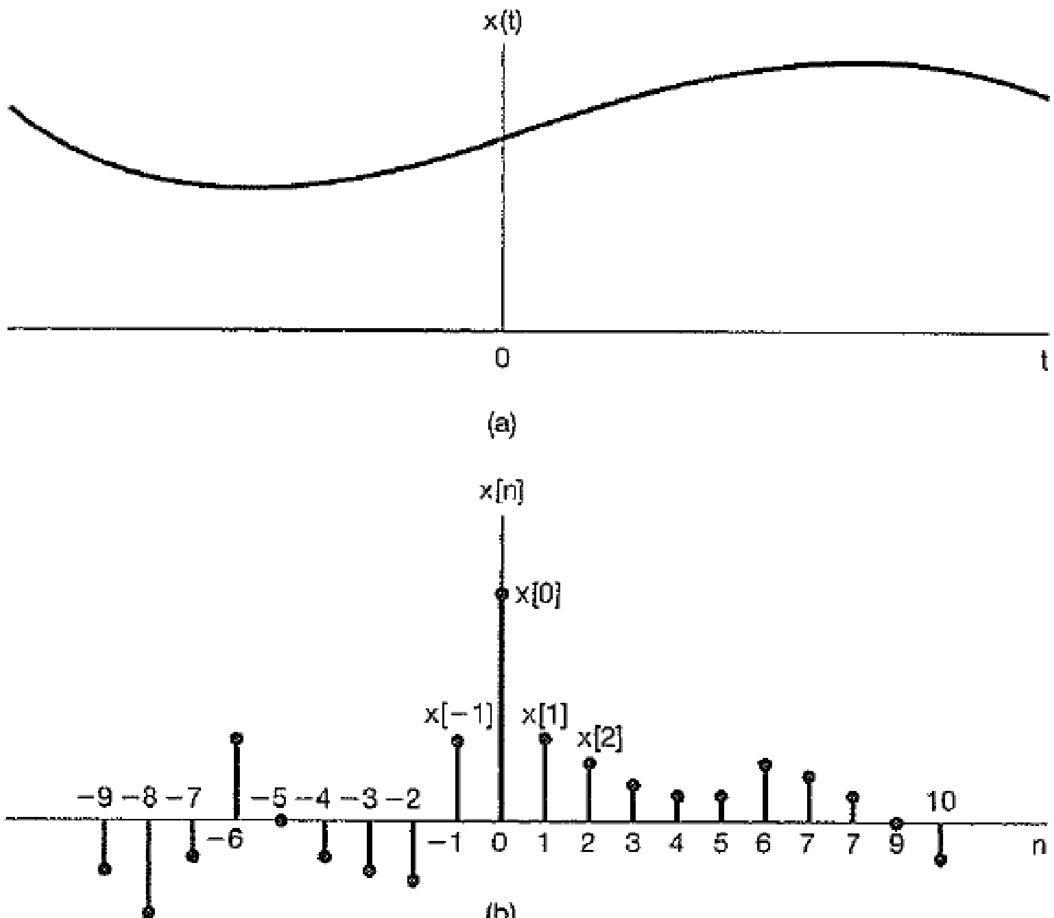
► A signal $x(t)$ is:

- **causal** if $x(t) = 0$ for $t < 0$ (starts at or after $t = 0$)
- **noncausal** if $x(t) \neq 0$ for any $t < 0$ (starts before $t = 0$)
- **anticausal** if $x(t) = 0$ for $t > 0$ (ends at or before $t = 0$) ◄

(a) Continuous-time signal

(b) Discrete-spatial signal

Continuous-time vs discrete-time



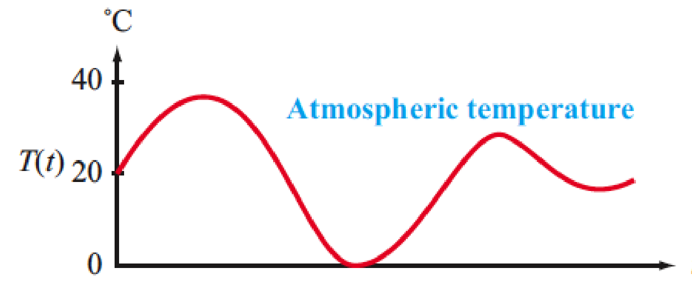
- notation: $x(t)$ - continuous; $x[n]$ - discrete
- integer n

Analog vs Digital

Why digital?

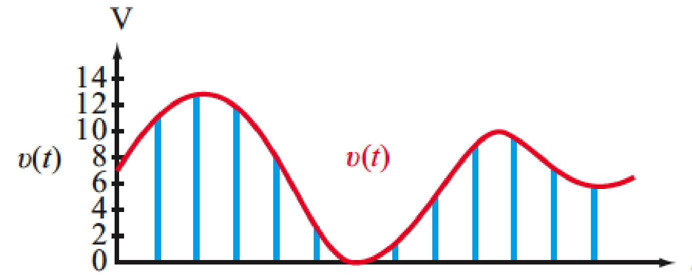
Digital signal processing can be done on a digital computer;

More immune to noise interference.



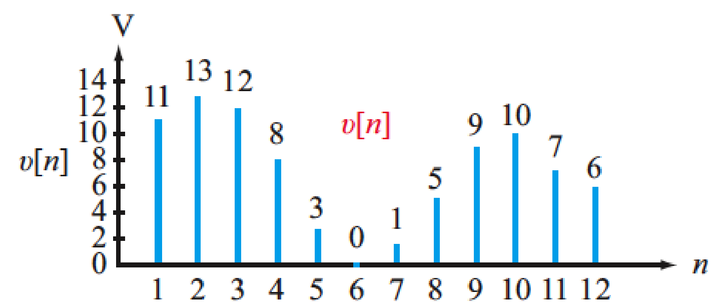
(a) Atmospheric temperature in ˚C

Sampling →



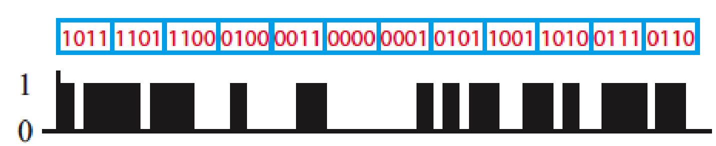
(b) Sensor voltage in volts

Discrete Signal →



(c) Discrete version of (b)

Digital Signal (binary) →

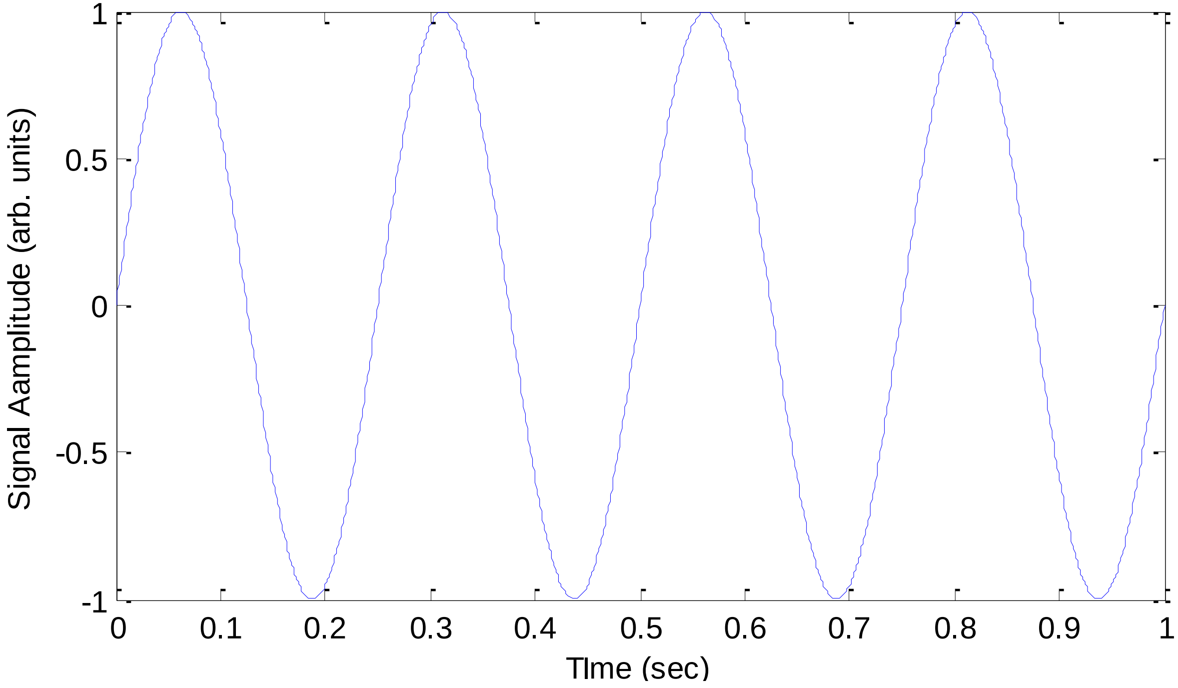


(d) Digital signal

Analog Signals : Continuous and discrete signals

What distinguishes “continuous” or “analog" signals is that they are defined for all times in some interval. They need not be continuous functions in the mathematical sense of the word (although real signals usually are).

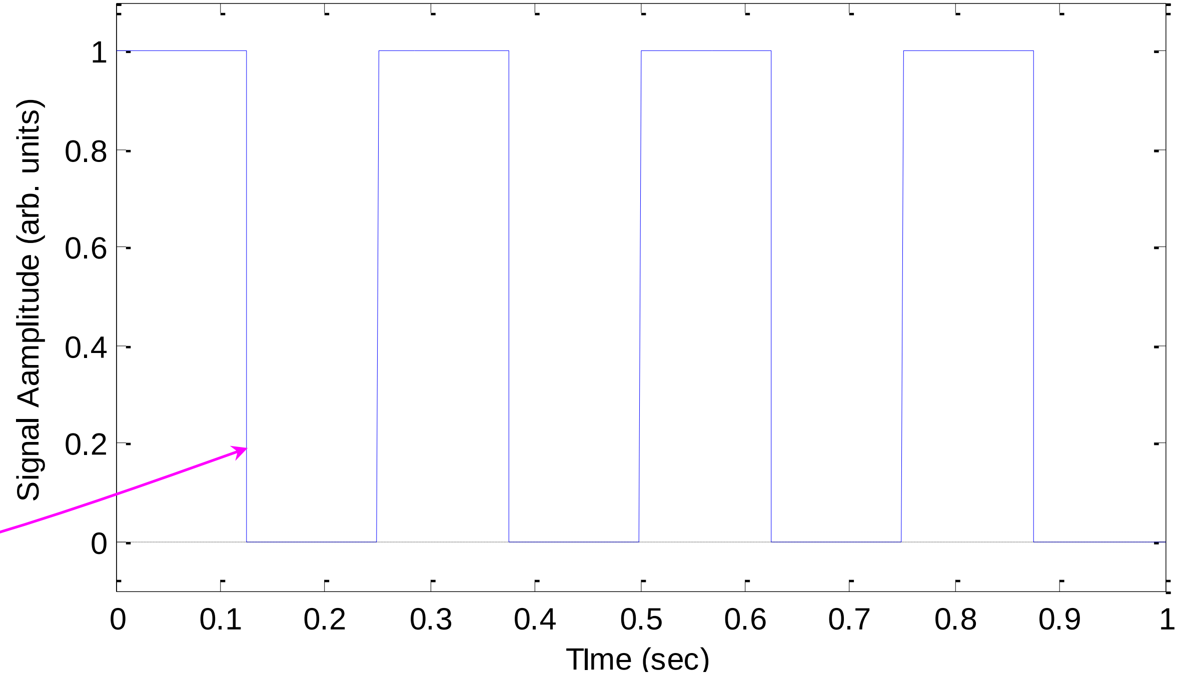
Analog signal which is also mathematically continuous (ie no sudden jumps).



Analog Signals : Continuous and discrete signals

What distinguishes “analog" signals is that they are defined for all times in some interval. They need not be continuous functions in the mathematical sense of the word (although real signals usually are).

Analog signal which is not mathematically continuous everywhere (ie there are sudden jumps).

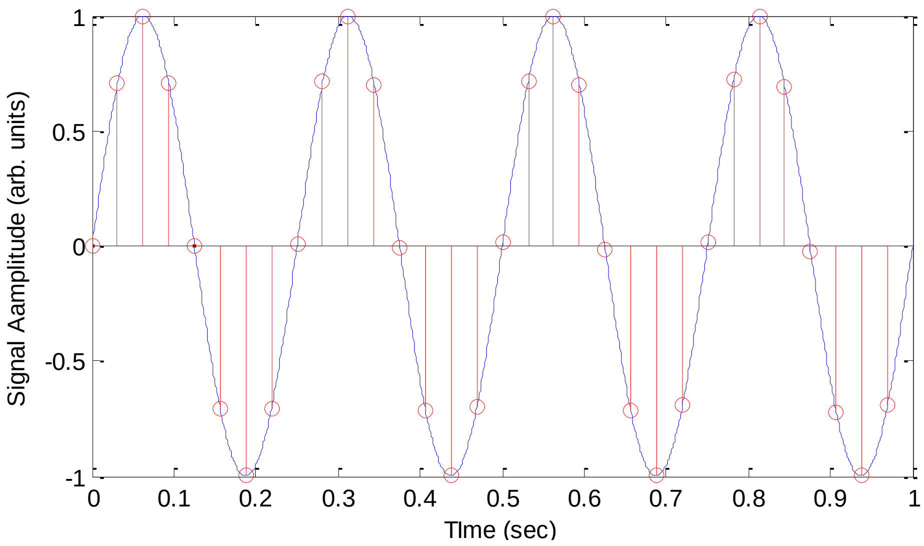


Analog Signals : Continuous and discrete signals

A discrete signal (regarded as a sampled signal for example) is defined only at certain instants separated by finite intervals. It is not zero between sampled values, it is simply undefined.

$$f(t) = A\sin(\omega t) \ : t \in [0, \infty)$$

For example the continuous signal shown has well defined values of all times from zero to infinity, whereas the discrete signal which results from sampling at intervals, T, is (strictly speaking) undefined between sampling instants.



$$f(n) = A\sin(n\omega T) \ ; \quad n = 0,1,2,3,\cdots$$

Analog Signals : Continuous and discrete signals

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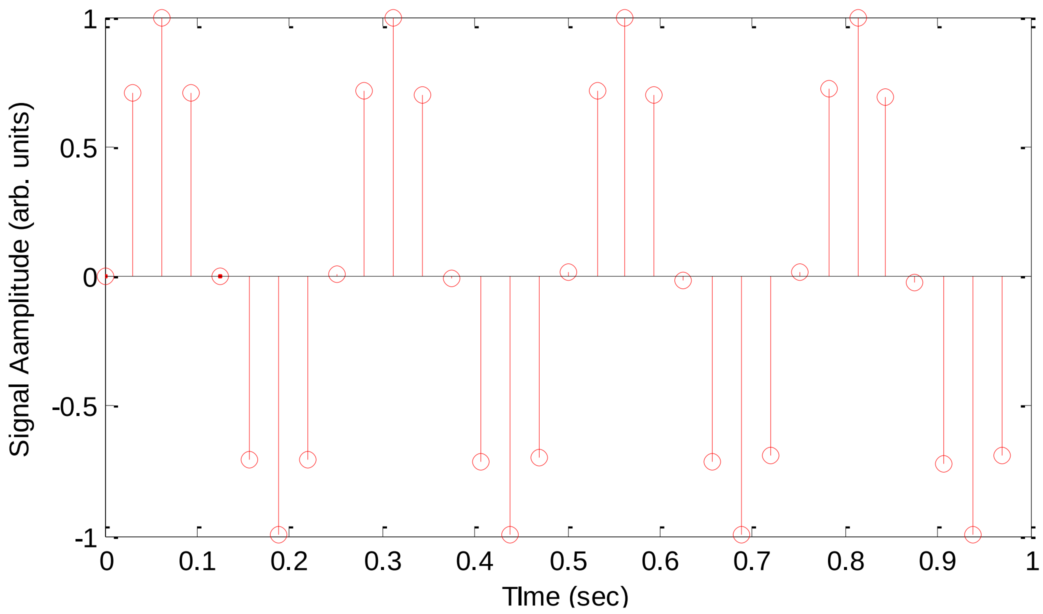


$$f(n) = A\sin(n\omega T)\ ; \quad n = 0,1,2,3,\cdots$$

Analog vs digital

The terms continuous-time, discrete-time, analog and digital can be summarized as follows:

- A signal $x(t)$ is analog and continuous-time if both x and t are continuous variables (infinite resolution). Most real world signals are analog and continuous time.
- A signal $x[n]$ is analog and discrete-time if the values of x are continuous but time n is discrete (integer-valued).
- A signal $x[n]$ is digital and discrete-time if the values of x are discrete (i.e. quantized) and time n also is discrete (integer-valued). Computers store and process digital discrete-time signals.
- A signal $x(t)$ is digital and continuous-time if $x(t)$ can only take on a finite number of values. An example is the class of logic signals, such as the output of a flip-flop, which can only take on values of 0 or 1.

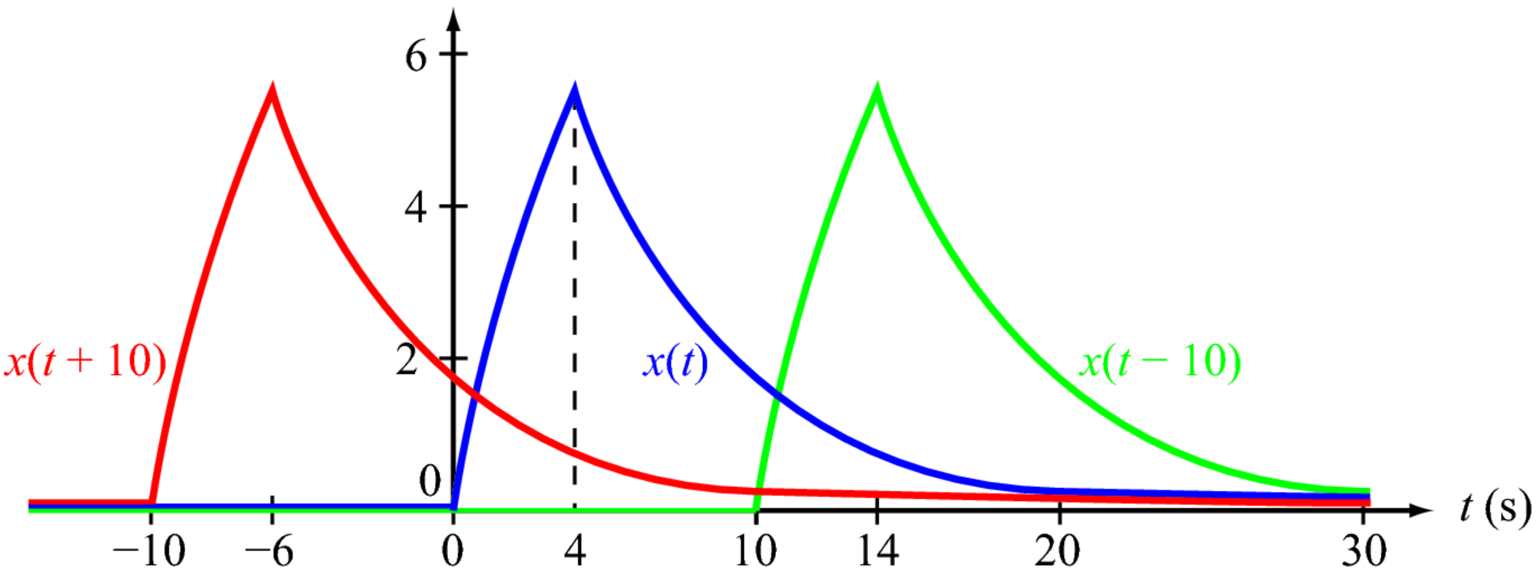
Signal Transformations

Time-shift Transformation

If $x(t)$ is a continuous-time signal, a ***time-shifted*** version with delay T is given by

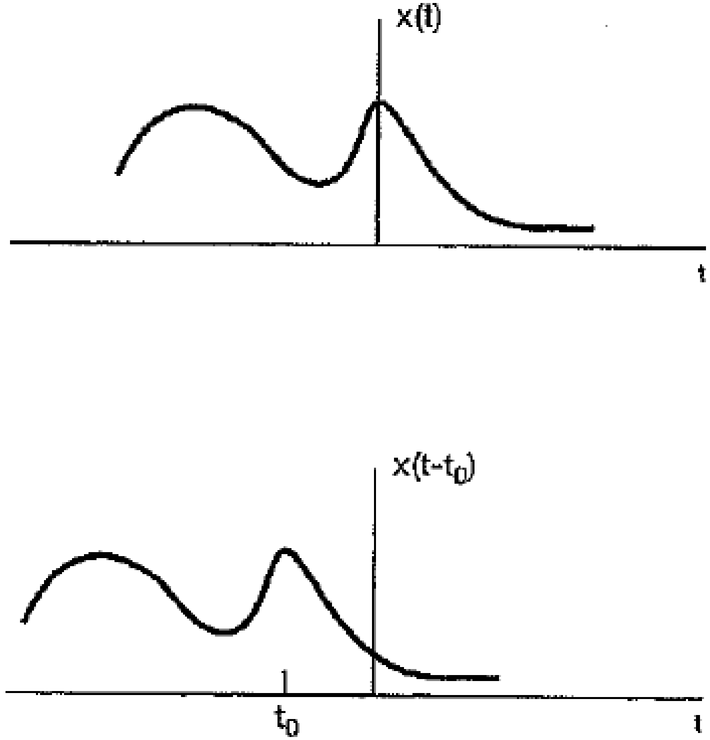
$$y(t) = x(t - T),$$

wherein t is replaced with $(t - T)$ everywhere in the expression



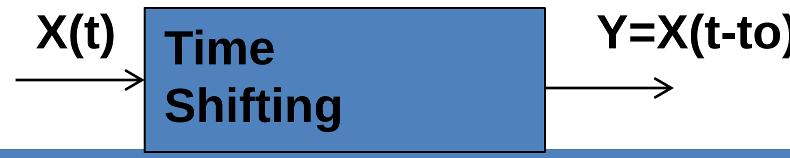
Waveforms of $x(t)$, $x(t - 10)$, and $x(t + 10)$.

Time shift



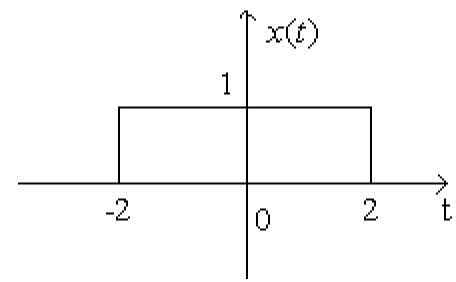
- continuous time signal
- here, however, $t_0 < 0$
- time shifts arise, for example, when multiple receivers in different locations observe a signal or a single receiver receives multiple reflections of a signal.

Time Shifting

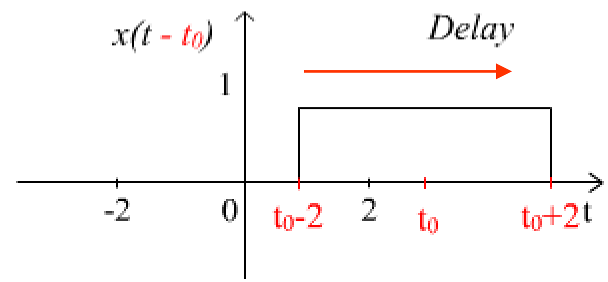


- The original signal $x(t)$ is shifted by an amount $t0$.

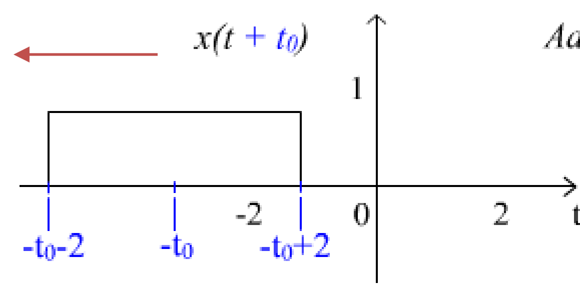
  Time Shift: y(t)=x(t-to)



- X(t)→X(t-to) // to>0 →Signal Delayed→ Shift to the right



- X(t)→X(t+to) // to<0 →Signal Advanced→ Shift to the left



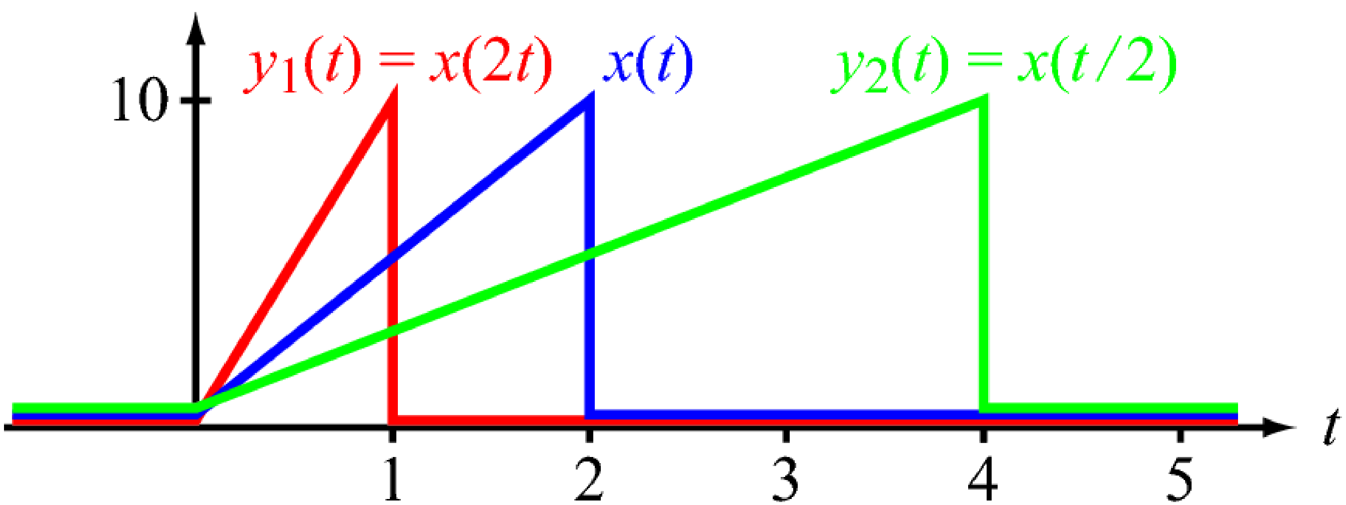
Signal Transformations

Time-scaling Transformation

Mathematically, the ***time-scaling transformation*** can be expressed as

$$y(t) = x(at),$$

where a is a compression or expansion factor depending on whether its absolute value is larger or smaller than 1,



Waveforms of $x(t)$, a compressed replica given by $y_1(t) = x(2t)$, and an expanded replica given by $y_2(t) = x(t/2)$.

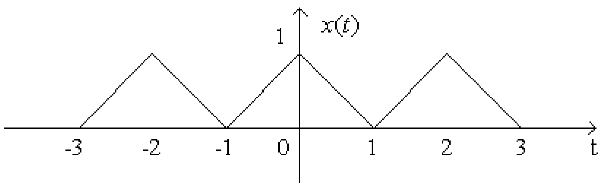
Time scaling

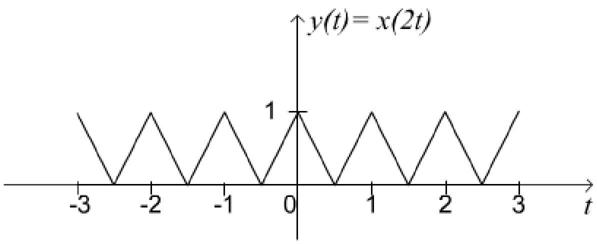
The signal $y(t) = x(at)$ is a time-scaled version of $x(t)$.

If $|a| > 1$, we are SPEEDING UP $x(t)$ by a factor of a.

If $|a| < 1$, we are SLOWING DOWN $x(t)$ by a factor of a.

The signal $y(t)$ has period $= \frac{T}{|a|}$, where T is the period of $x(t)$.





Example: Given $x(t)$, find $y(t) = x(2t)$. This SPEEDS UP $x(t)$ (the graph is shrinking)

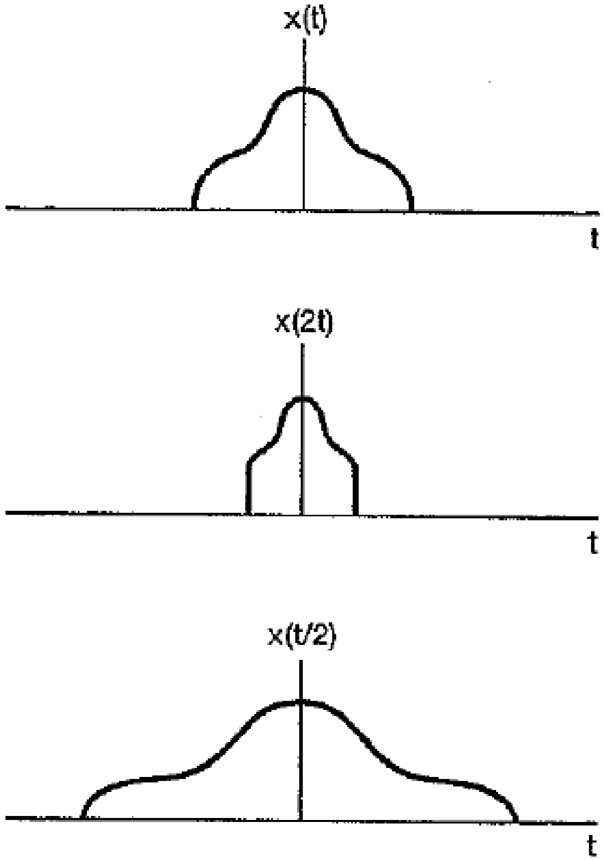
The period decreases!

What happens to the period T?

The period of $x(t)$ is 2 and the period of $y(t)$ is 1,

a>1 → Speeds up → Smaller period → Graph shrinks!
a<1 → slows down → Larger period → Graph expands

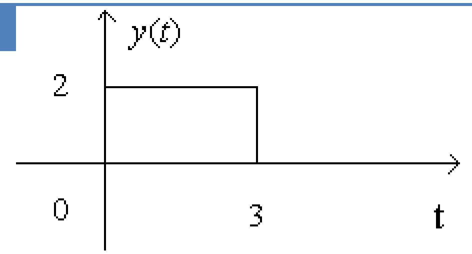
Time Scaling

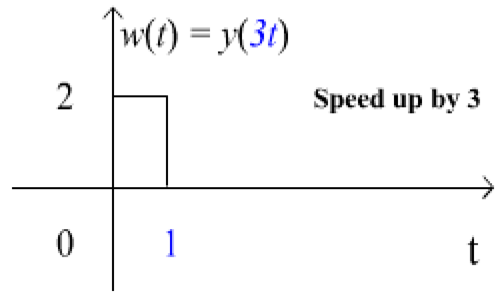


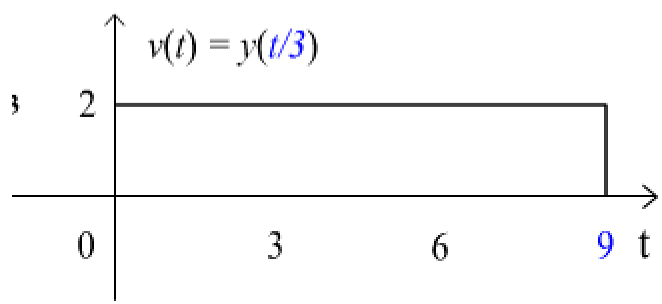
- $x(\alpha t)$: stretched by α if $\alpha < 1$
- $x(\alpha t)$: compressed by α if $\alpha > 1$

Time scaling

- Given $y(t)$,
  - find $w(t) = y(3t)$
  - $v(t) = y(t/3)$.



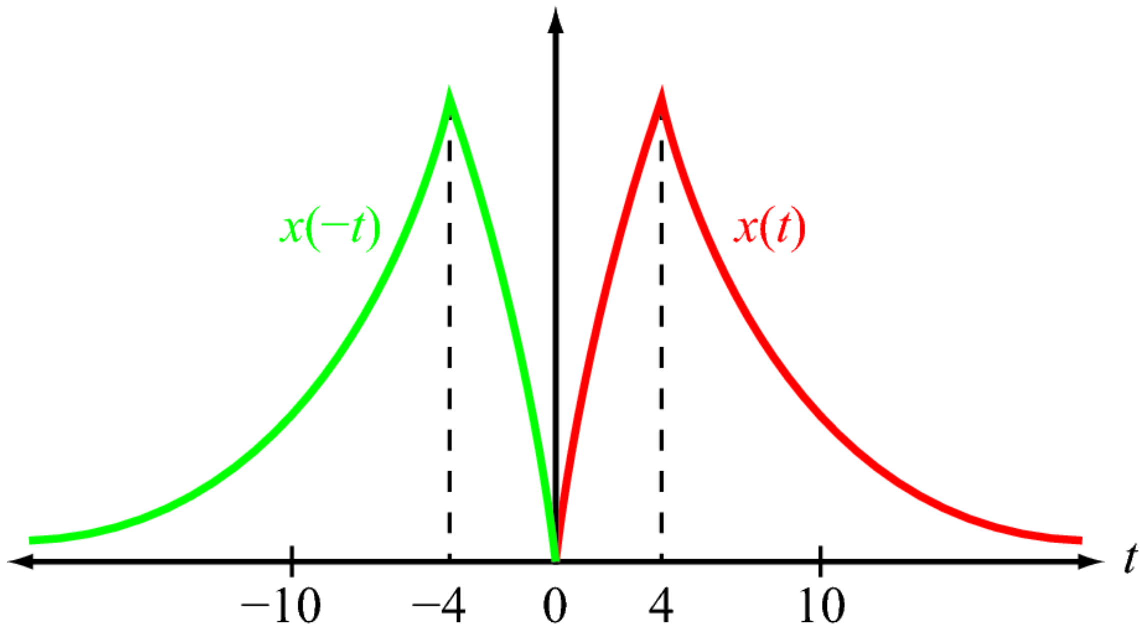




Signal Transformations

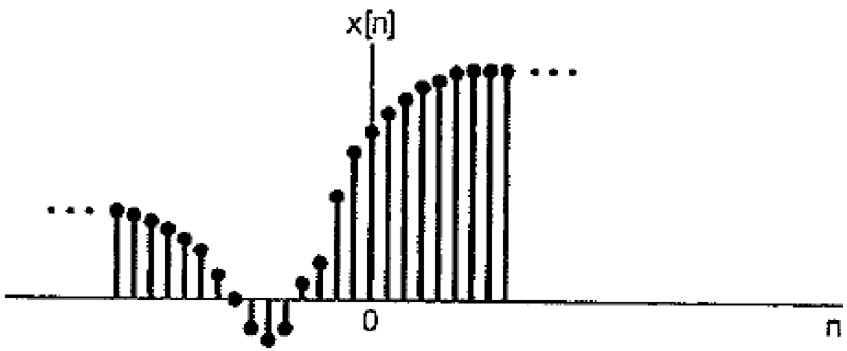
Time-Reversal Transformation

► Replacing t with $-t$ in $x(t)$ generates a signal $y(t)$ whose waveform is the mirror image of that of $x(t)$ with respect to the vertical axis. ◄

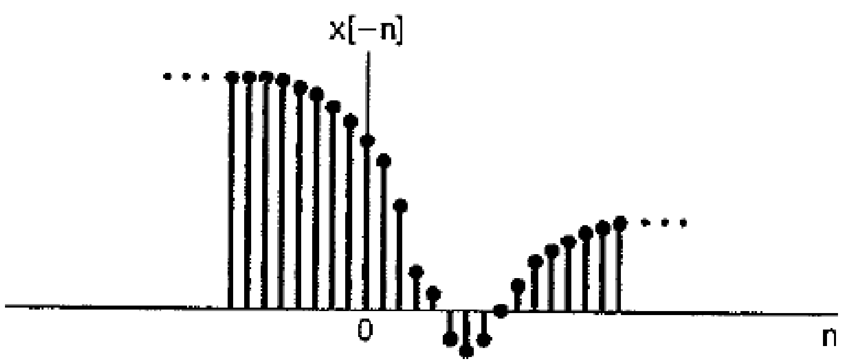


Waveforms of $x(t)$ and its time reversal $x(-t)$.

Time Reversal

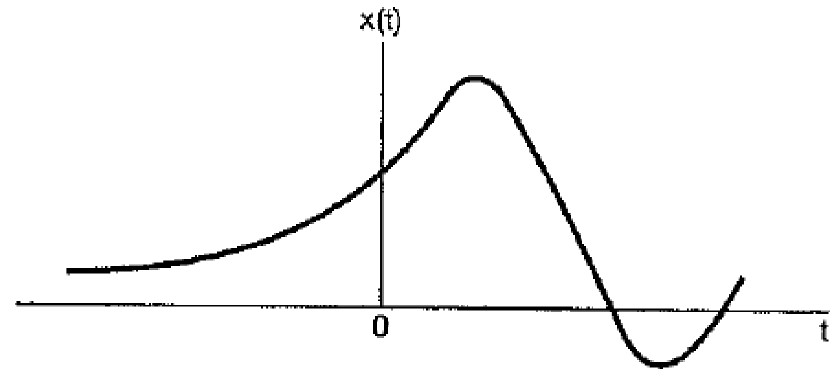


(a)

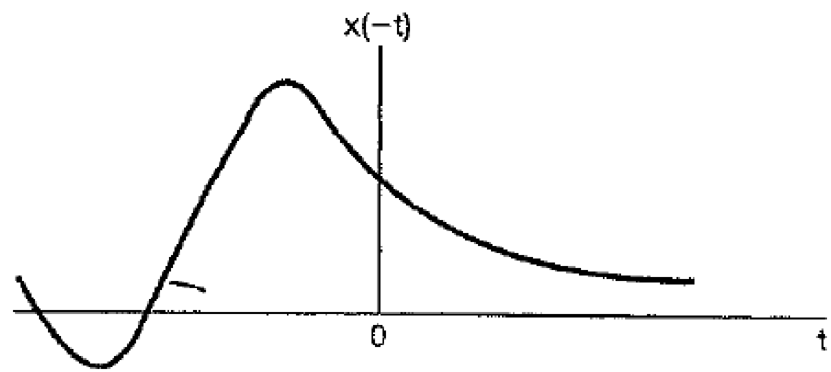


(b)

DT: reflection about $n = 0$



(a)



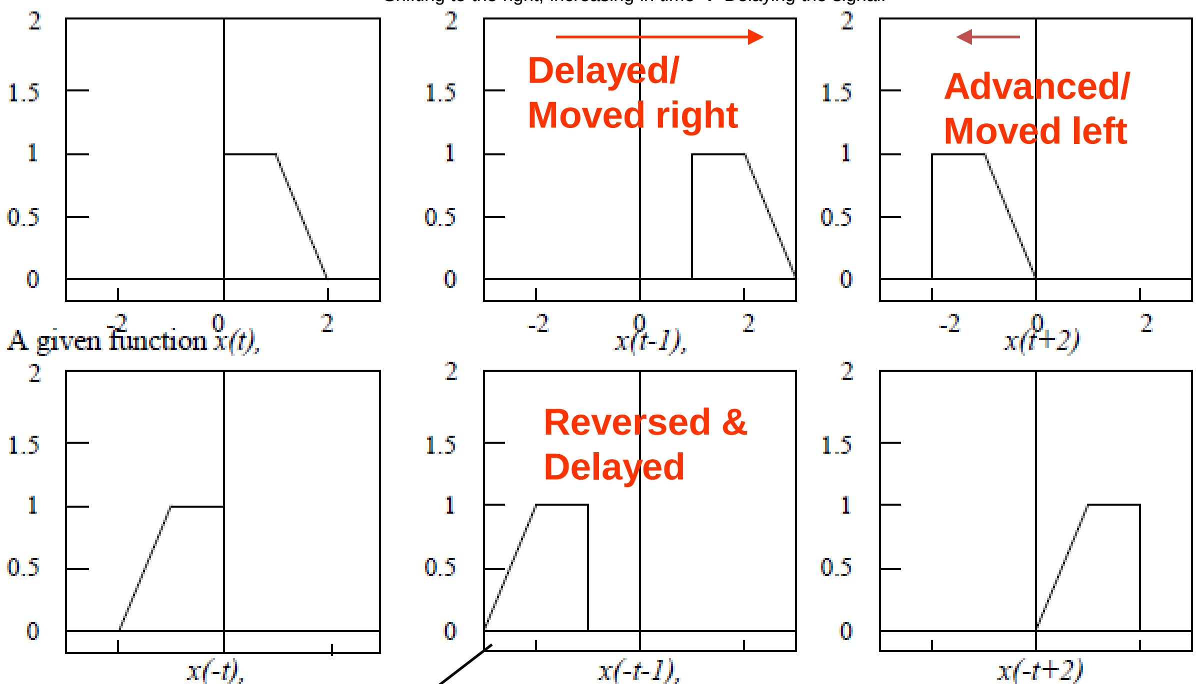
(b)

CT: reflection about $t = 0$

Summary

shift to the **left** of t=0 by two units!

Shifting to the right; increasing in time → Delaying the signal!



Or rewrite as: X[-(t+1)]
Hence, reverse the signal in time.
Then shift to the left of t=0
by one unit!

Or rewrite as: X[-(t-2)]
Hence, reverse the signal in time.
Then shift to the right of t=0
by two units!

This is really:
X(-(t+1))

See Notes

Signal Transformations

Combined (Multiple) Transformation

The three aforementioned transformations can be combined into a generalized transformation:

$$y(t) = x(at - b) = x\left(a\left(t - \frac{b}{a}\right)\right) = x(a(t - T)), \tag{1.5}$$

where $T = b/a$. We recognize T as the time shift and a as the compression/expansion factor. Additionally, the sign of a ($-$ or $+$) denotes whether or not the transformation includes a time-reversal transformation.

Signal Transformation Procedure

The procedure for obtaining $y(t) = x(a(t-T))$ from $x(t)$ is as follows:

(1) Scale time by a:

- If $|a| < 1$, then $x(t)$ is expanded.
- If $|a| > 1$, then $x(t)$ is compressed.
- If $a < 0$, then $x(t)$ is also reflected.

This results in $z(t) = x(at)$.

(2) Time shift by T:

- If $T > 0$, then $z(t)$ shifts to the right.
- If $T < 0$, then $z(t)$ shifts to the left.

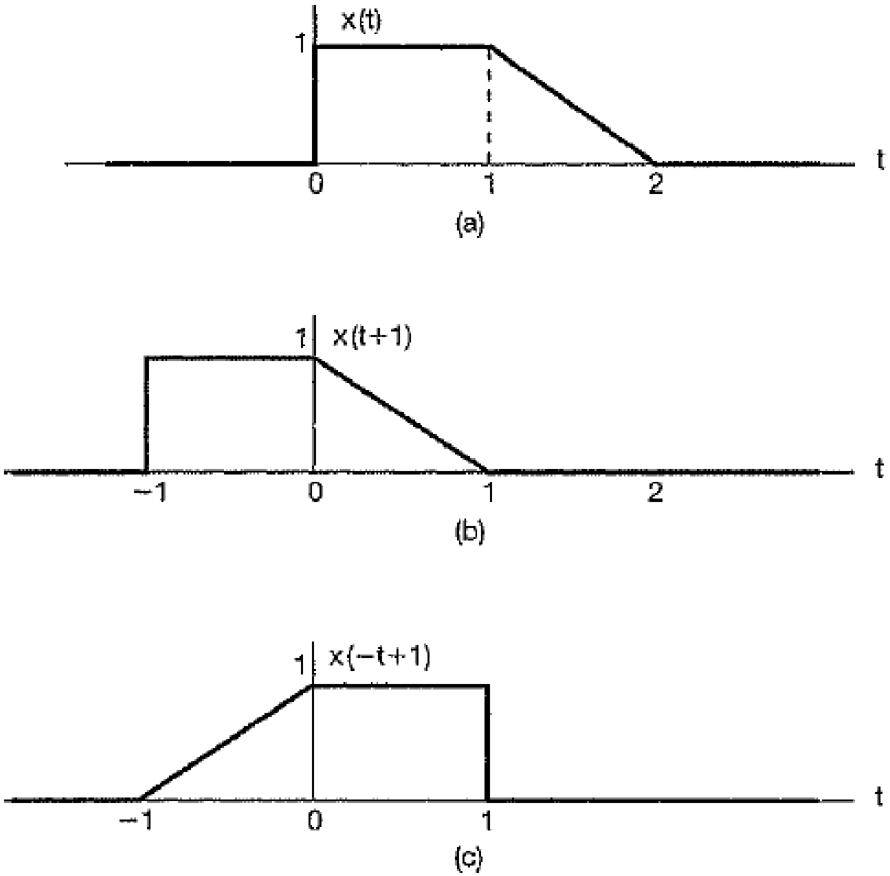
This results in $z(t-T) = x(a(t-T)) = y(t)$.

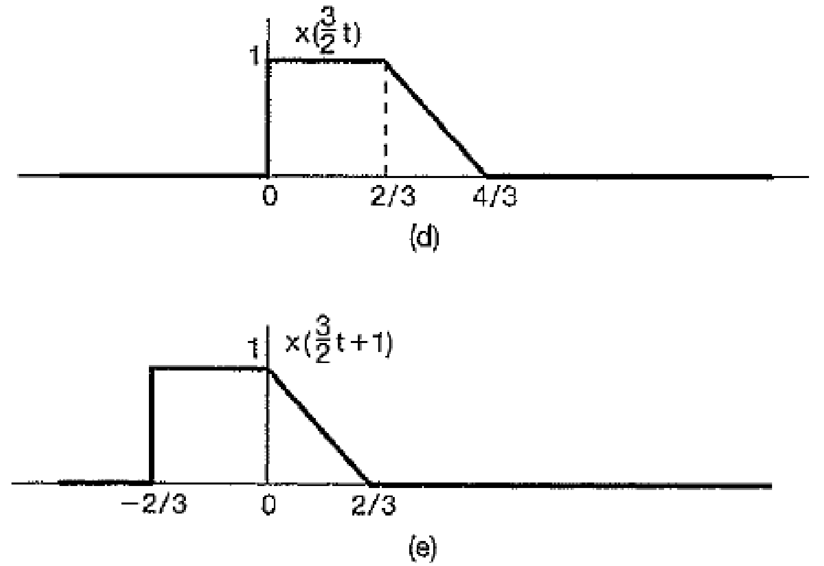
The procedure for obtaining $y(t) = x(at-b)$ from $x(t)$ reverses the order of time scaling and time shifting:

(1) Time shift by b.

(2) Time scale by a.

Multiple Transformations





- NOTE:
  $x(\alpha t + \beta) = x\left(\alpha\left(t + \beta/\alpha\right)\right)$;
  shift by β/α, then scale by α.

Example 1-1: Multiple Transformations

For signal $x(t)$ profiled in Fig. 1-10(a), generate the corresponding profile of $y(t) = x(-2t + 6)$.

Solution:
We start by recasting the expression for the dependent variable into the standard form given by Eq. (1.5),

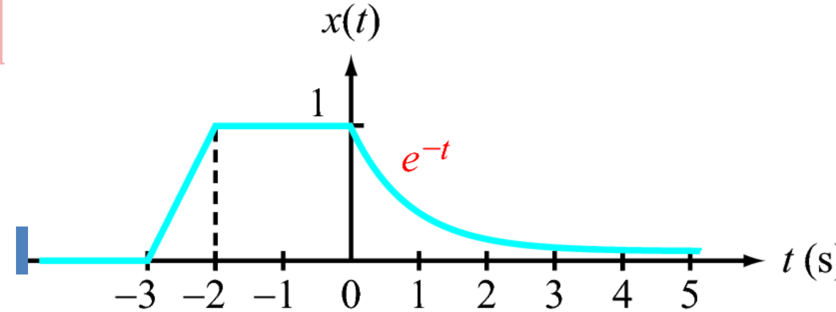
$$y(t) = x\left(-2\left(t - \frac{6}{2}\right)\right)$$
$$= x(-2(t - 3)).$$

Reversal Compression factor Time-shift

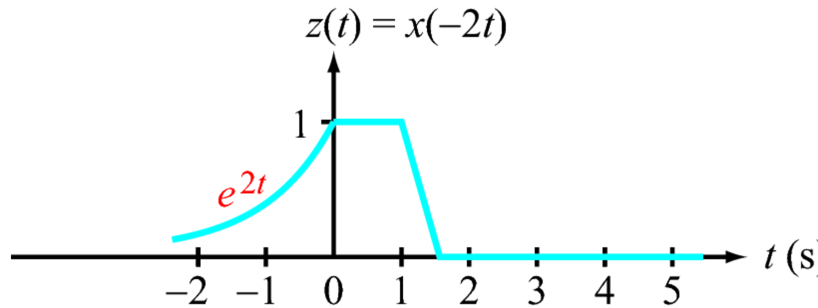
We need to apply the following transformations:

(1) Scale time by $-2t$: This causes the waveform to reflect around the vertical axis and then compresses time by a factor of 2. These steps can be performed in either order. The result, $z(t) = x(-2t)$, is shown in Fig. 1-10(b).

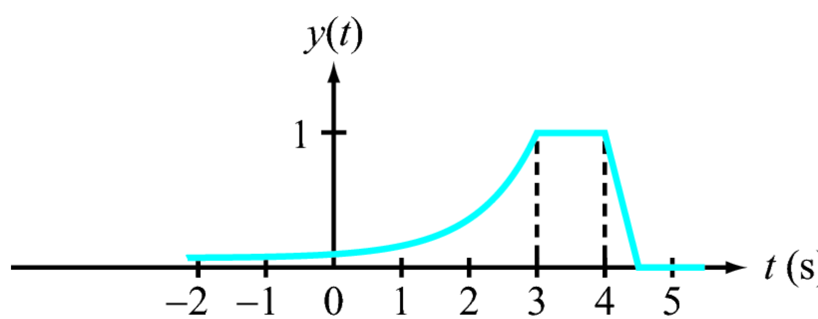
(2) Delay waveform $z(t)$ by 3 s: This shifts the waveform to the right by 3 s (because the sign of the time shift is negative). The result, $y(t) = z(t - 3) = x(-2(t - 3))$, is displayed in Fig. 1-10(c).



(a) $x(t)$



(b) $z(t)$



(c) $y(t)$

Amplitude Operations

In general:

$y(t)=Ax(t)+B$

B>0 → Shift up

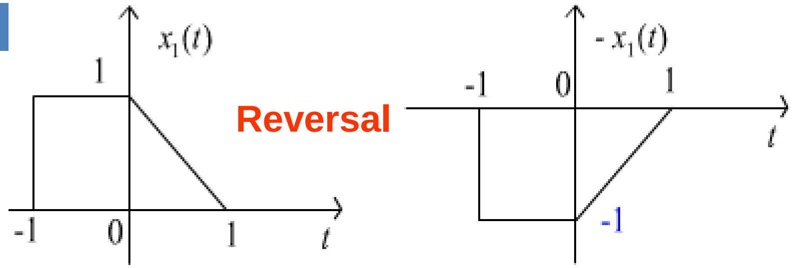
B<0 → Shift down

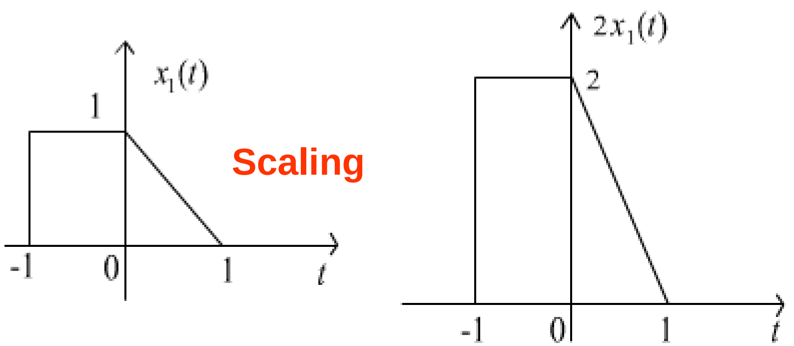
|A|>1→ Gain

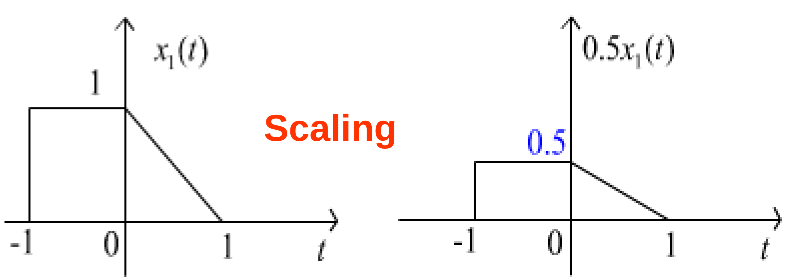
|A|<1→ Attenuation

A>0→NO reversal

A<0→ reversal



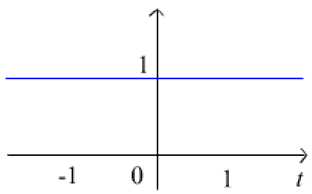


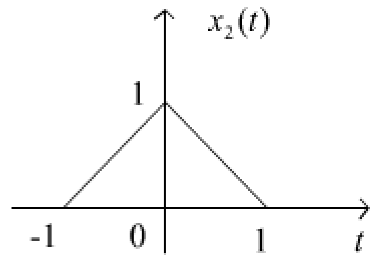


Amplitude Operations

Given $x_2(t)$, find $1 - x_2(t)$.

Remember:
This is y(t) =1

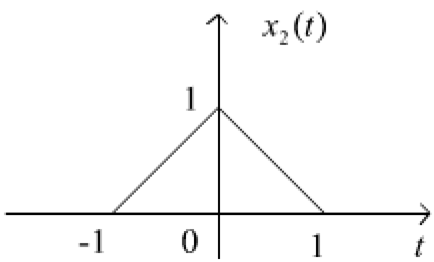


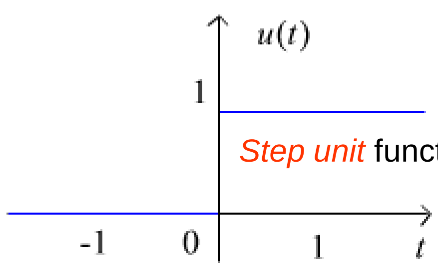


Ans.

Multiplication of two signals:$x_2(t)u(t)$

Find $x_2(t)u(t)$





Step unit function

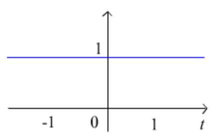
Ans.

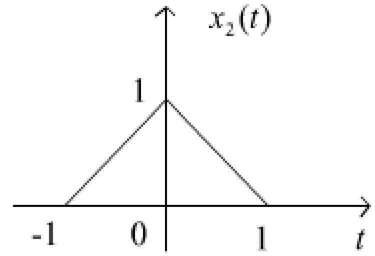
Signals can be added or multiplied

Amplitude Operations

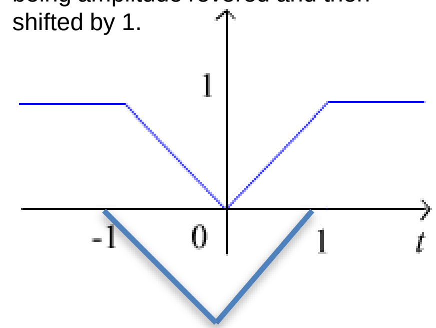
Given $x_2(t)$, find $1 - x_2(t)$.

Remember: This is y(t) =1



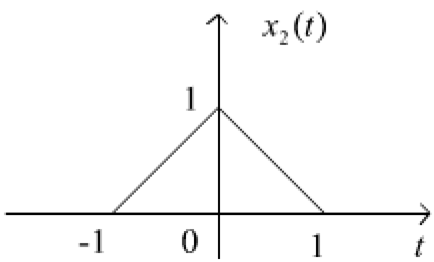


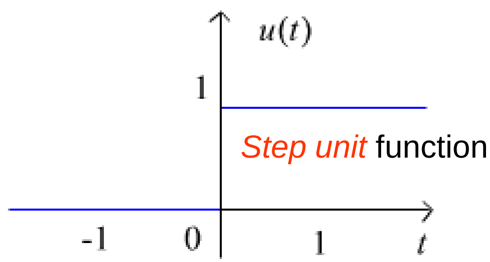
Note: You can also think of it as $X_2(t)$ being amplitude revered and then shifted by 1.

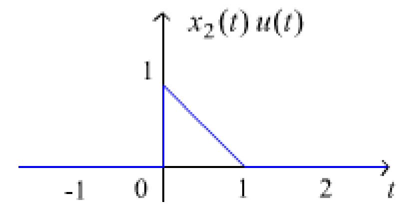


Multiplication of two signals: $x_2(t)u(t)$

Find $x_2(t)u(t)$







Signals can be added or multiplied → e.g., we can *filter* parts of a signal!

Odd/even symmetry

Even symmetry

▶ A signal $x(t)$ exhibits ***even symmetry*** if its waveform is symmetrical with respect to the vertical axis. ◀

$$x(t) = x(-t) \qquad \textbf{(even symmetry)}.$$

Odd symmetry

▶ A signal exhibits odd symmetry if the shape of its waveform on the left-hand side of the vertical axis is the *inverted* mirror image of the waveform on the right-hand side. ◀

$$x(t) = -x(-t) \qquad \textbf{(odd symmetry)}.$$

Even/Odd Synthesis

$$x(t) = x_e(t) + x_o(t),$$

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)],$$

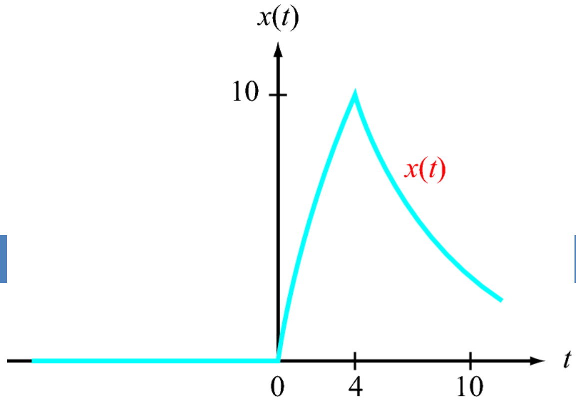
$$x_o(t) = \frac{1}{2}[x(t) - x(-t)].$$

$$(\text{even}) \times (\text{even}) = \text{even},$$

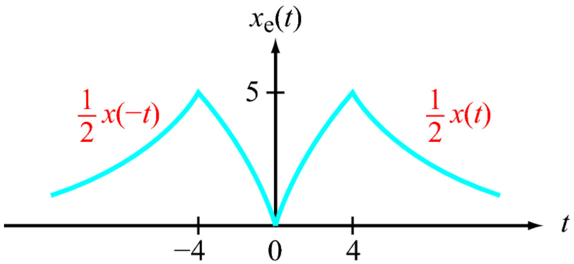
$$(\text{even}) \times (\text{odd}) = \text{odd},$$

and

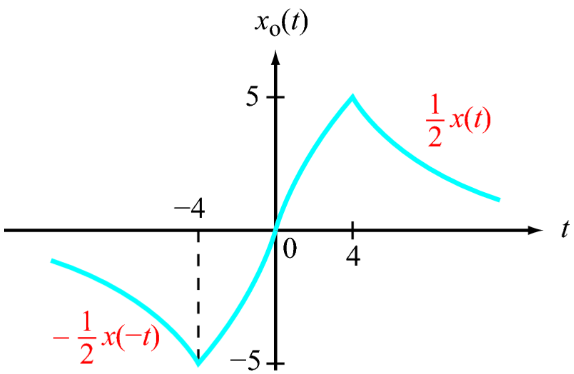
$$(\text{odd}) \times (\text{odd}) = \text{even}.$$



(a) $x(t)$



(b) $x_e(t)$



(c) $x_o(t)$

Even and Odd Signals

- Consider the *symmetry* of a signal
- Continuous time signal is *even* if

$$x(-t) = x(t)$$

- Discrete time signal is even if

$$x[-n] = x[n]$$

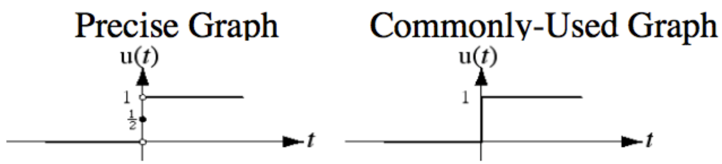
- signal is *odd* if

$$x(-t) = -x(t)$$
$$x[-n] = -x[n]$$

Signal Characteristics:
Find $u_o(t)$ and $u_e(t)$

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \\ \text{undefined}, & t = 0 \end{cases}$$

Remember:

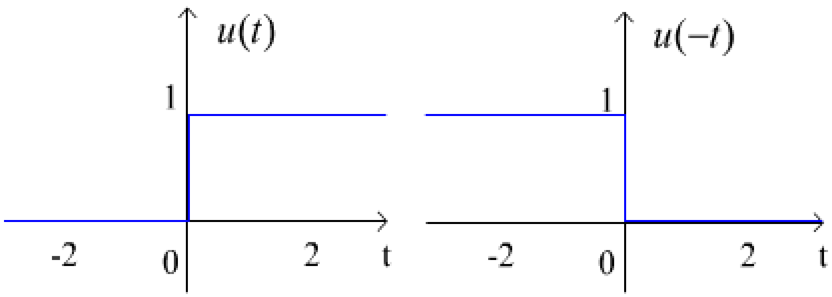


Note: The signal is discontinuous at zero but is an analog signal

Note: The product signal g(t)u(t) for any g(t) can be thought of as the signal g(t) "turned on" at time t = 0.

find $u_e(t)$ and $u_o(t)$

Given:



Signal Characteristics

An even signal is symmetric across the vertical axis.
An odd signal is anti-symmetric across the vertical axis.

$$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right)$$

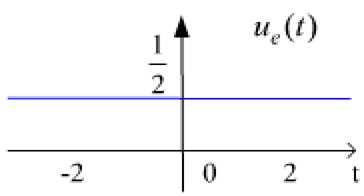
$$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right)$$

given the *unit step function* (a discontinuous continuous-time signal),

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \\ \text{undefined}, & t = 0 \end{cases}$$

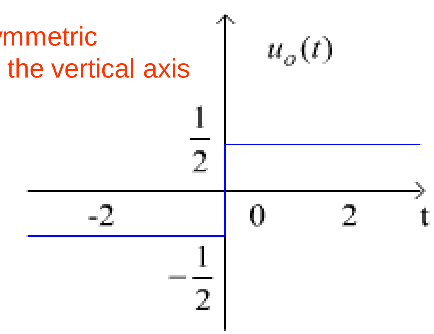
Symmetric across the vertical axis

$$u_e(t) = \frac{1}{2}\left(u(t) + u(-t) \right)$$



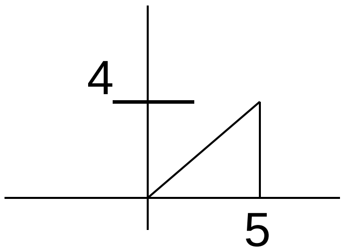
Anti-symmetric across the vertical axis

$$u_o(t) = \frac{1}{2}\left(u(t) - u(-t) \right)$$



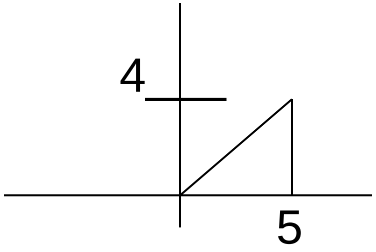
Example

- Given x(t) find $x_e(t)$ and $x_o(t)$



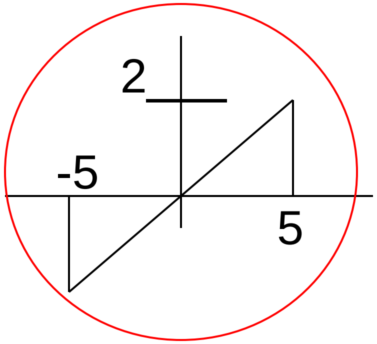
Example

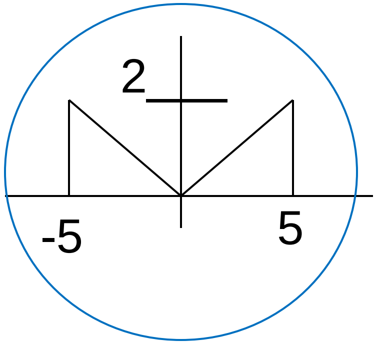
- Given x(t) find $x_e(t)$ and $x_o(t)$



$$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right)$$

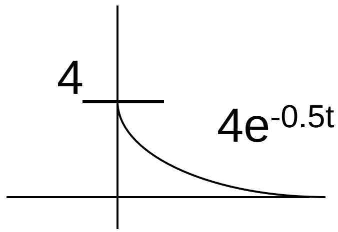
$$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right)$$





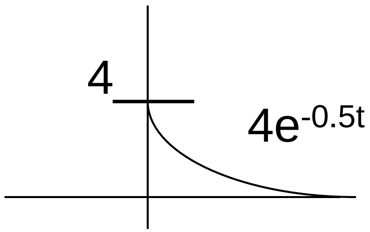
Example

- Given $x(t)$ find $x_e(t)$ and $x_o(t)$



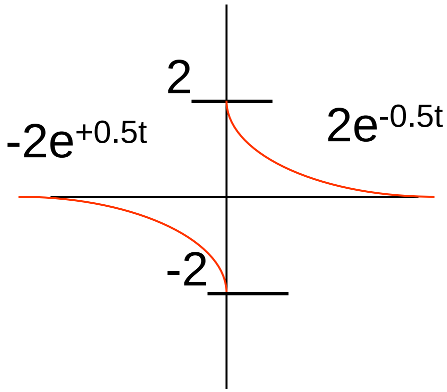
Example

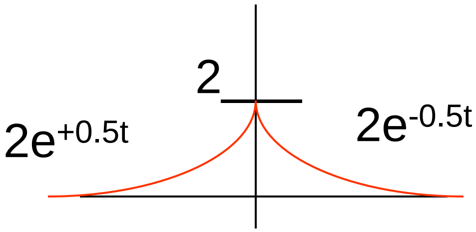
- Given x(t) find $x_e(t)$ and $x_o(t)$



$$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right)$$

$$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right)$$





Summary so far…

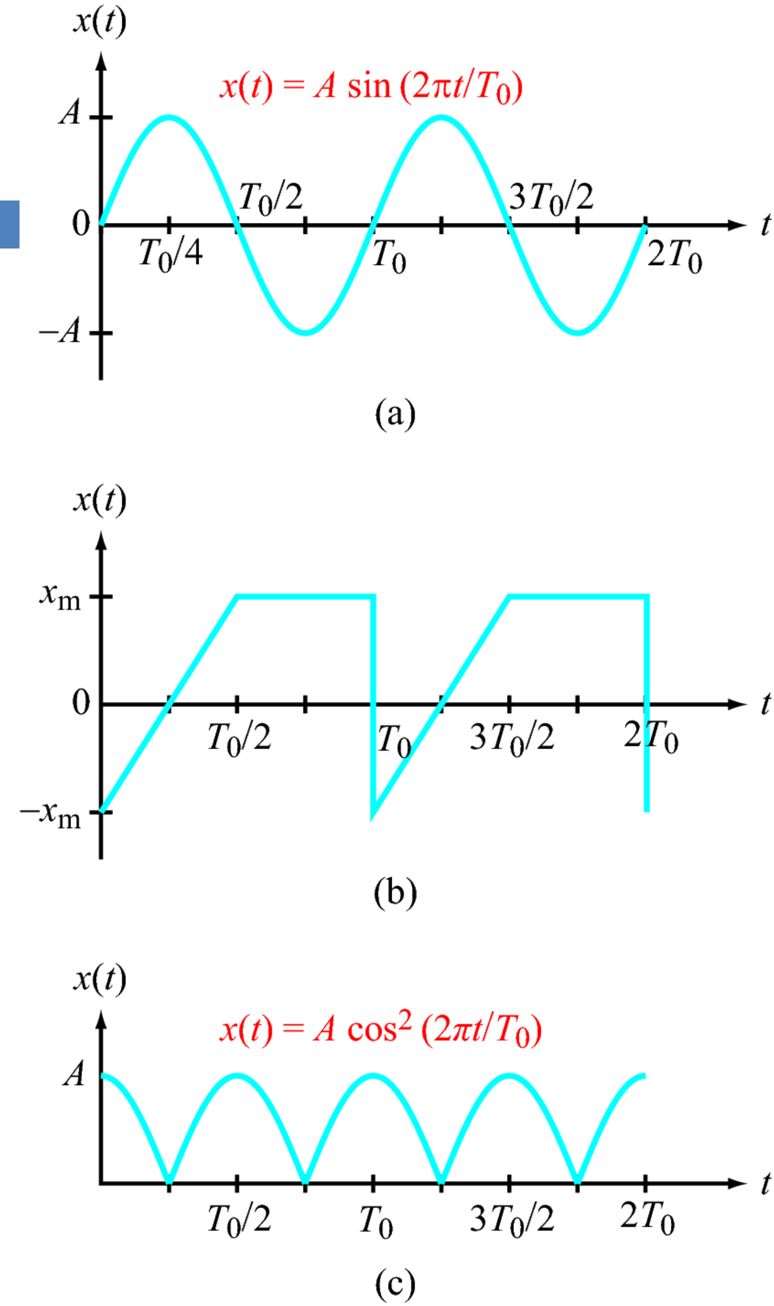
Transformation	Expression	Consequence
Time shift	$y(t) = x(t - T)$	Waveform is shifted along $+t$ direction if $T > 0$ and along $-t$ direction if $T < 0$.
Time scaling	$y(t) = x(at)$	Waveform is compressed if $\lvert a \rvert > 1$ and expanded if $\lvert a \rvert < 1$.
Time reversal	$y(t) = x(-t)$	Waveform is mirror-imaged relative to vertical axis.
Generalized	$y(t) = x(at - b)$	Shift by b, then scale by a, time reversal if $a < 0$.
Even / odd synthesis	$x(t) = x_e(t) + x_o(t)$	$x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}, \ x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$.

Periodic Signals

▶ A periodic signal $x(t)$ of period T_0 satisfies the ***periodicity*** property:

$$x(t) = x(t + nT_0)$$

for all integer values of n and all times t. ◀



Periodic Signals

- There is a positive value of T such that

$$x(t) = x(t+T) \quad \forall t$$

- Implies that can also write

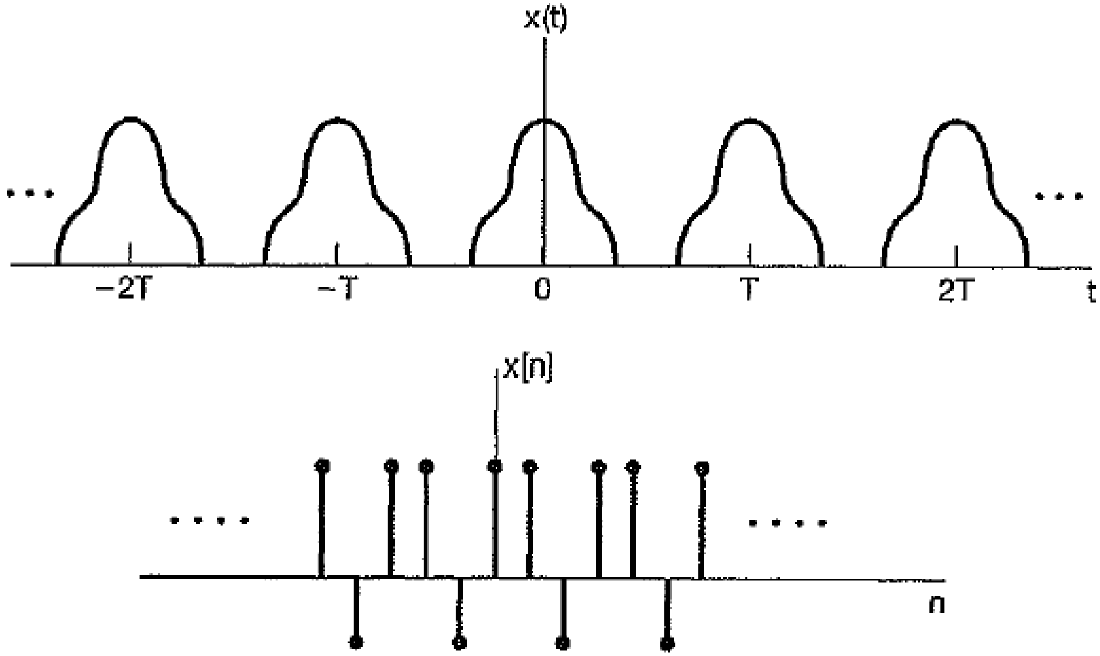
$$x(t) = x(t+mT) \quad \forall t, m \in \mathbb{Z}$$

- Smallest value of T is the *fundamental period*, denoted T_0
- Discrete time signals:

$$x[n] = x[n+N] \quad \forall n$$

- *fundamental period* N_0

Periodic Signals



Periodic sinusoids

The most important family of periodic signals are sinusoids. A sinusoidal signal $x(t)$ has the form

$$x(t) = A\cos(\omega_0 t + \theta), \qquad -\infty < t < \infty,$$

where

$$A = \text{amplitude}; \quad x_{\max} = A \quad \text{and} \quad x_{\min} = -A,$$

$$\omega_0 = \text{angular frequency in rad/s},$$

$$\theta = \text{phase-angle shift in radians or degrees}.$$

Related quantities include

$$f_0 = \omega_0/2\pi = \text{circular frequency in Hertz},$$

$$T_0 = 1/f_0 = \text{period of } x(t) \text{ in seconds}.$$

Periodic complex exponential

Another important periodic signal is the complex exponential given by

$$x(t) = Ae^{j\omega_0 t} = |A|e^{j(\omega_0 t+\theta)},$$

where, in general, A is a complex amplitude given by

$$A = |A|e^{j\theta}.$$

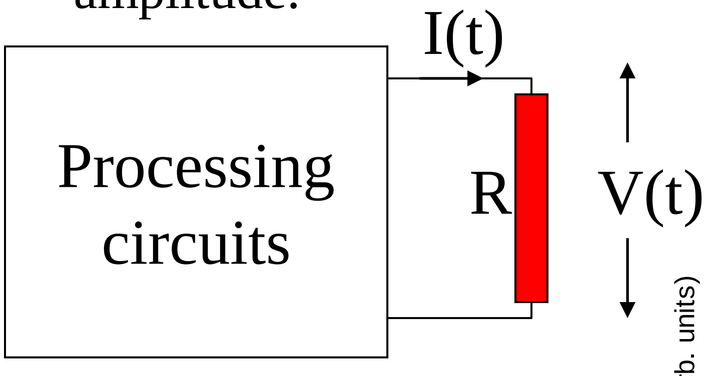
By Euler's formula,

$$x(t) = |A|e^{j(\omega_0 t+\theta)} = |A|\cos(\omega_0 t + \theta) + j|A|\sin(\omega_0 t + \theta).$$

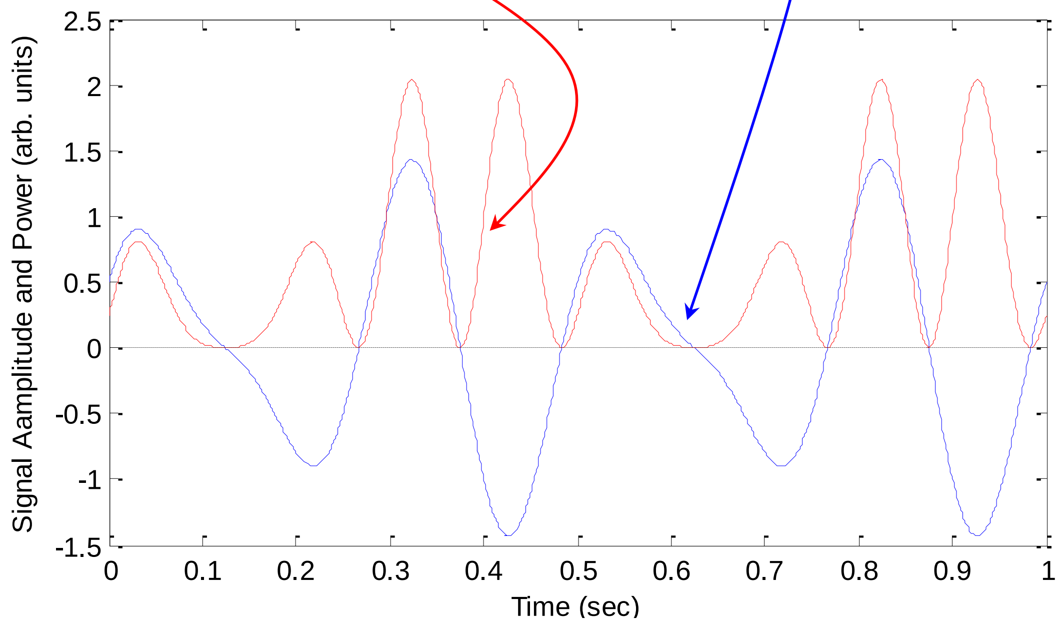
Hence, the complex exponential is periodic with period $T_0 = 2\pi/\omega_0$.

Analog Signals : Energy and Power of Signals

For voltage (or current) signals, the instantaneous power delivered to a load is (generally speaking) proportional to the square of the signal amplitude.



$$P(t)=\frac{V^2(t)}{R}=I^2(t)R$$

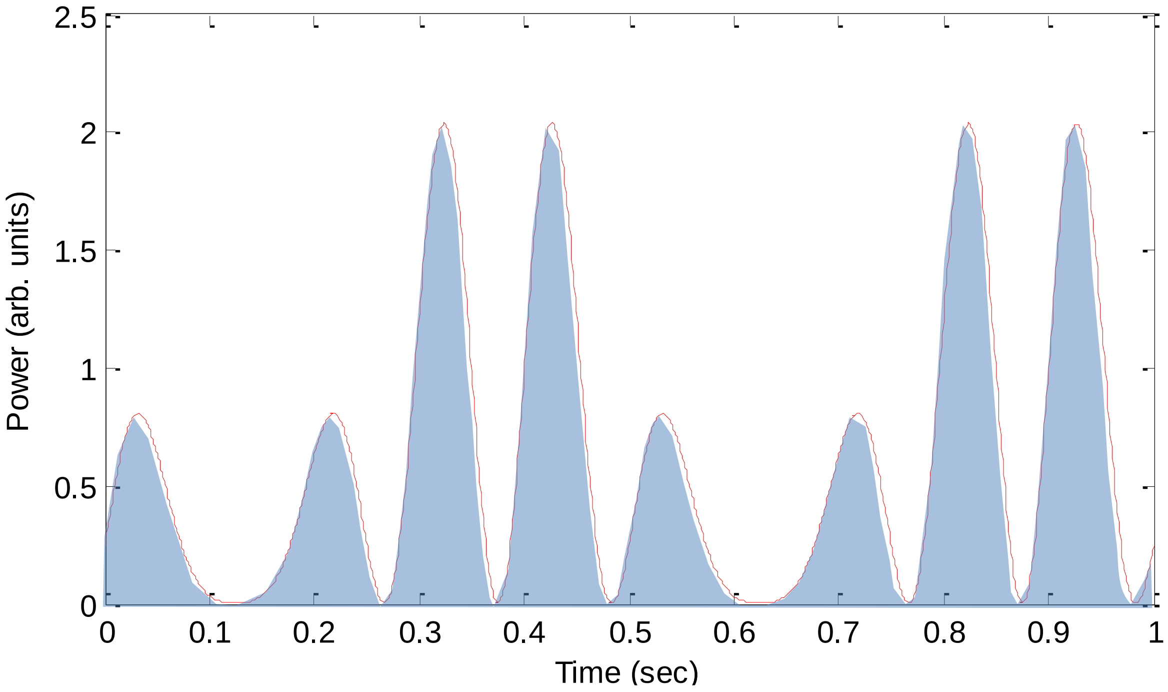


Analog Signals : Energy and Power of Signals

The total energy (= power x time) in the signal is then proportional to the area under the power vs time curve

$$E \propto \int_{-\infty}^{\infty} V^2(t)\,dt$$

$$E \propto \int_{-\infty}^{\infty} I^2(t)\,dt$$

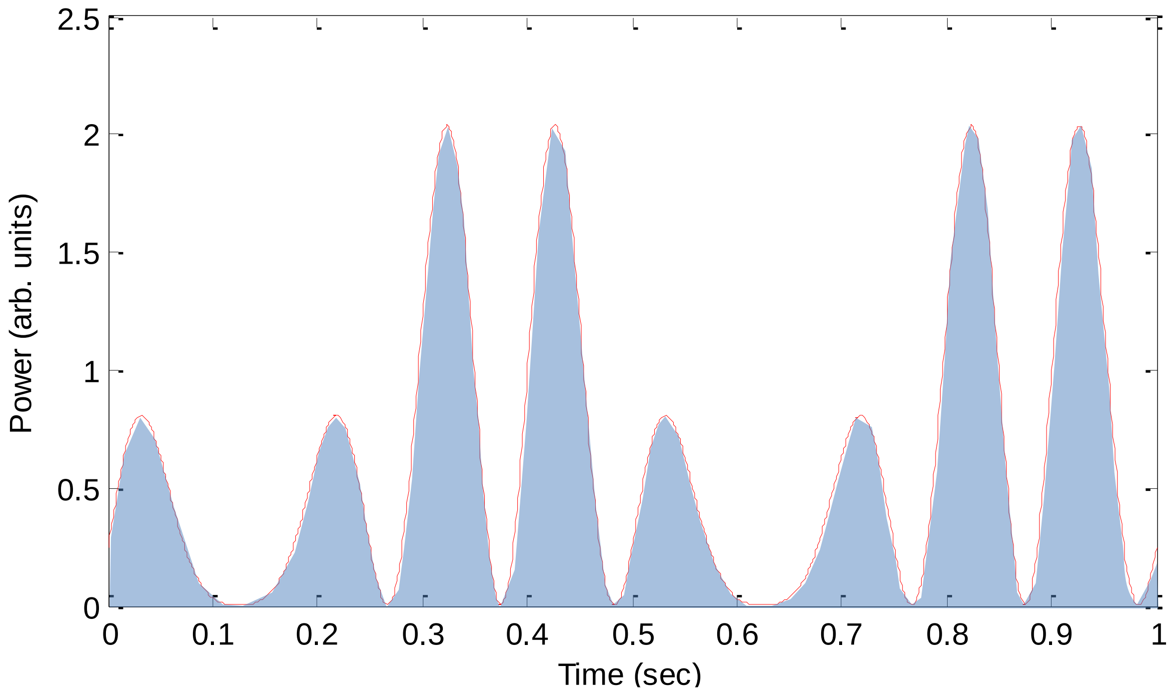


Analog Signals : Energy and Power of Signals

This relation is used as a convenient definition for all signals, regardless of their physical nature.

Thus the total energy of a signal, *f(t)*, is defined to be

$$E = \int_{-\infty}^{\infty} \left| f(t) \right|^2 dt$$



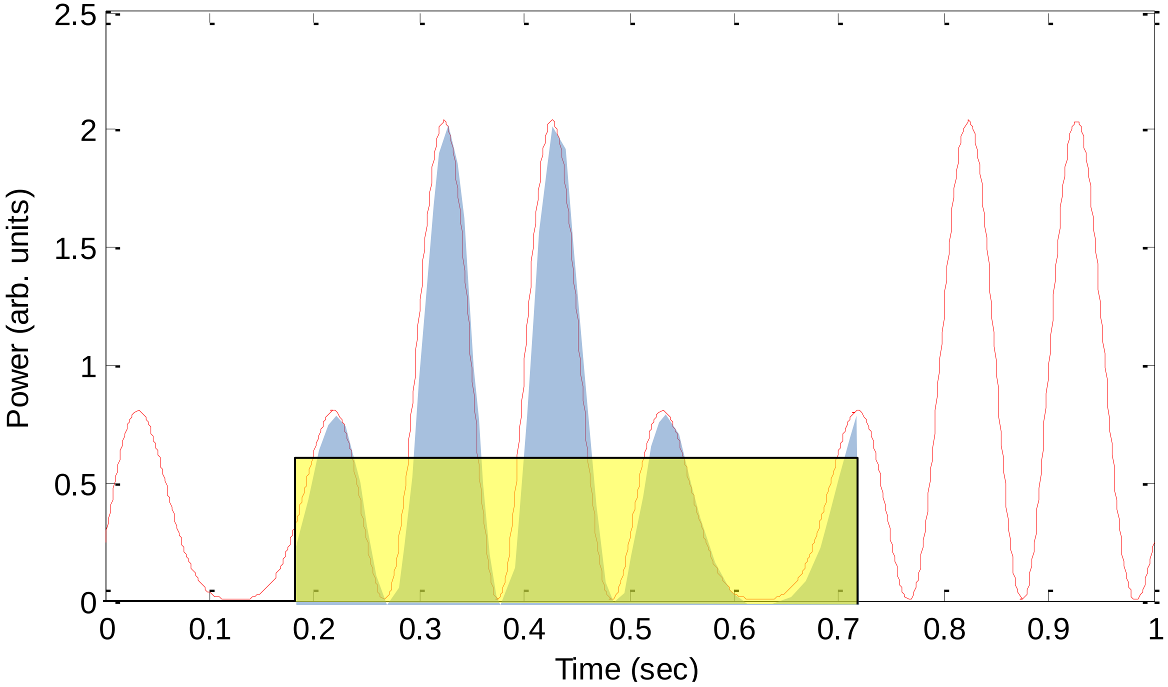
The magnitude in this expression allows for the case in which *f(t)* is complex valued.

Analog Signals : Energy and Power of Signals

The average power in a signal over a given interval of duration T centred on the point t , is defined to be the energy in the time interval T divided by the duration of the interval.

$$\overline{P}(t) = \frac{1}{T} \int_{t-\frac{1}{2}T}^{t+\frac{1}{2}T} \left| f(t') \right|^2 dt'$$

The average power is the instantaneous power of a constant signal with the same energy in the interval as the signal.

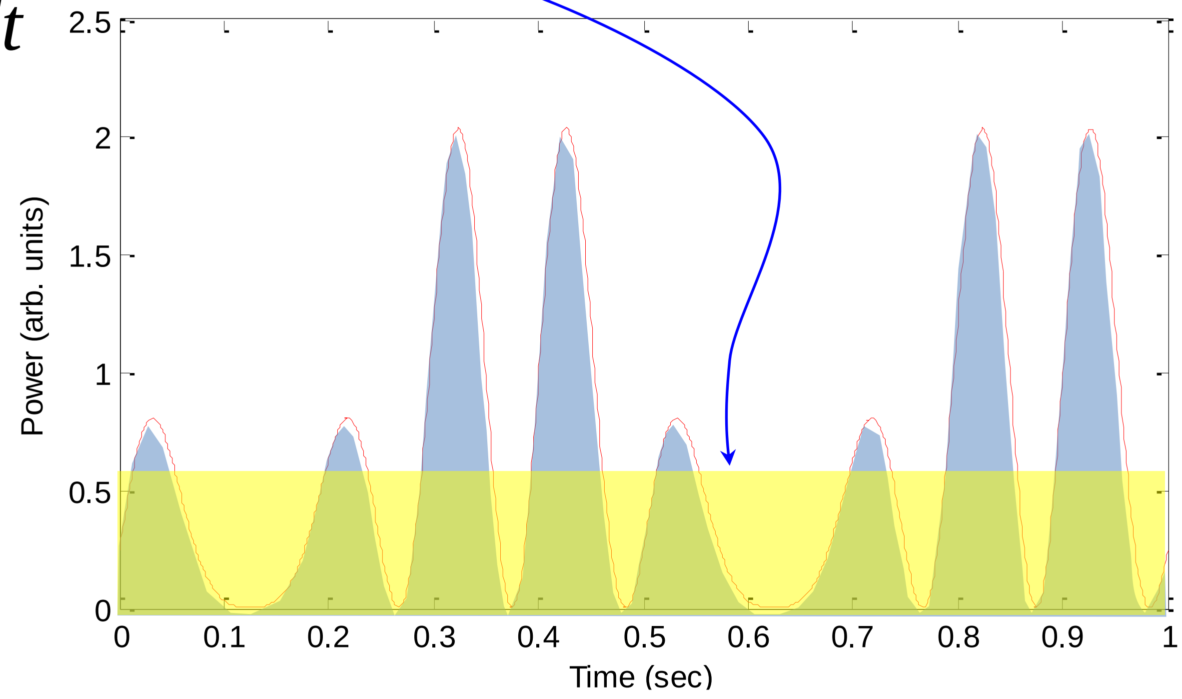


Analog Signals : Energy and Power of Signals

The total average power in a signal is defined as the limiting value of the average power over an interval as the duration of the interval goes to infinity.

$$\overline{P} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} \left| f(t) \right|^2 dt$$

Note that there is no guarantee that the limit will exist, although for real physical signals it should.



Analog Signals : Finite Energy Signals

Pulse-like signals form an important sub-class of analog signals. One mathematical generalisation is the class of finite energy signals.

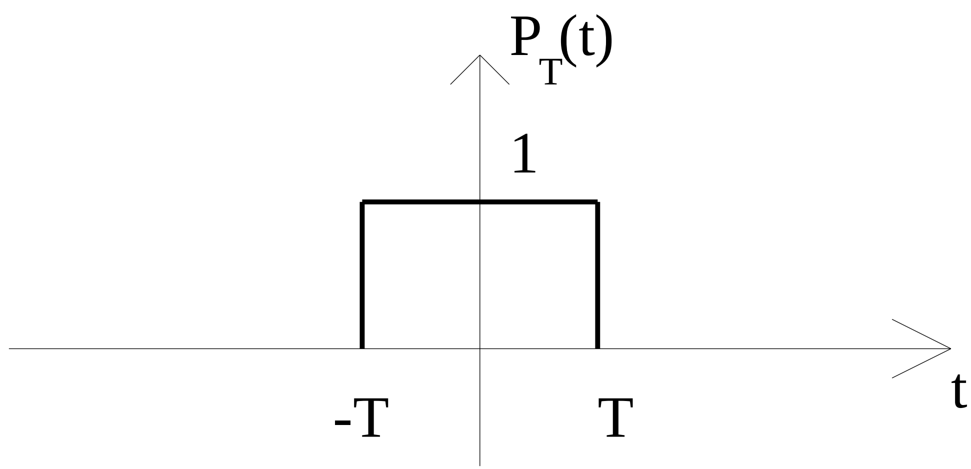
For finite energy signals the total energy of the signal is finite.

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty$$

Analog Signals : Finite Energy Signals

For example the unit height rectangular pulse of half-width T, which we will write as $P_T(t)$, is defined by

$$P_T(t) = \begin{cases} 1 \quad ; \ |t| \leq T \\ 0 \quad ; \ |t| > T \end{cases}$$



Analog Signals : Finite Energy Signals

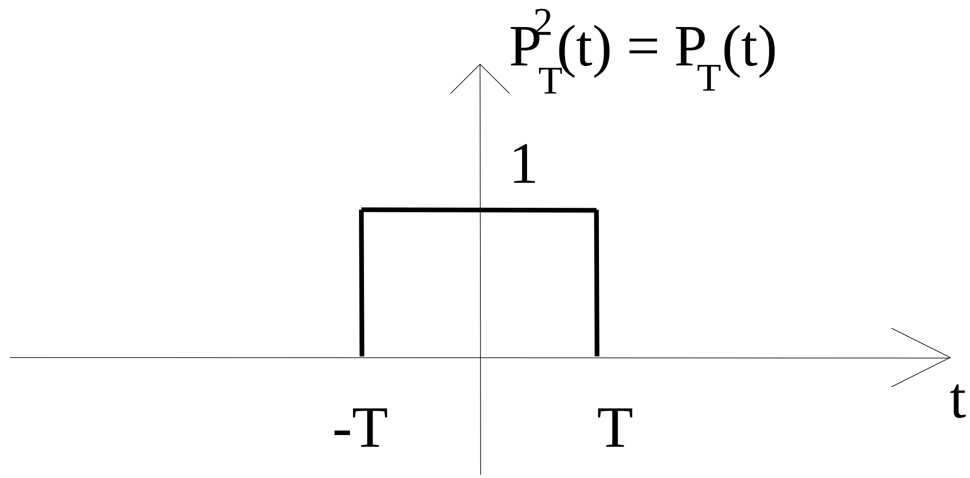
This obviously has finite energy, since

$$\mathrm{E} = \int_{-\infty}^{\infty} f^2(t)dt$$

$$= \int_{-\infty}^{\infty} P_T^2(t)dt$$

$$= \int_{-T}^{T} dt$$

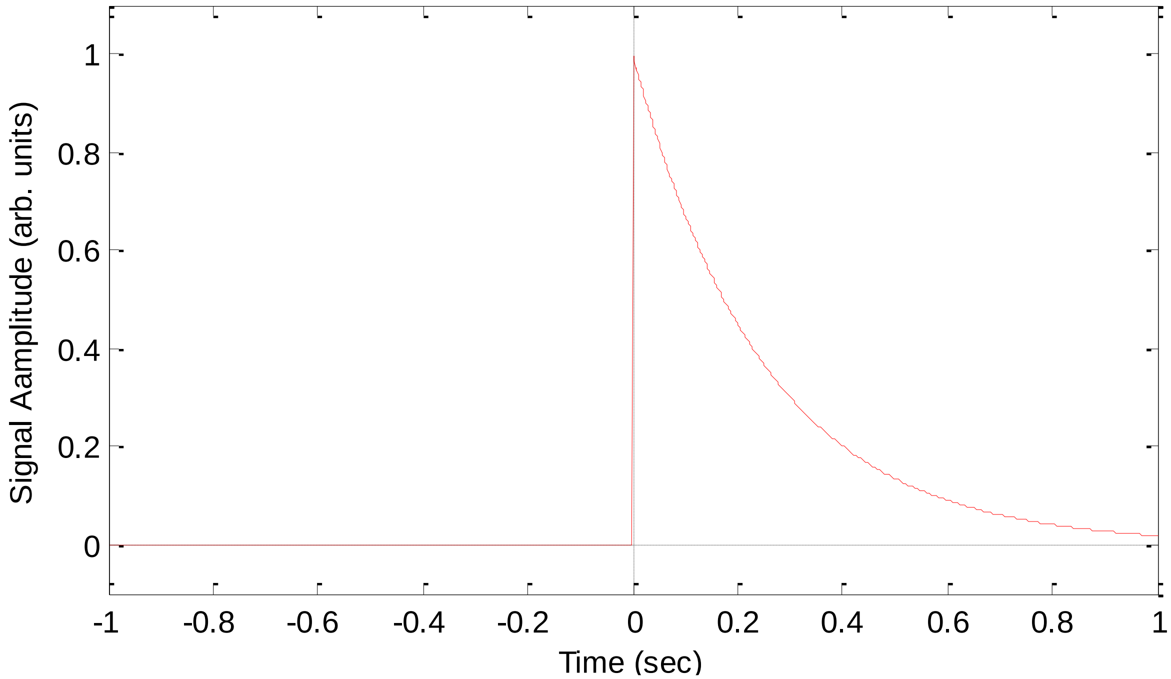
$$= 2T < \infty$$



Analog Signals : Finite Energy Signals

Not all finite energy signals are of "compact support" (ie. zero for all t outside some finite interval).
An important example is the "causal exponential" :

$$f(t) = \begin{cases} 0 & : t < 0 \\ e^{-t/T} & : \mathrm{t} \geq 0 \end{cases}$$



Analog Signals : Finite Energy Signals

This signal has non-zero values for all t > 0, but the energy is finite :

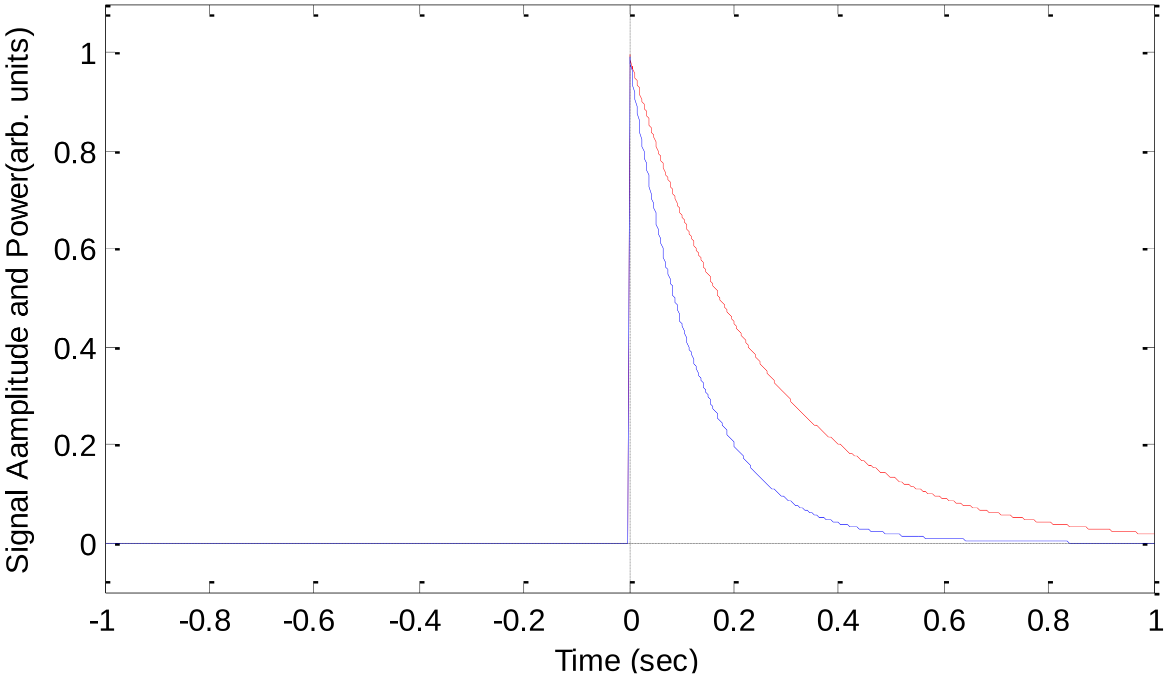
$$f^2(t) = \begin{cases} 0 & : t < 0 \\ e^{-2t/T} & : t \geq 0 \end{cases}$$

$$\mathrm{E} = \int_{-\infty}^{\infty} f^2(t)dt$$

$$= \int_{0}^{\infty} e^{-2t/T} dt$$

$$= -\frac{T}{2}\left[e^{-2t/T}\right]_0^{\infty}$$

$$= \frac{T}{2} < \infty$$

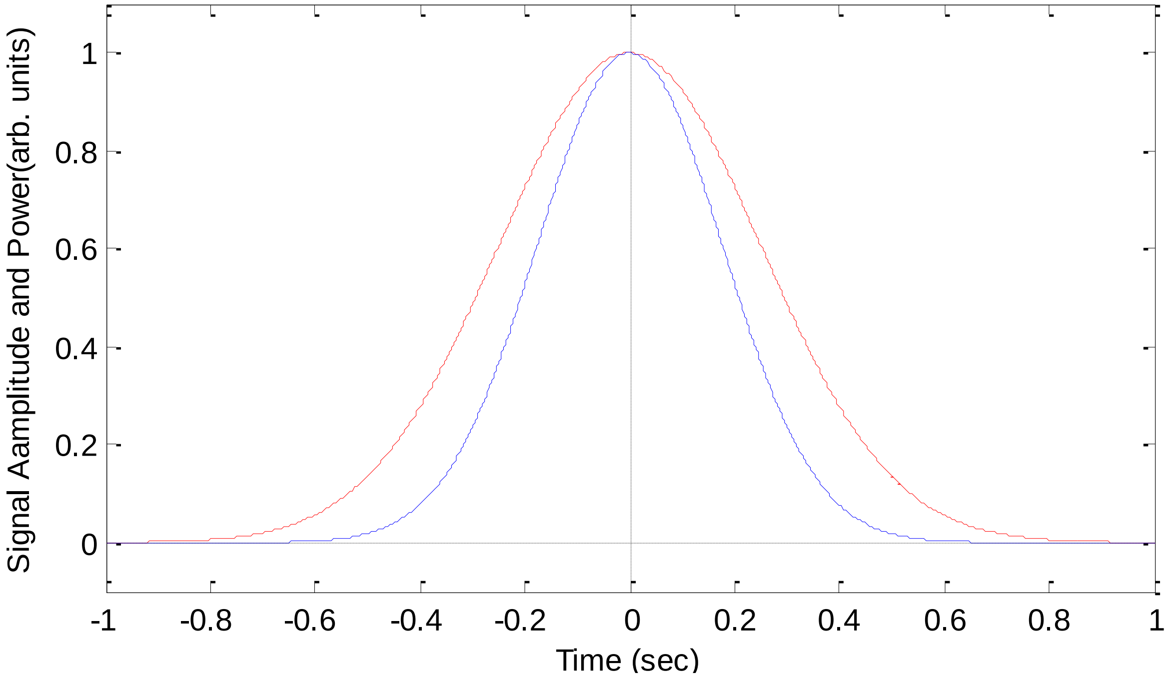


Analog Signals : Finite Energy Signals

The Gaussian pulse is also an important pulse-like signal frequently encountered in signal processing.

$$f(t) = e^{-\frac{1}{2}(t/T)^2}$$

$$f^2(t) = e^{-(t/T)^2}$$



Analog Signals : Finite Energy Signals

That this is finite energy follows from the well-known result:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2}\,dt = \sqrt{2\pi}$$

which leads to :

$$\int_{-\infty}^{\infty} e^{-\left(\frac{t}{T}\right)^2}\,dt = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{t}{\left(T/\sqrt{2}\right)}\right)^2}\,dt$$

$$= \left(T/\sqrt{2}\right)\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{t}{\left(T/\sqrt{2}\right)}\right)^2}\,d\left(\frac{t}{T/\sqrt{2}}\right)$$

$$= \left(T/\sqrt{2}\right)\int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2}\,ds$$

$$= T\sqrt{\pi}$$

$$< \infty$$

Analog Signals : Finite Power Signals

In reality all physical signals must be finite energy, but many signals are conveniently *idealized* by functions which do not have the finite energy property.

These signals are rather like periodic signals and noise, in that they neither decay or explode at time increases.

The mathematical property that defines this class of signals is that the average power of the signal is finite.

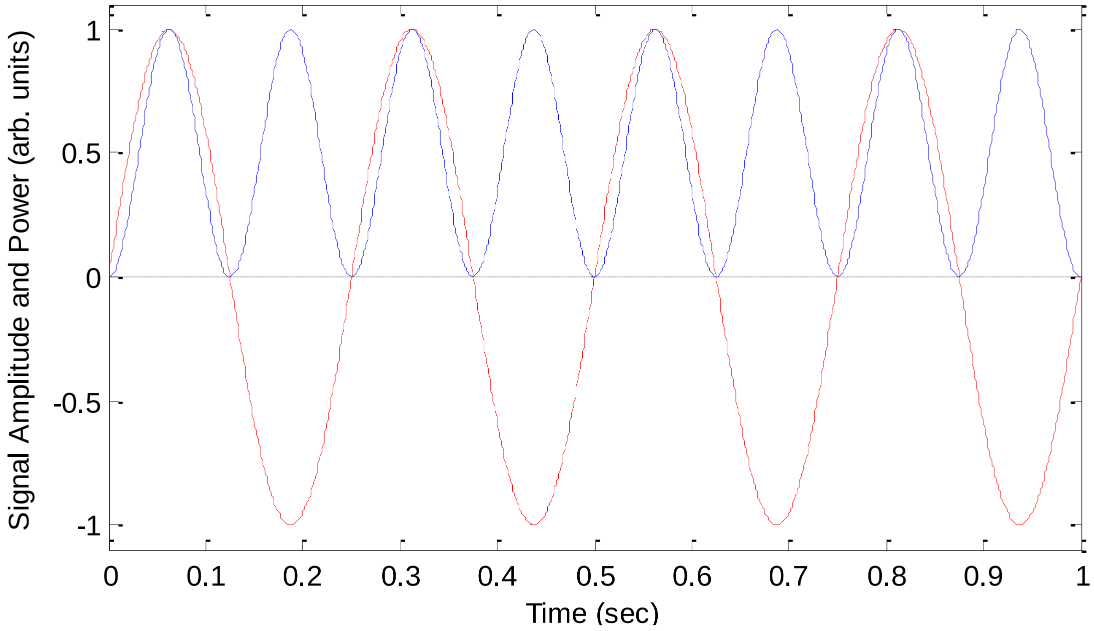
$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f^2(t)dt < \infty$$

Analog Signals : Finite Power Signals

Periodic signals are good examples of finite power signals.
For example the sinusoidal oscillation:

$$f(t) = \sin(\omega t); -\infty < t < \infty$$

$$f^2(t) = \sin^2(\omega t); -\infty < t < \infty$$



Analog Signals : Finite Power Signals

This is clearly not finite energy since the energy on any interval grows in proportion to the length of the interval:

$$\mathrm{E}_T = \int_{-T}^{T} f^2(t)\,dt$$

$$= \int_{-T}^{T} \sin^2(\omega t)\,dt$$

$$= \int_{-T}^{T} \tfrac{1}{2}(1-\cos(2\omega t))\,dt$$

$$= \tfrac{1}{2}\left[t - \frac{\sin(2\omega t)}{2\omega}\right]_{-T}^{T} = T - \frac{\sin(2\omega T)}{2\omega}$$

Analog Signals : Finite Power Signals

When the energy on the interval is divided by the length of the interval to give the average power, the result is:

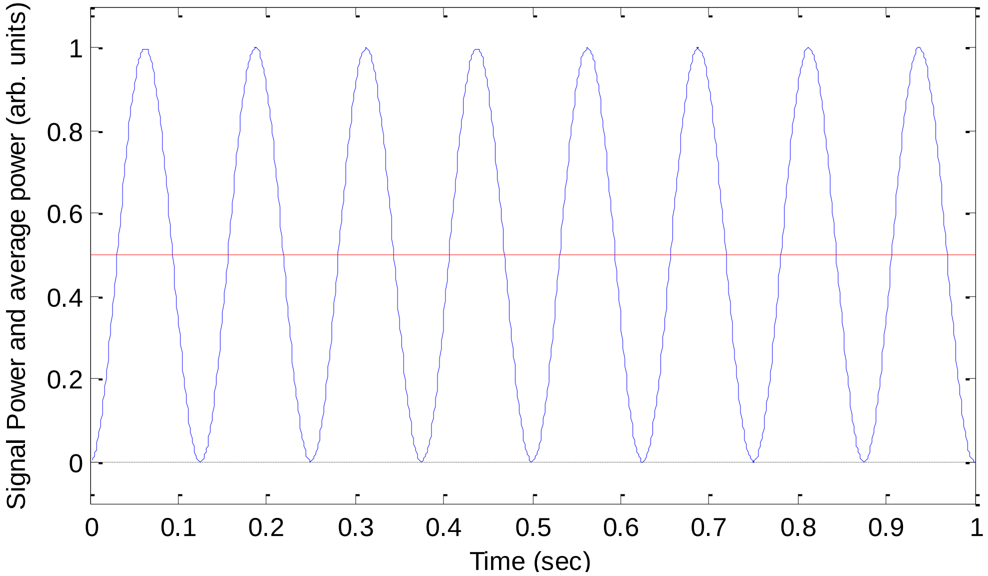
$$\mathrm{P}_T = \frac{1}{2T}\int_{-T}^{T} f^2(t)dt = \frac{1}{2} - \frac{\sin(2\omega T)}{4\omega T}$$

If we let T become arbitrarily large, the average power approaches a finite limit:

$$\overline{P} = \lim_{T\to\infty} P_T = \frac{1}{2} < \infty$$

Analog Signals : Finite Power Signals

$$\overline{P} = \lim_{T \to \infty} P_T = \frac{1}{2} < \infty$$



In fact any bounded periodic function is finite power.
The converse is not true, not all finite power signals are periodic.
Noise is an important example of a finite power signal which is not periodic.

Signal Power and Energy

For a signal x(t),

Average Power:

$$P_{\text{av}} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} p(t)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt.$$

(Eq 1.35)

Total Energy:

$$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |x(t)|^2\, dt,$$

$$P_{\text{av}} = \lim_{T\to\infty} \frac{E}{T}$$

(Eq 1.36)

- P_{av} and E define three classes of signals:

(a) ***Power signals***: P_{av} is finite and $E \to \infty$

(b) ***Energy signals***: $P_{\text{av}} = 0$ and E is finite

(c) ***Non-physical signals***: $P_{\text{av}} \to \infty$ and $E \to \infty$

Signal Energy and Power

- consider energy and power for *any* (generic) signal
- many signals are complex valued
- *signal energy* for a continuous-time signal

$$E_x = \int_{t_1}^{t_2} |x(t)|^2 \, dt$$

- signal energy for a discrete-time signal

$$E_x = \sum_{n=n_1}^{n_2} |x[n]|^2$$

- dividing by $t_1 - t_2$ and $n_2 - n_1 + 1$, gives the (time-averaged) signal power
- In many cases, *physical* energy is related to *signal* energy by a simple constant (in the above examples R and $1/R$)

- Energy over infinite time interval

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- Power over infinite time interval

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

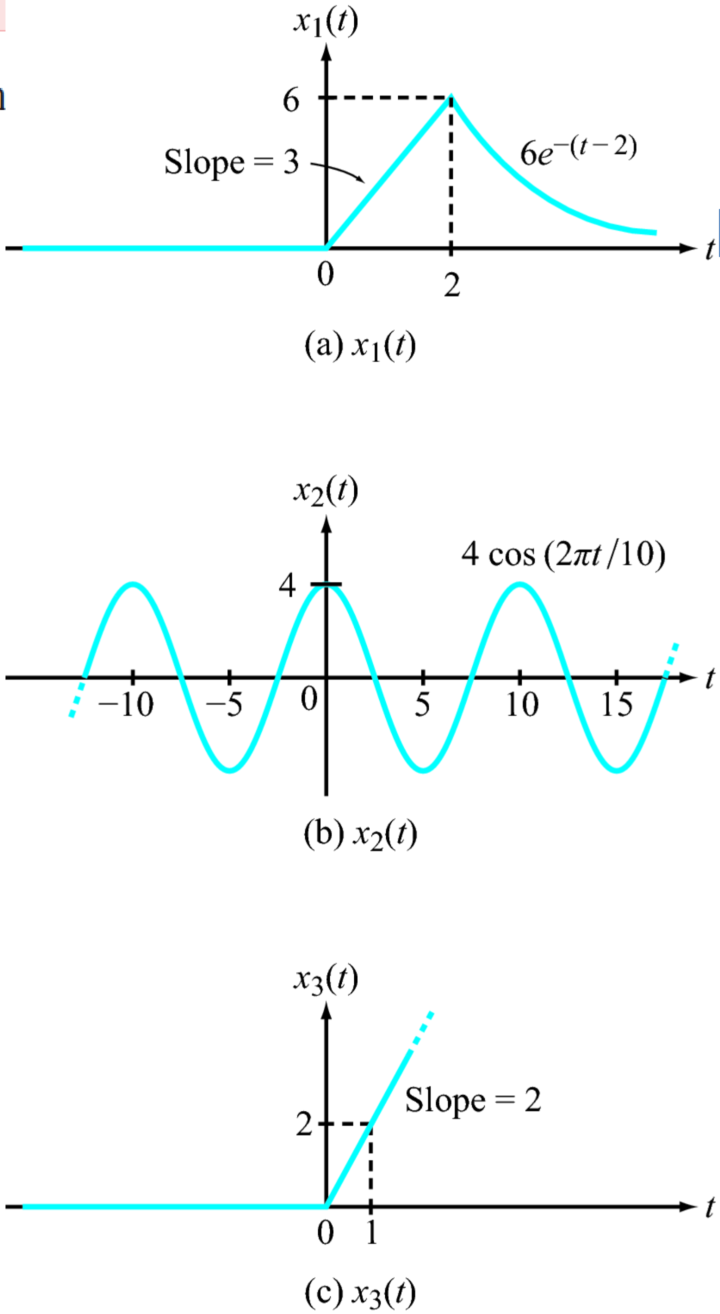
Example 1-7: Power and Energy

Evaluate P_{av} and E for each of the three signals displayed in Fig. 1-23.

Solution:

(a) Signal $x_1(t)$ is given by

$$x_1(t) = \begin{cases} 0 & \text{for } t \le 0, \\ 3t & \text{for } 0 \le t \le 2, \\ 6e^{-(t-2)} & \text{for } t \ge 2. \end{cases}$$



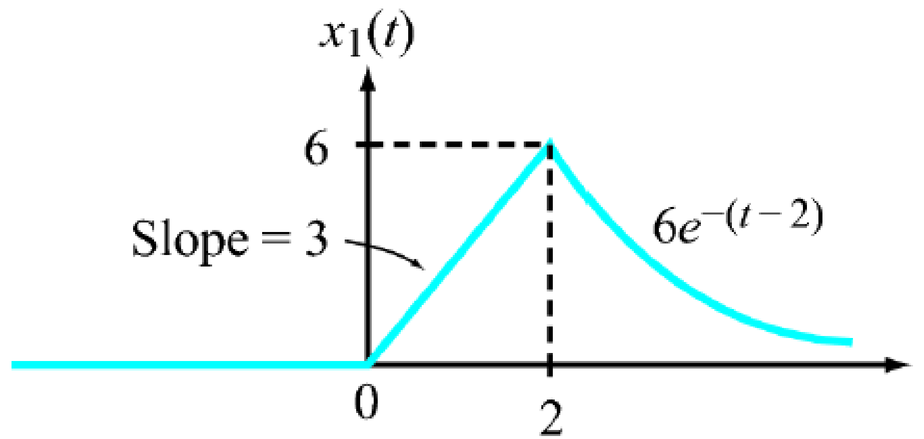
$$E_1 = \int_0^2 (3t)^2\,dt + \int_2^\infty [6e^{-(t-2)}]^2\,dt$$

$$= \int_0^2 9t^2\,dt + \int_2^\infty 36e^{-2(t-2)}\,dt$$

$$= \left.\frac{9t^3}{3}\right|_0^2 + 36e^4 \int_2^\infty e^{-2t}\,dt$$

$$= 24 + 36e^4 \left(\left.\frac{-e^{-2t}}{2}\right|_2^\infty \right)$$

$$= 42.$$

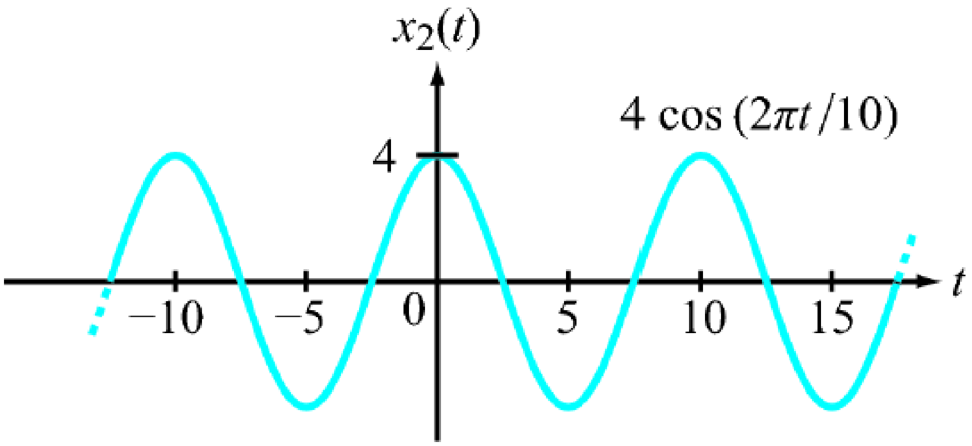


(a) $x_1(t)$

$$P_{av} = \lim_{T\to\infty} \frac{E}{T}$$

Since E_1 is finite, it follows from Eq. (1.35) that $P_{av_1} = 0$.

b)



(b) $x_2(t)$

$$x_2(t) = 4\cos\left(\frac{2\pi t}{10}\right).$$

From the argument of $\cos(2\pi t/10)$, the period is 10 s. Hence, application of Eq. (1.36) leads to

$$P_{\mathrm{av}_2} = \frac{1}{10}\int_{-5}^{5}\left[4\cos\left(\frac{2\pi t}{10}\right)\right]^2 dt$$

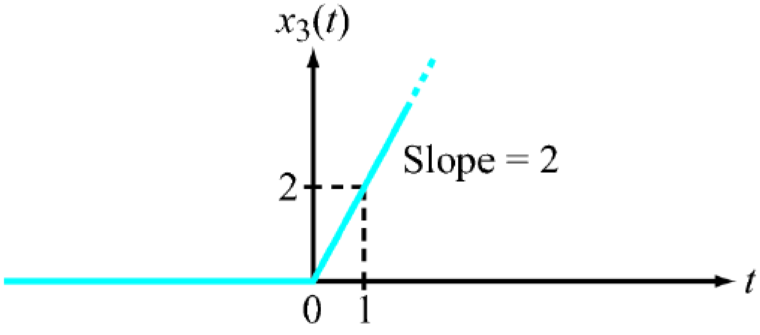
$$= \frac{1}{10}\int_{-5}^{5} 16\cos^2\left(\frac{2\pi t}{10}\right) dt$$

$$= 8.$$

Since P_{av_2} is finite, it follows that $E_2 \to \infty$.

(c) Signal $x_3(t)$ is given by

$$x_3(t) = 2r(t) = \begin{cases} 0 & \text{for } t \le 0, \\ 2t & \text{for } t \ge 0. \end{cases}$$



(c) $x_3(t)$

The time-averaged power associated with $x_3(t)$ is

$$P_{\text{av}_3} = \lim_{T\to\infty} \frac{1}{T} \int_0^{T/2} 4t^2\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \left[\frac{4t^3}{3} \bigg|_0^{T/2} \right]$$

$$= \lim_{T\to\infty} \left[\frac{1}{T} \times \frac{4T^3}{24} \right]$$

$$= \lim_{T\to\infty} \left[\frac{T^2}{6} \right] \to \infty.$$

Moreover, $E_3 \to \infty$ as well.