LECTURE BLOCK 1B: SIGNALS

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Revision

Earlier, we have looked at:

- Types of signals,
- Signal transformations,
- Odd/even, periodic signals,
- Signal power and energy.

Today we will look at some non-periodic signals.

And DT and CT exponential signals.

Step Function

Ideal Step Function

$$u(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t > 0. \end{cases}$$

Realistic Step Function

$$u(t) = \lim_{\epsilon \to 0} \begin{cases} 0 & \text{for } t \leq -\epsilon \\ \left[\frac{1}{2} \left(\frac{t}{\epsilon} + 1 \right) \right] & \text{for } -\epsilon \leq t \leq \epsilon \\ 1 & \text{for } t \geq \epsilon, \end{cases}$$

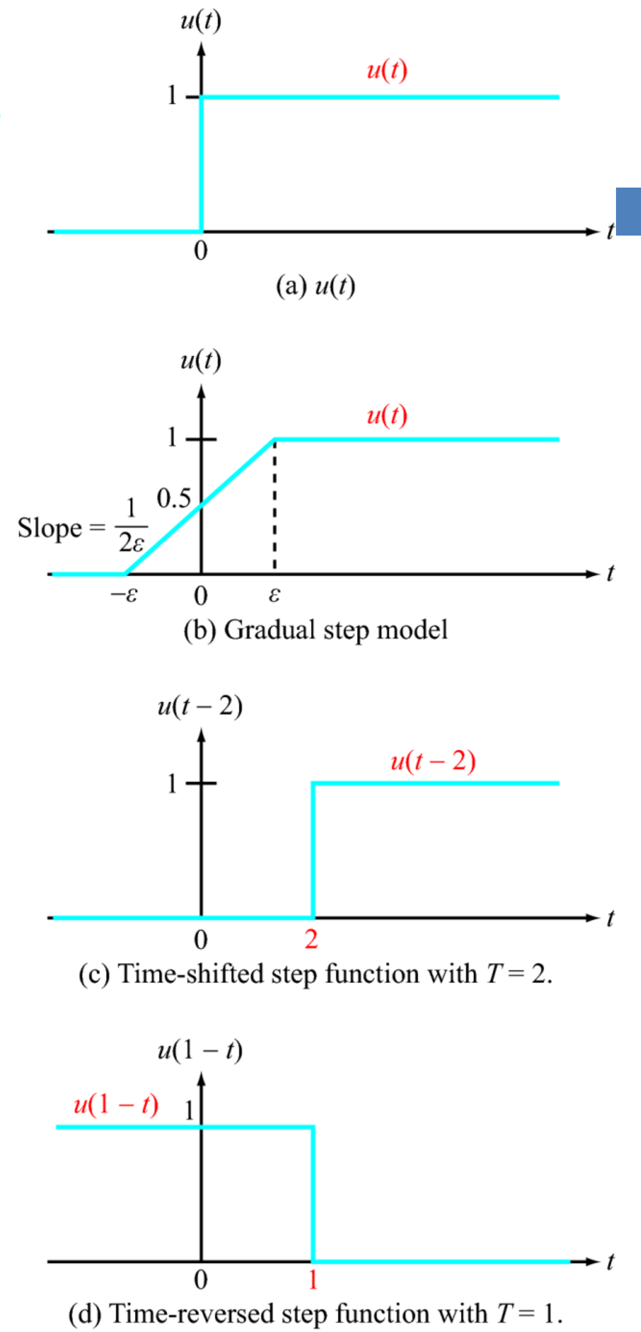
Time-Shifted Step Function

$$u(t - T) = \begin{cases} 0 & \text{for } t < T, \\ 1 & \text{for } t > T. \end{cases}$$

Time-Reversed Step Function

$$u(T - t) = \begin{cases} 1 & \text{for } t < T, \\ 0 & \text{for } t > T. \end{cases}$$

Step Functions



Ramp Function

Unit Ramp Function

$$r(t) = \begin{cases} 0 & \text{for } t \le 0, \\ t & \text{for } t \ge 0, \end{cases}$$

and

$$r(t-T) = \begin{cases} 0 & \text{for } t \le T, \\ (t-T) & \text{for } t \ge T. \end{cases}$$

Unit Time-shifted Ramp Function

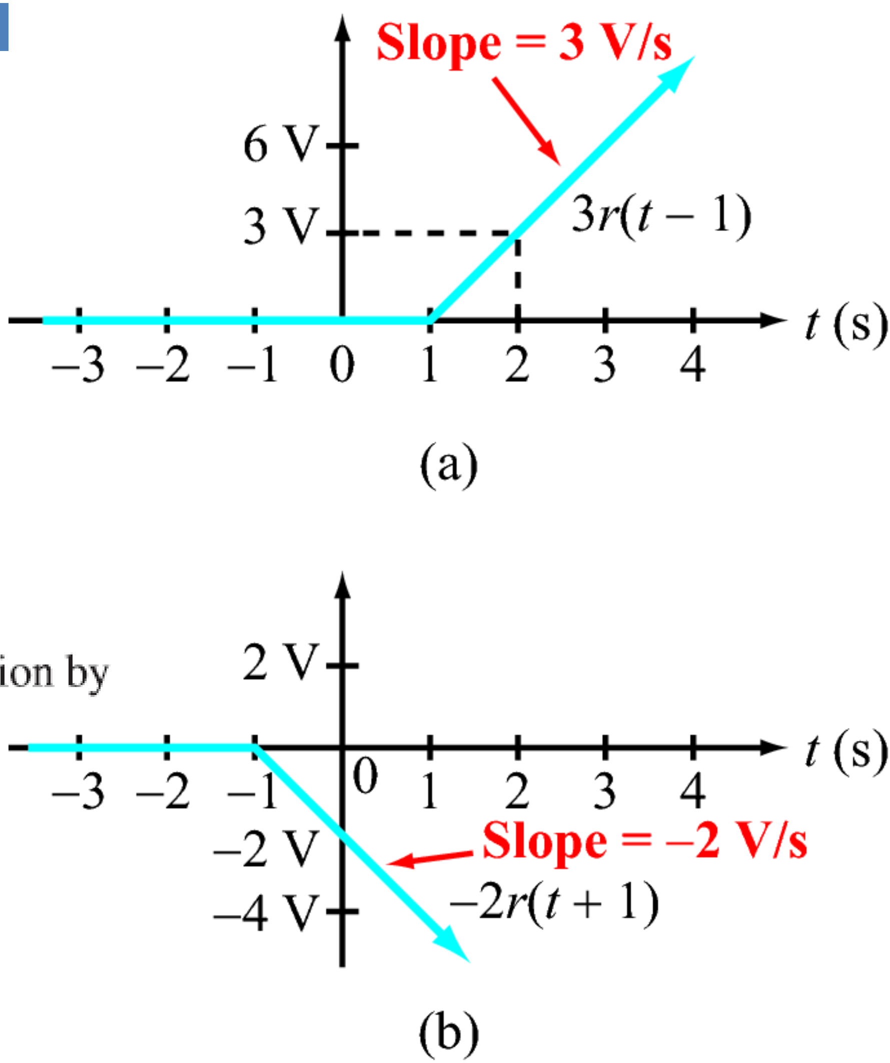
The unit ramp function is related to the unit step function by

$$r(t) = \int_{-\infty}^{t} u(\tau)\, d\tau = t\, u(t),$$

and for the time-shifted case,

$$r(t-T) = \int_{-\infty}^{t} u(\tau - T)\, d\tau = (t-T)\, u(t-T).$$

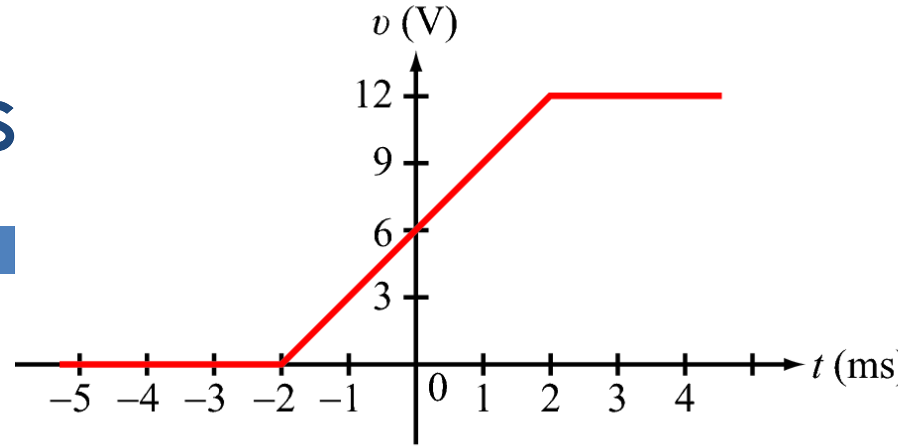
Ramp Functions



Waveform Synthesis

v(t) as sum of two time-shifted ramp functions

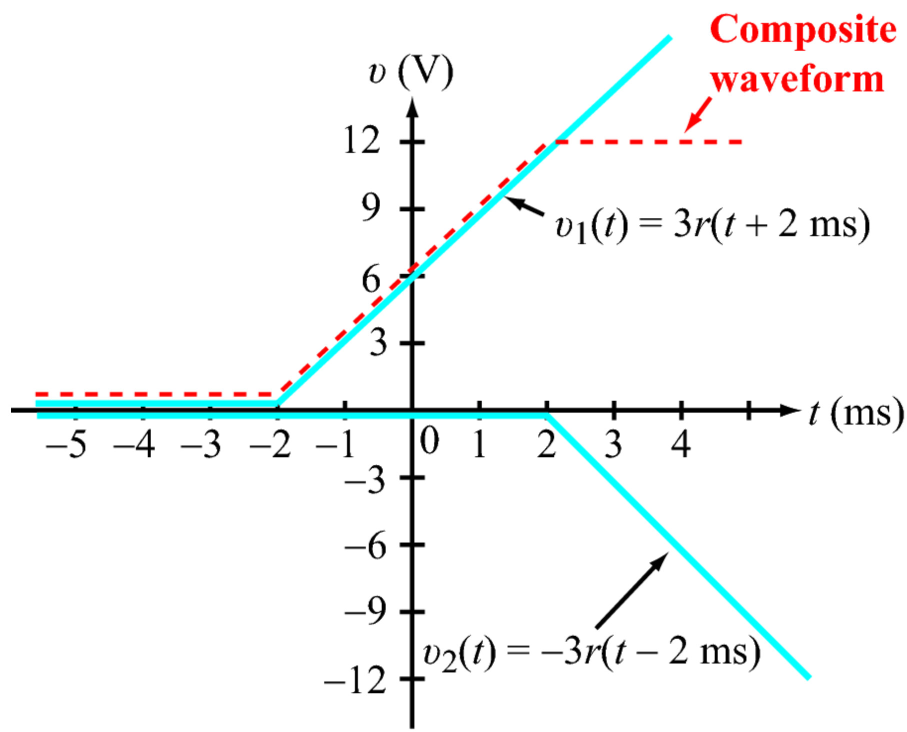
$$\upsilon(t) = \upsilon_1(t) + \upsilon_2(t)$$
$$= 3r(t + 2\text{ ms}) - 3r(t - 2\text{ ms})$$



(a) Original function

Equivalently:

$$\upsilon(t) = 3(t + 2\text{ ms})\, u(t + 2\text{ ms})$$
$$- 3(t - 2\text{ ms})\, u(t - 2\text{ ms})$$



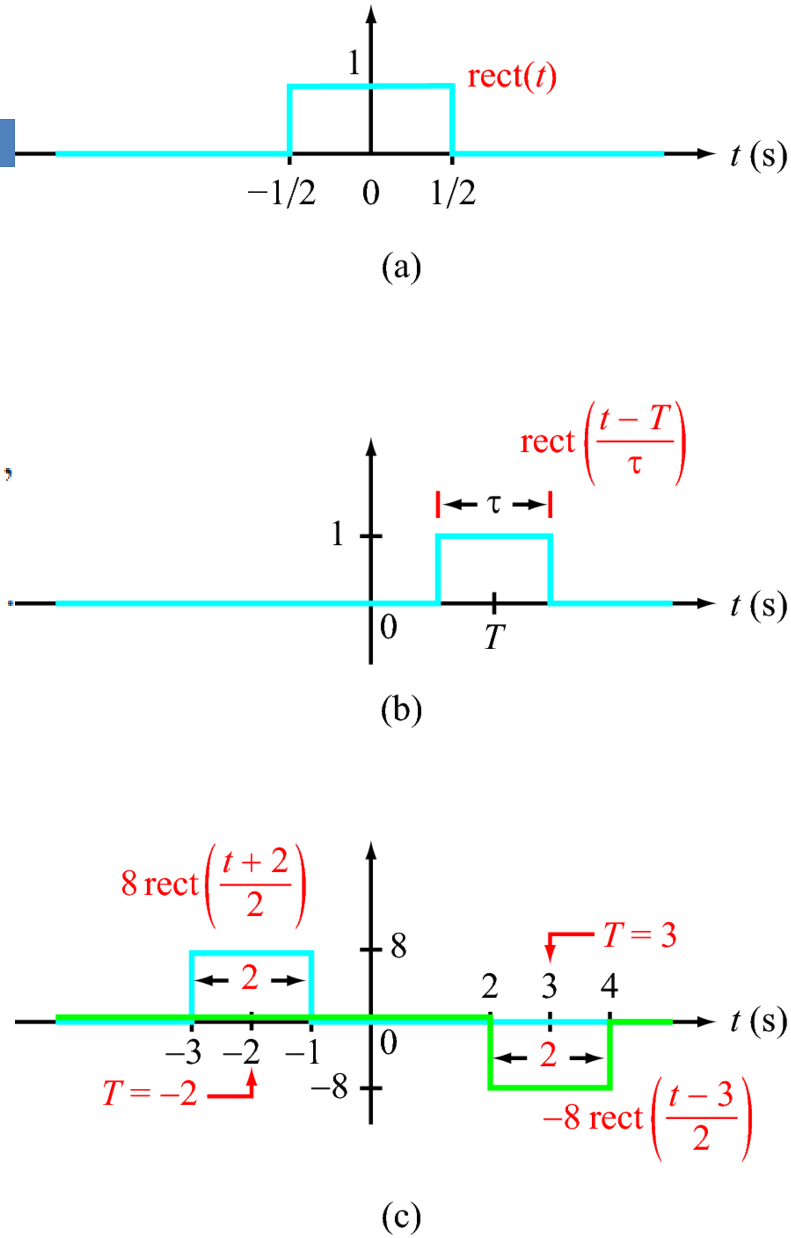
(b) As sum of two time-shifted ramp functions

Rectangular Function (Pulse)

$$\mathrm{rect}\left(\frac{t-T}{\tau}\right) = \begin{cases} 0 & \text{for } t < (T-\tau/2), \\ 1 & \text{for } (T-\tau/2) < t < (T+\tau/2), \\ 0 & \text{for } t > (T+\tau/2). \end{cases}$$

(1.

Rectangular Pulses



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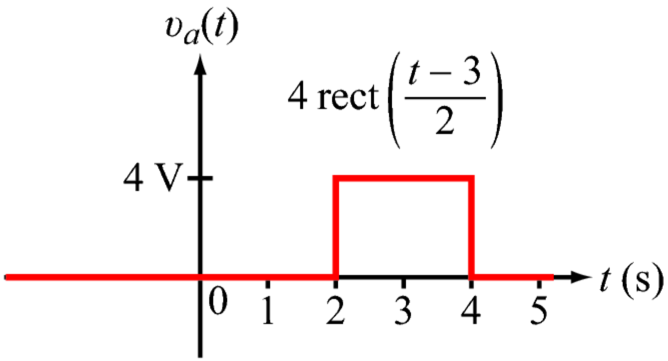
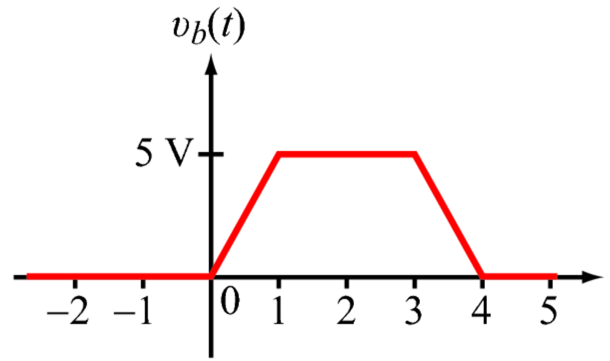
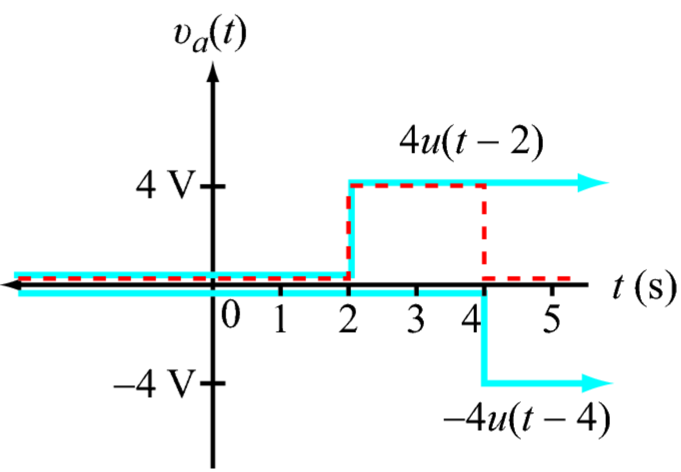


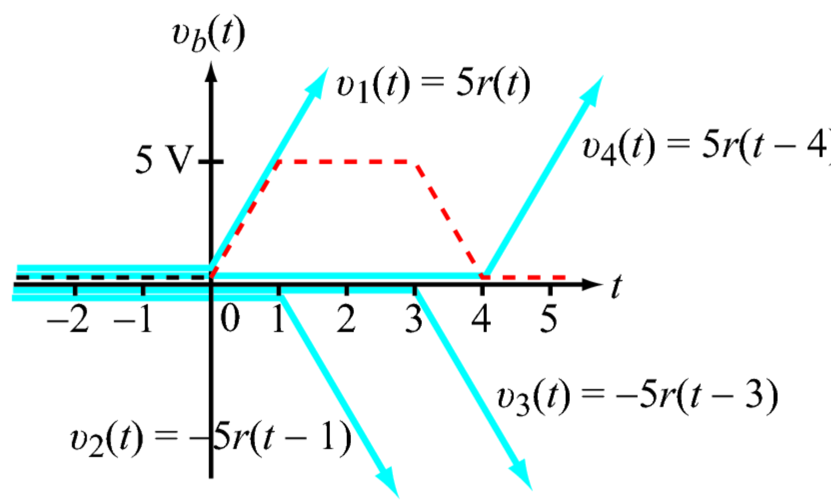
(a) Rectangular pulse



(b) Trapezoidal pulse



(c) $v_a(t) = 4u(t-2) - 4u(t-4)$



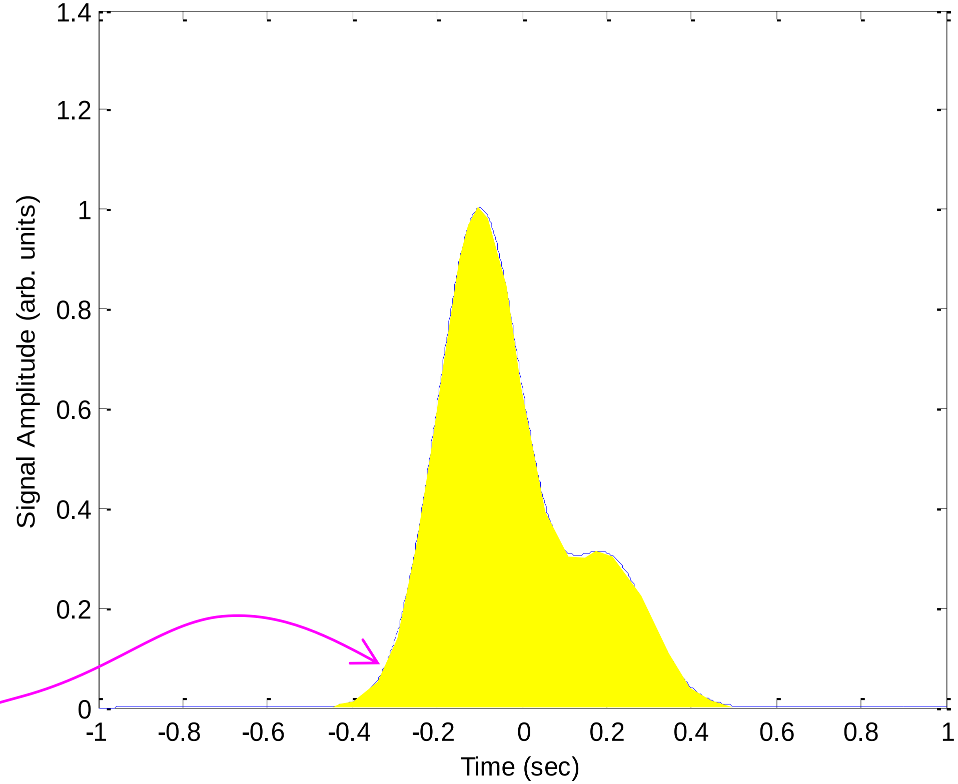
(d) $v_b(t) = v_1(t) + v_2(t) + v_3(t) + v_4(t)$

Analog Signals : Impulsive Signals

Energy is one physical quantity that is useful to generalise to signals in general. A second physical quantity that turns out to be of considerable significance is the impulse of a signal.

The analogy is with the effect of a time varying force, F(t), on the momentum of a particle:

$$\Delta p = \int_{-\infty}^{\infty} F(t)\,dt$$



Analog Signals : Impulsive Signals

In general the *Impulse* of a signal is the area under the signal amplitude vs time curve.

$$I = \int_{-\infty}^{\infty} f(t)dt$$

An impulsive (or impulse) signal is an idealization of a pulse that has a finite "impulse" but is of too short a duration for its exact shape to be of importance. Impulse-type signals also have a number of useful mathematical properties that will be exploited in subsequent lectures .

Analog Signals : Impulsive Signals

The basic impulse signal is the Dirac delta function . Very loosely speaking this is a pulse with the following properties :

$$\delta(t) = \begin{cases} 0 & : \quad t \neq 0 \\ \infty & : \quad t = 0 \end{cases}$$

The pulse is zero everywhere except at the origin.

$$\int_a^b \delta(t)\,dt = 1 \text{ for all } a < 0 \text{ and } b > 0$$

The impulse is one for any interval including the origin.

ie. the Dirac delta function is a "unit impulse" pulse of vanishing duration.

Impulse Function

Unit impulse function or Dirac delta function

$$\delta(t - T) = 0 \qquad \text{for } t \neq T$$

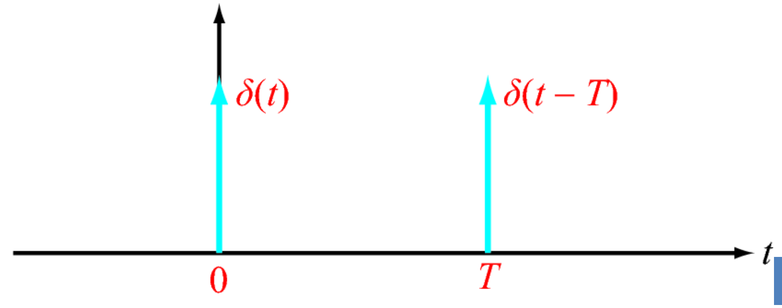
and

$$\int_{-\infty}^{\infty} \delta(t - T)\, dt = 1.$$

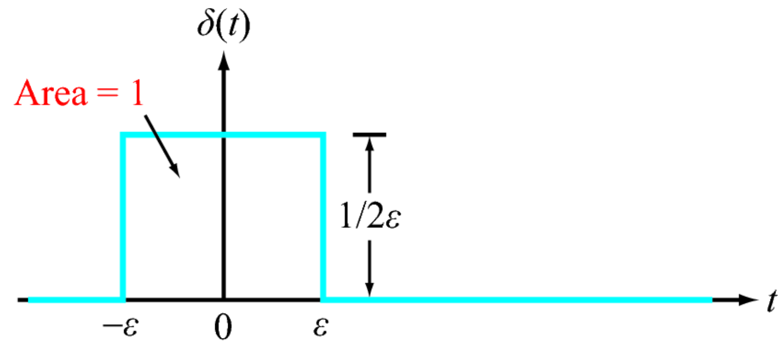
Relationship to u(t)

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau,$$

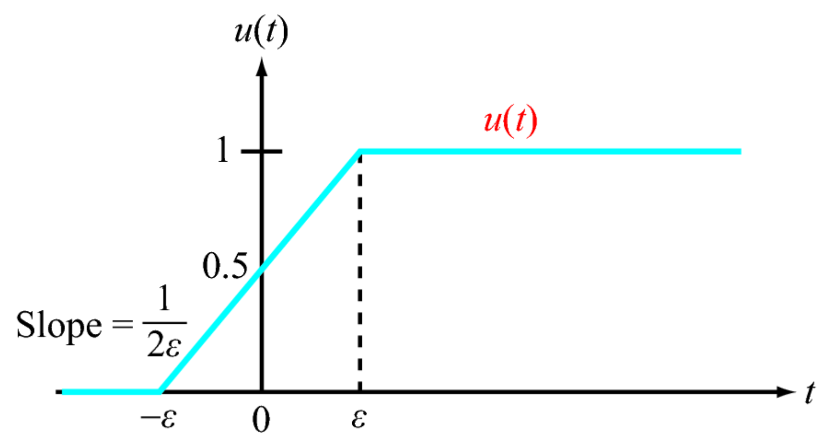
$$\frac{du(t)}{dt} = \delta(t).$$



(a) $\delta(t)$ and $\delta(t - T)$



(b) Rectangle model for $\delta(t)$

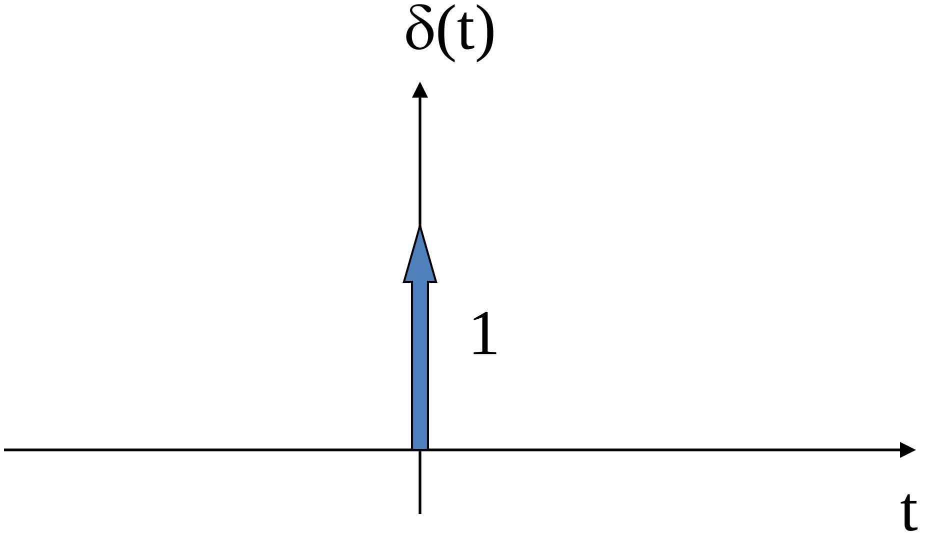


(c) Gradual step model for $u(t)$

Analog Signals : Impulsive Signals

The physical analogy of a Dirac delta function is a sharp blow that delivers a certain amount of momentum to a body.
The idea is that the impulse is important but the exact time dependence of the pulse is not.

A delta function is usually represented as an vertical arrow that has a length proportional to the impulse (not the amplitude) of the pulse.



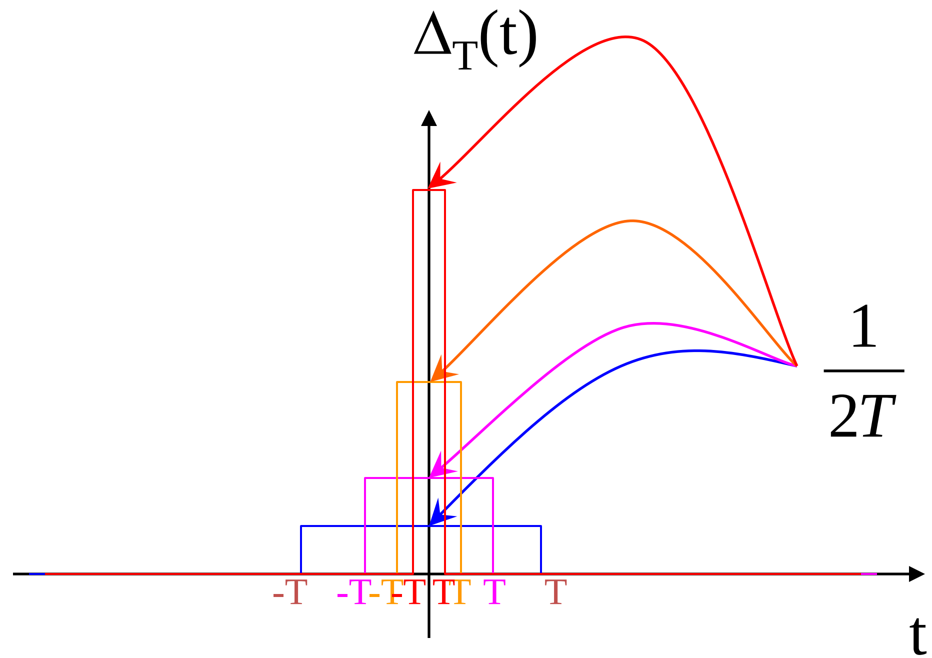
Analog Signals : Impulsive Signals

It is often convenient to think of a delta function as the limiting form of a unit area pulse as the pulse width goes to zero.

For example if we use a rectangular pulse

$$\Delta_T(t) = \frac{1}{2T} P_T(t)$$

As the pulse width decreases, the pulse height increases in such a way that the area under the curve (the *impulse* of the pulse) remains constant.



Unit impulse function

What signal qualifies as a unit impulse function?

1) be zero everywhere except at t=0,

2) be infinite at t=0,

3) be even,

4) have a unit area.

Analog Signals : Impulsive Signals

Clearly such a pulse has the properties:

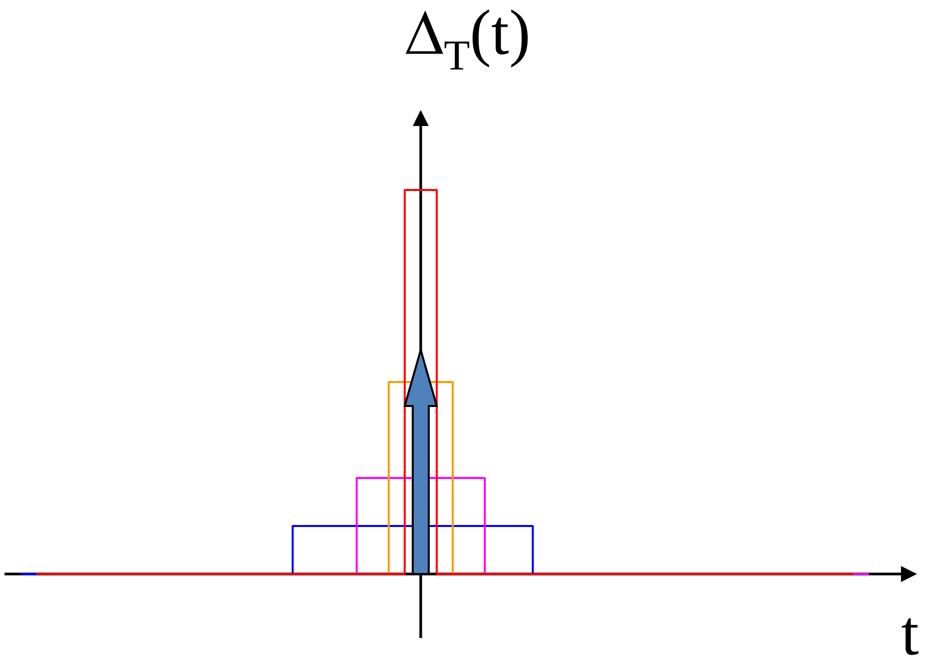
$$\Delta_T(t) = 0 \quad : |t| > T$$

and

$$\int_a^b \Delta_T(t)dt = 1 \quad : \ a < -T \text{ and } b > T$$

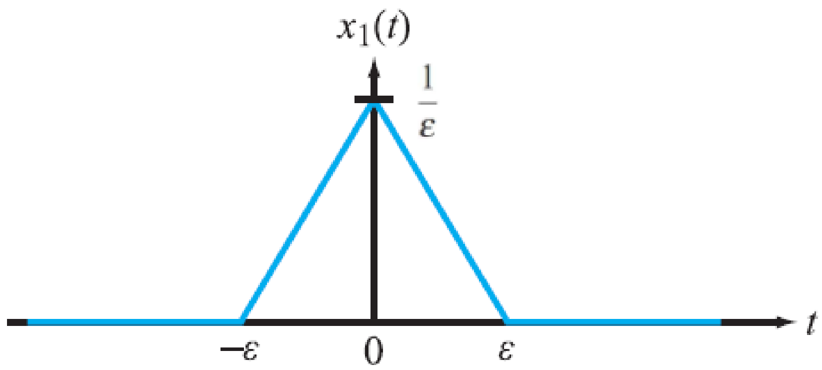
In the limit as T approaches zero these formulas reduce to the informal defining properties of a delta function.

In fact any reasonable unit area pulse, sharply peaked about t = 0 will produce the same results in the limit as the pulse width vanishes.



Example

Triangular model x(t)



(1) As $\epsilon \to 0$, $x_1(t)$ is indeed zero everywhere except at $t = 0$.

(2) $\lim_{\epsilon \to 0} x_1(0) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} = \infty$; hence infinite at $t = 0$.

(3) $x_1(t)$ is clearly an even function

(4) Area of triangle $= \frac{1}{2}\left(2\epsilon \times \frac{1}{\epsilon}\right) = 1$, regardless of the value of ϵ.

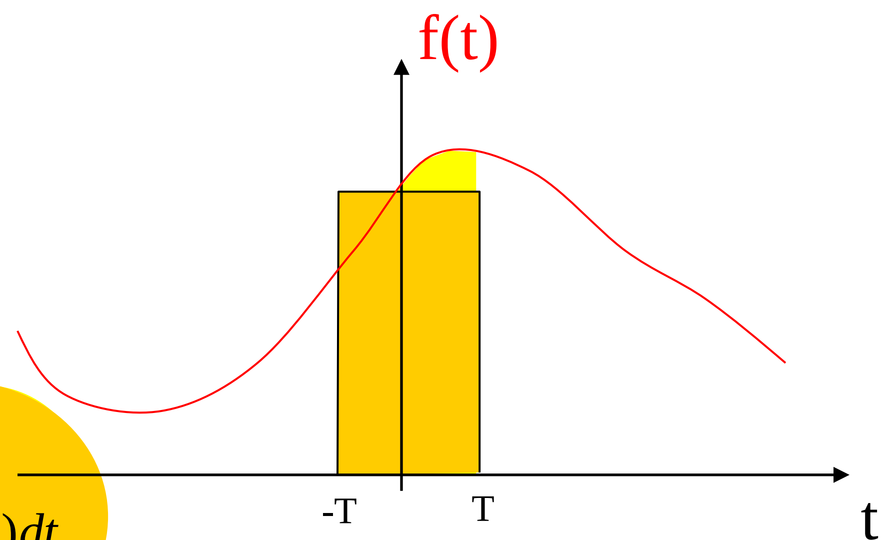
x(t) does qualify as a unit impulse function.

Analog Signals : Properties of Delta Functions

The Dirac delta function must be used with some care, and many of its properties are only rigorously true when discussing integrals involving delta functions.

1. The defining property: for all functions *f(t)* continuous at *t = 0*.

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0)$$



It is easy to establish this property from the informal definition using (say) the $\Delta_T(t)$ functions:

$$\int_{-\infty}^{\infty} \Delta_T(t) f(t) dt = \frac{1}{2T} \int_{-\infty}^{\infty} P_T(t) f(t) dt = \frac{1}{2T} \int_{-T}^{T} f(t) dt$$

= average value of f(t) on the interval (-T,T)

Analog Signals : Properties of Delta Functions

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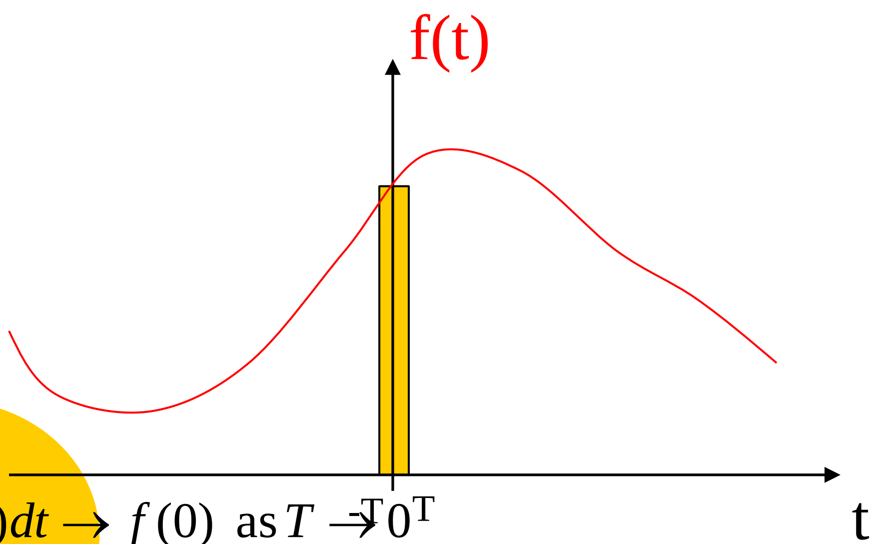
1. The defining property: for all functions *f(t)* continuous at *t = 0*.

$$\int_{-\infty}^{\infty} \delta(t) f(t)\,dt = f(0)$$

As the interval shrinks to zero, the average value on the interval approaches the actual value at t = 0 (provided the function is continuous at the origin.



$$\int_{-\infty}^{\infty} \Delta_T(t) f(t)\,dt = \frac{1}{2T}\int_{-\infty}^{\infty} P_T(t) f(t)\,dt = \frac{1}{2T}\int_{-T}^{T} f(t)\,dt \to f(0) \ \text{as}\ T \to 0$$

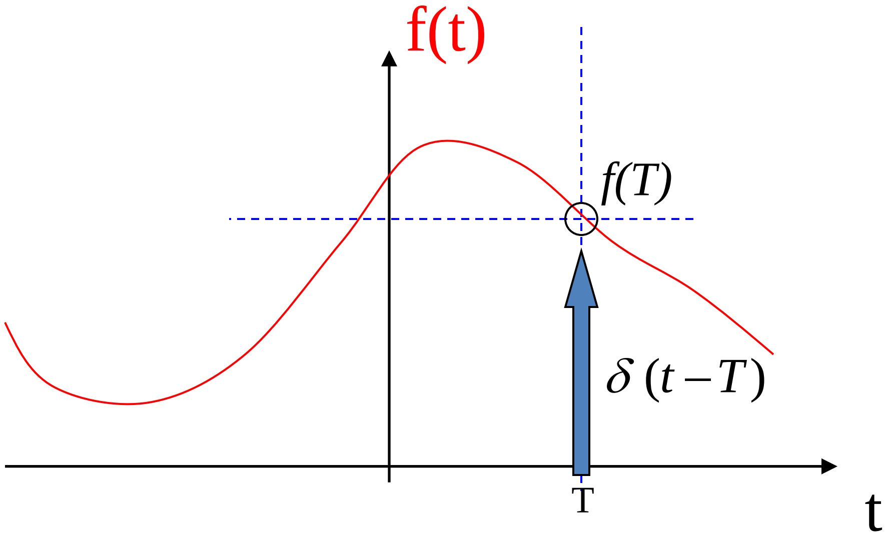
Analog Signals : Properties of Delta Functions

If the delta function in the defining property is shifted to a new position, t = T, then we get a related property

2. The shift property: for all functions *f(t)* continuous at *t = T*.

$$\int_{-\infty}^{\infty} \delta(t-T)\, f(t)\, dt = f(T)$$

Integrating with a delta function picks out the value of the function at the point where the delta function diverges.



Analog Signals : Properties of Delta Functions

Properties of delta functions outside integrals actually refer to their effect when used inside integrals. Thus the fact that a change of variable from t to –t inside the integral in the defining property does not change the result leads to :

3. The symmetry property:

$$\delta\,(t) = \delta\,(-t)$$

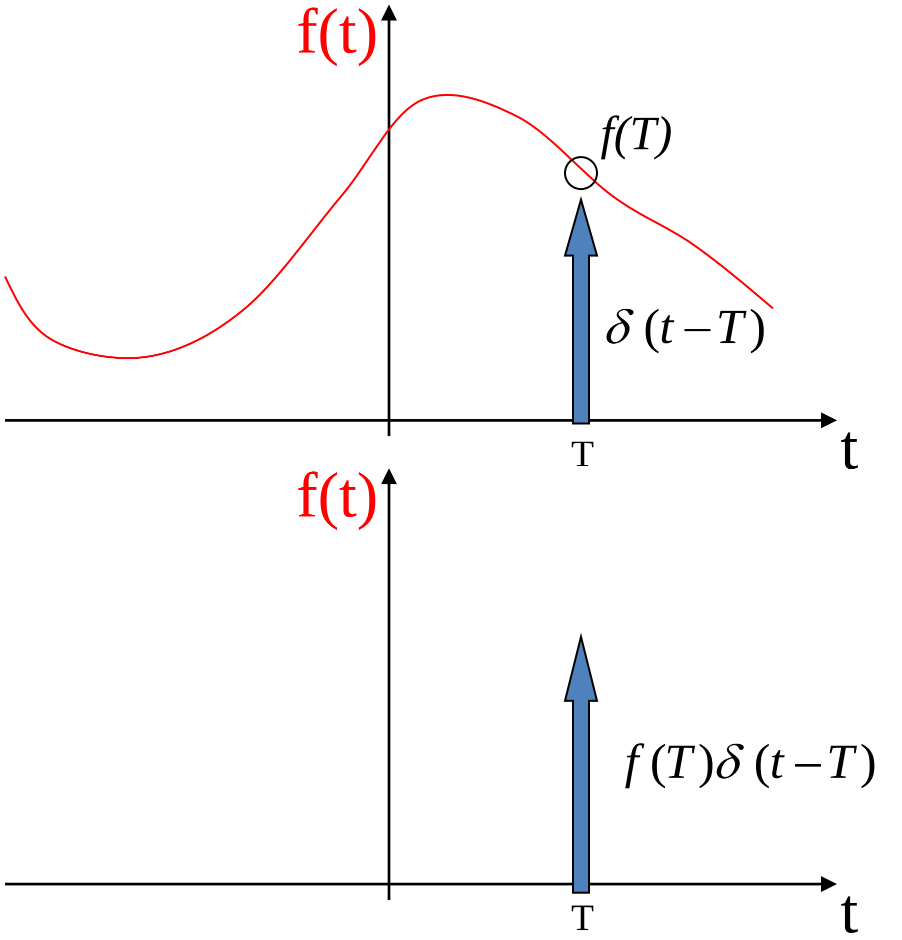
In effect a delta function can be considered to be an even function of t.

Analog Signals : Properties of Delta Functions

By similarly considering the effect on the shift property we come to

4. The multiplication property:

$$f(t)\delta(t-T) = f(T)\delta(t-T)$$



Intuitively, since the delta function is zero at all points except t = T, the value of the multiplying function at any other point is irrelevant.

Sampling Property of $\delta(t)$

Multiplication of a signal x(t) by a unit impulse response,

$$x(t)\,\delta(t) = x(0)\,\delta(t),$$

Multiplication of a signal x(t) by a time-shifted unit impulse response,

$$x(t)\,\delta(t-T) = x(T)\,\delta(t-T).$$

This is the sampling or shifting property of impulse function:-

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-T)\,dt = x(T).$$

(sampling property)

Example 1-6: impulse integral

Evaluate $\int_1^2 t^2\,\delta(2t-3)\,dt$.

Solution: Using the time-scaling property, the impulse function can be expressed as

$$\delta(2t-3) = \delta\left(2\left(t-\frac{3}{2}\right)\right) = \frac{1}{2}\,\delta\left(t-\frac{3}{2}\right).$$

Hence,

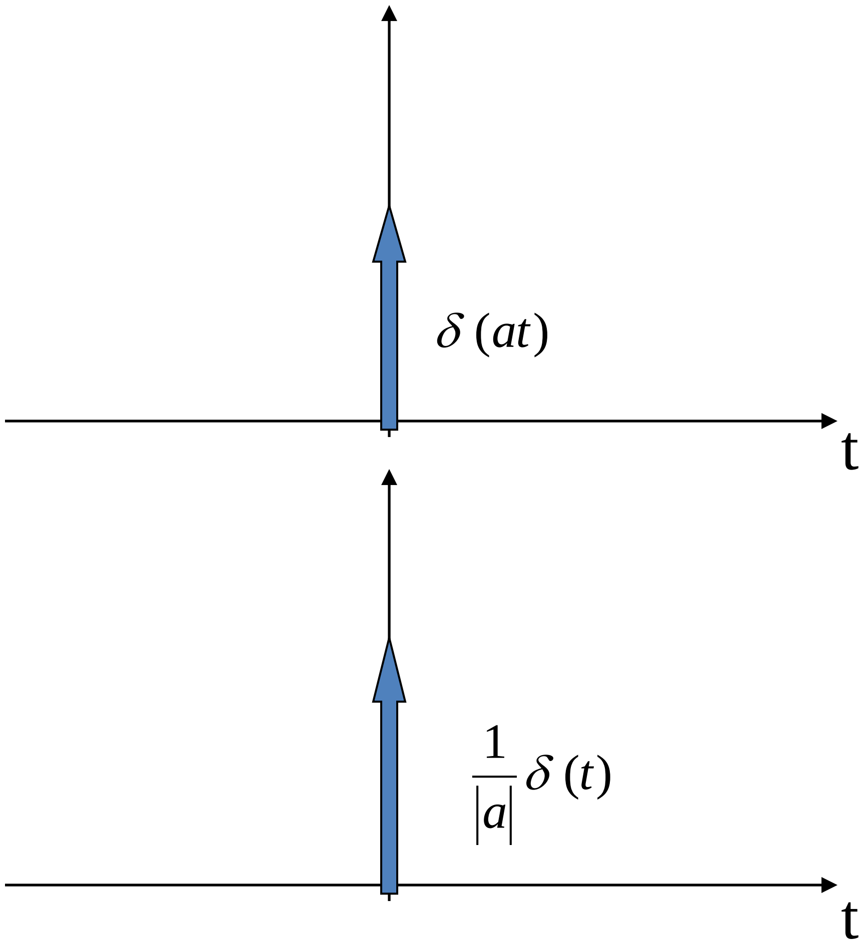
$$\int_1^2 t^2\,\delta(2t-3)\,dt = \frac{1}{2}\int_1^2 t^2\,\delta\left(t-\frac{3}{2}\right)\,dt = \frac{1}{2}\left(\frac{3}{2}\right)^2 = \frac{9}{8}.$$

Analog Signals : Properties of Delta Functions

Considering the effect of scaling under the integral in the defining property leads to

5. The scaling property:

$$\delta(at) = \frac{1}{|a|}\delta(t) \;:\; a \neq 0$$



Intuitively, the dimensions (ie units) of a delta function are the reciprocal of the dimensions (units) of the variable of integration.

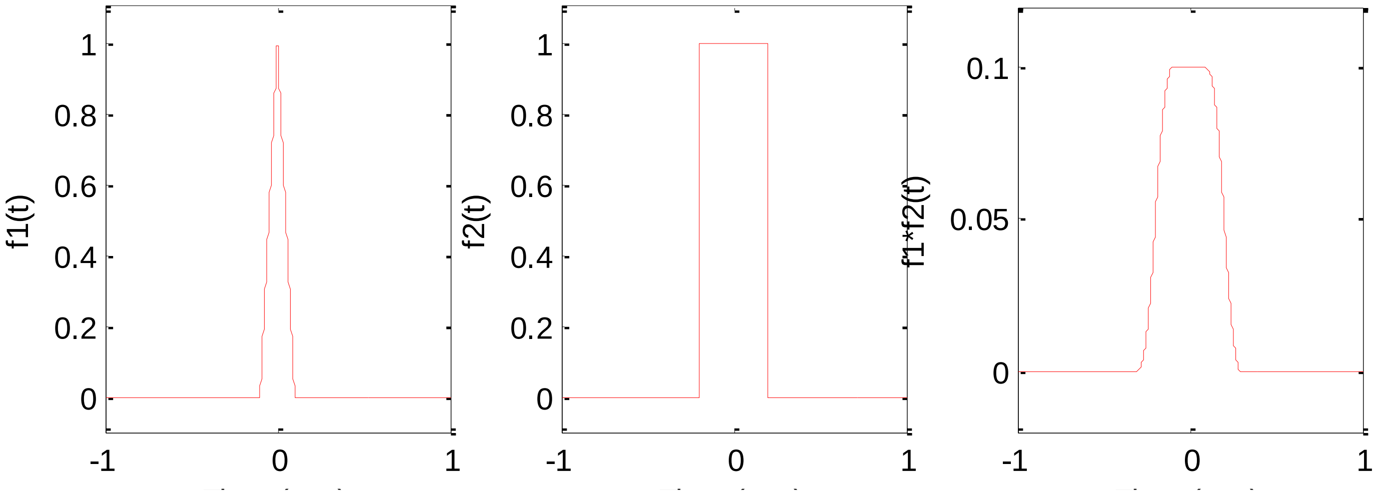
For example, if the variable is time the delta function has the dimensions of frequency, and scales as you would expect of a frequency (doubling the period halves the frequency etc.).

Analog Signals : Properties of Delta Functions

A *convolution* is a *smoothing* of one function by another.
Although at first sight rather mysterious, the convolution of two functions is of crucial importance in signal processing.
For two functions f(t) and g(t), the convolution is defined by:

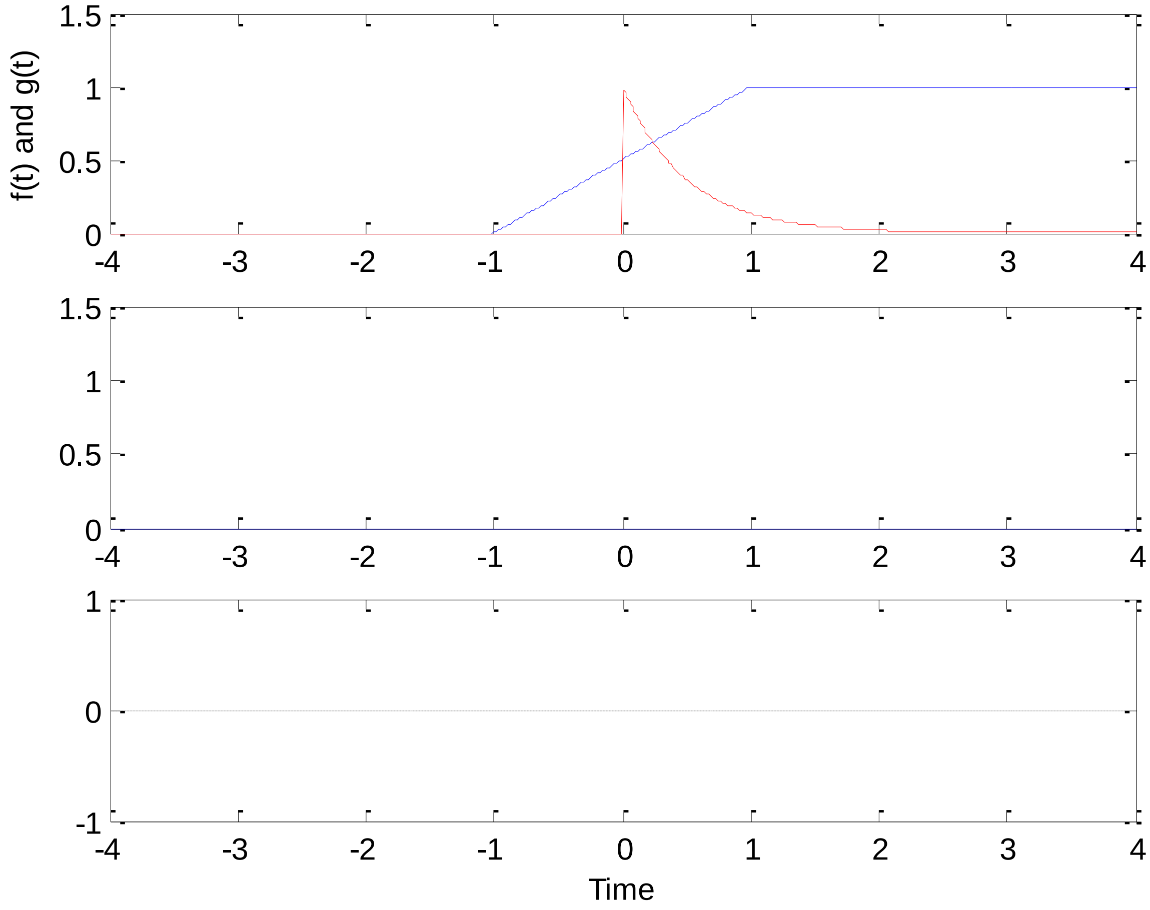
$$f * g(t) = g * f(t) = \int_{-\infty}^{\infty} f(t')g(t-t')dt'$$

Note that the convolution of two functions of time is a function of time in its own right.



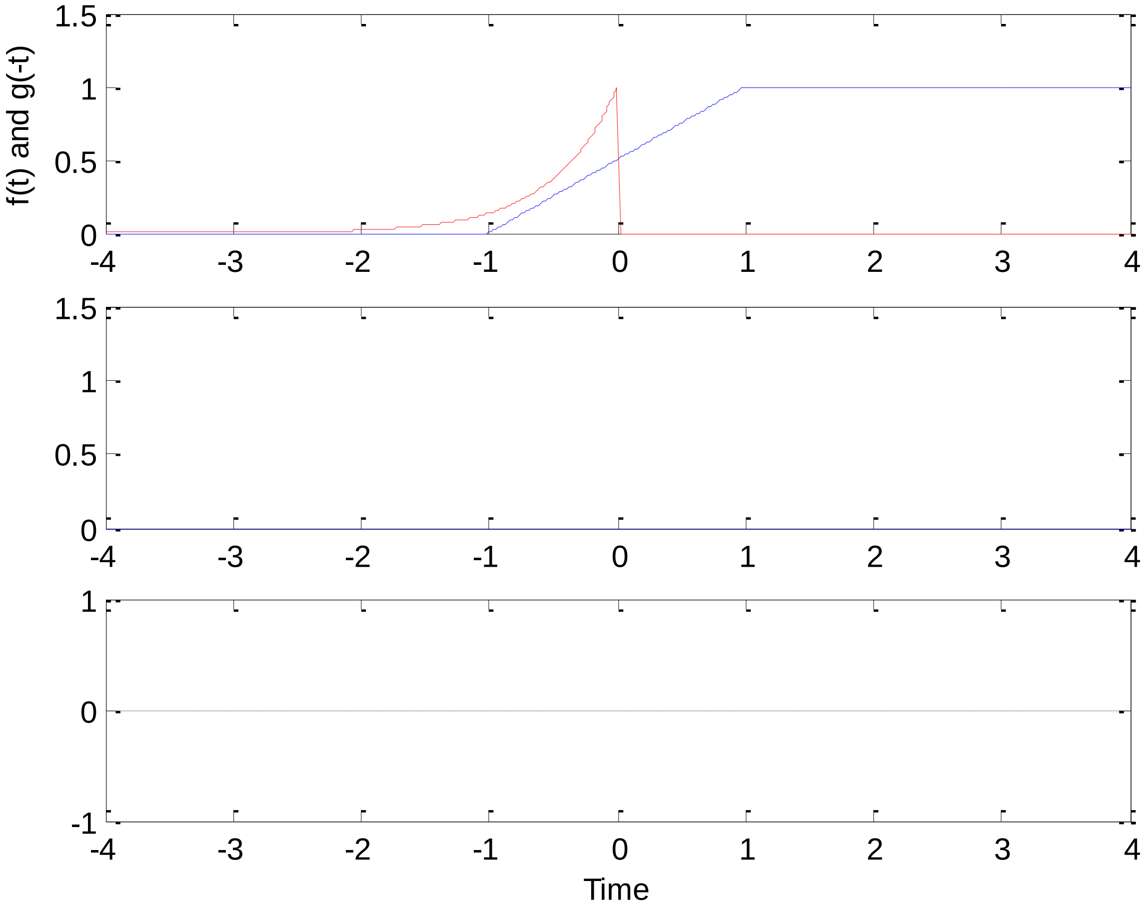
Analog Signals : Properties of Delta Functions

When one of the functions is pulse like, c*onvolution* can be pictured as a moving window average.



Analog Signals : Properties of Delta Functions

Suppose *g(t)* is the pulse like function, then the moving window is *g(-t)*, ie. the mirror image of *g(t)*.

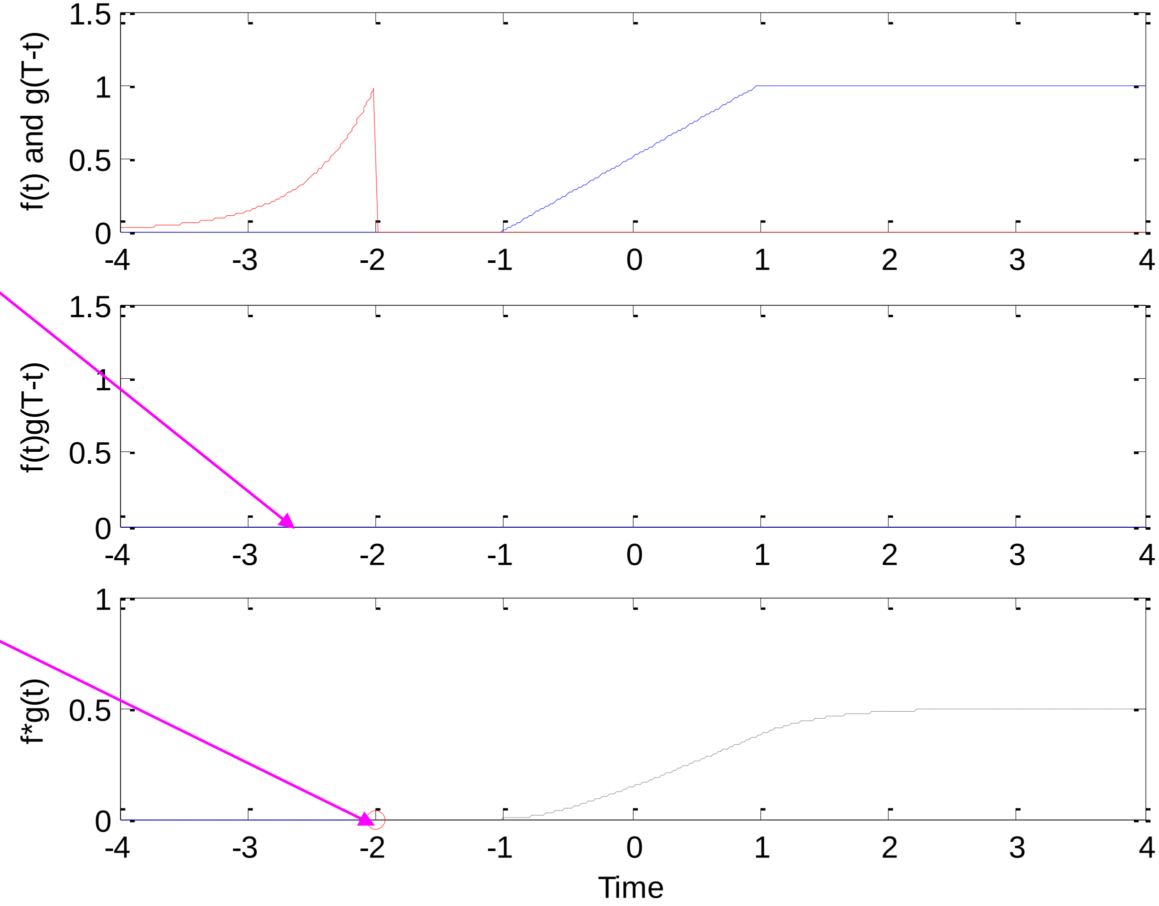


Analog Signals : Properties of Delta Functions

The window function g(-t') is then shifted to give g(-(t'-t)) = g(t-t')

The shifted pulse and the stationary function are then multiplied to give f(t')g(t-t') as a function of t'.

And this product function is integrated over t' to give the value of the convolution at the time t.



Analog Signals : Properties of Delta Functions

The process is repeated for each value of t to give the convolution of f and g as a function of t.

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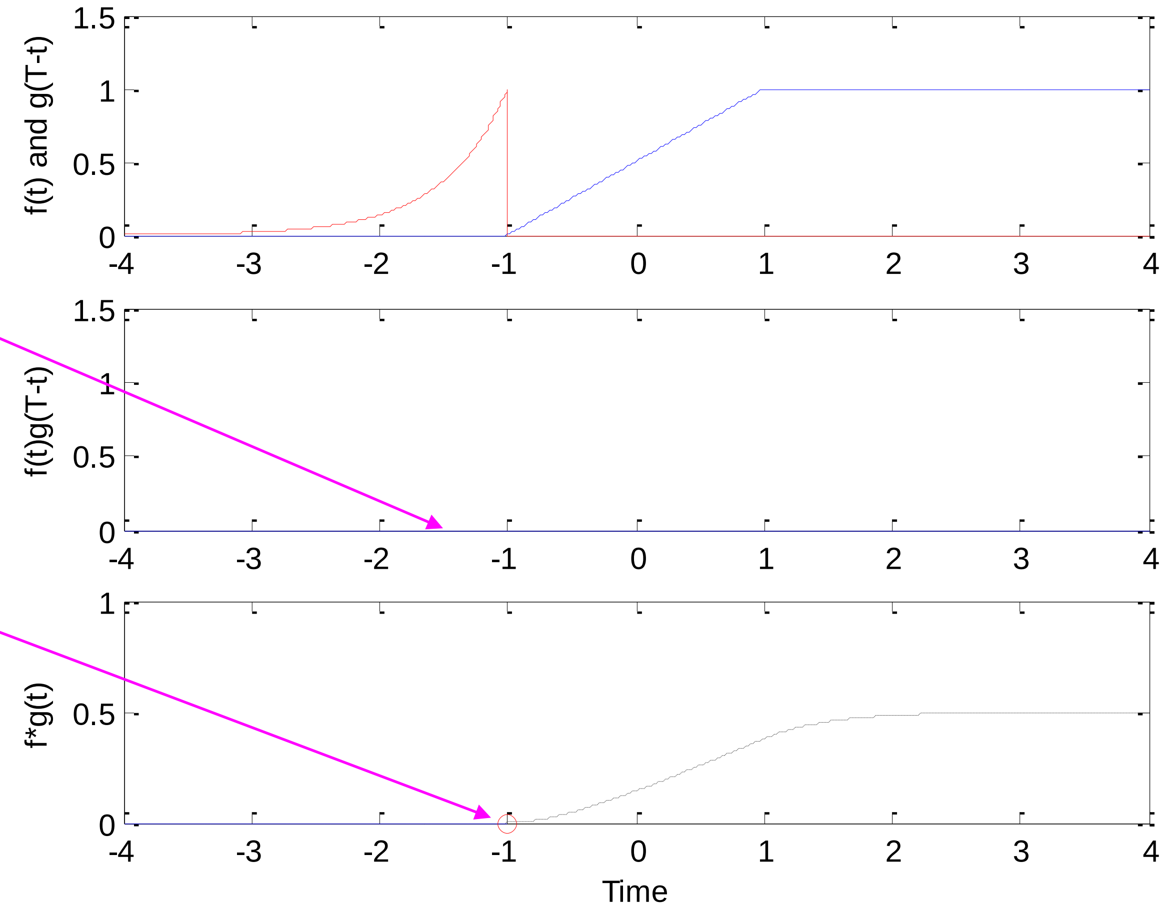

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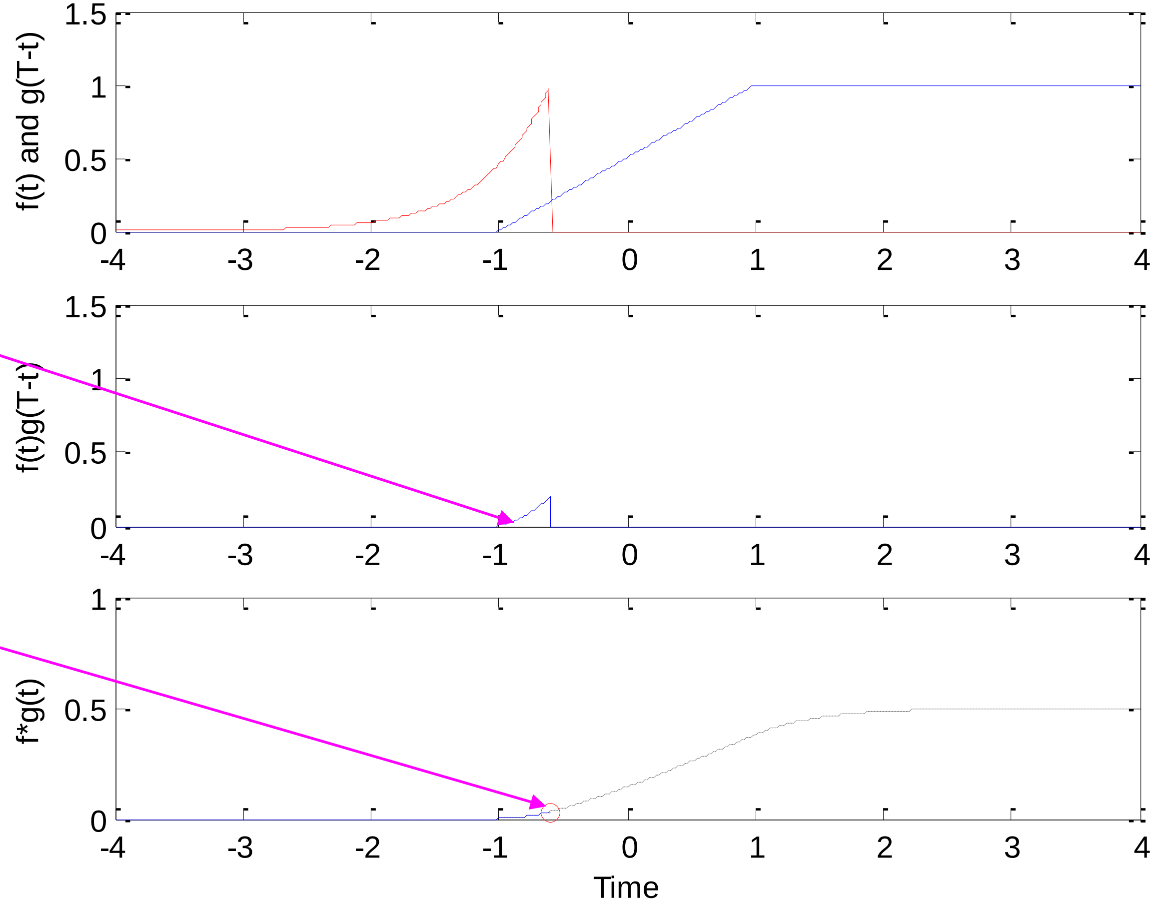

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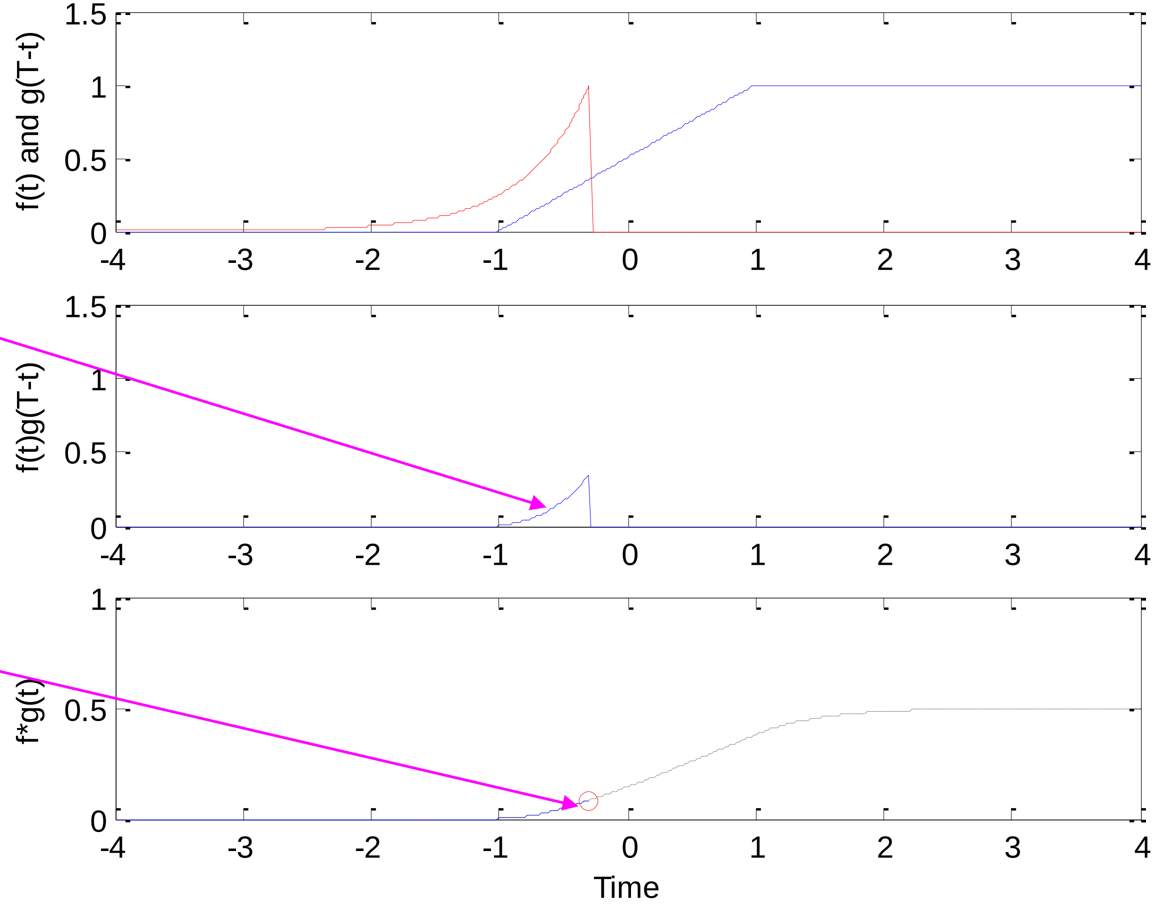

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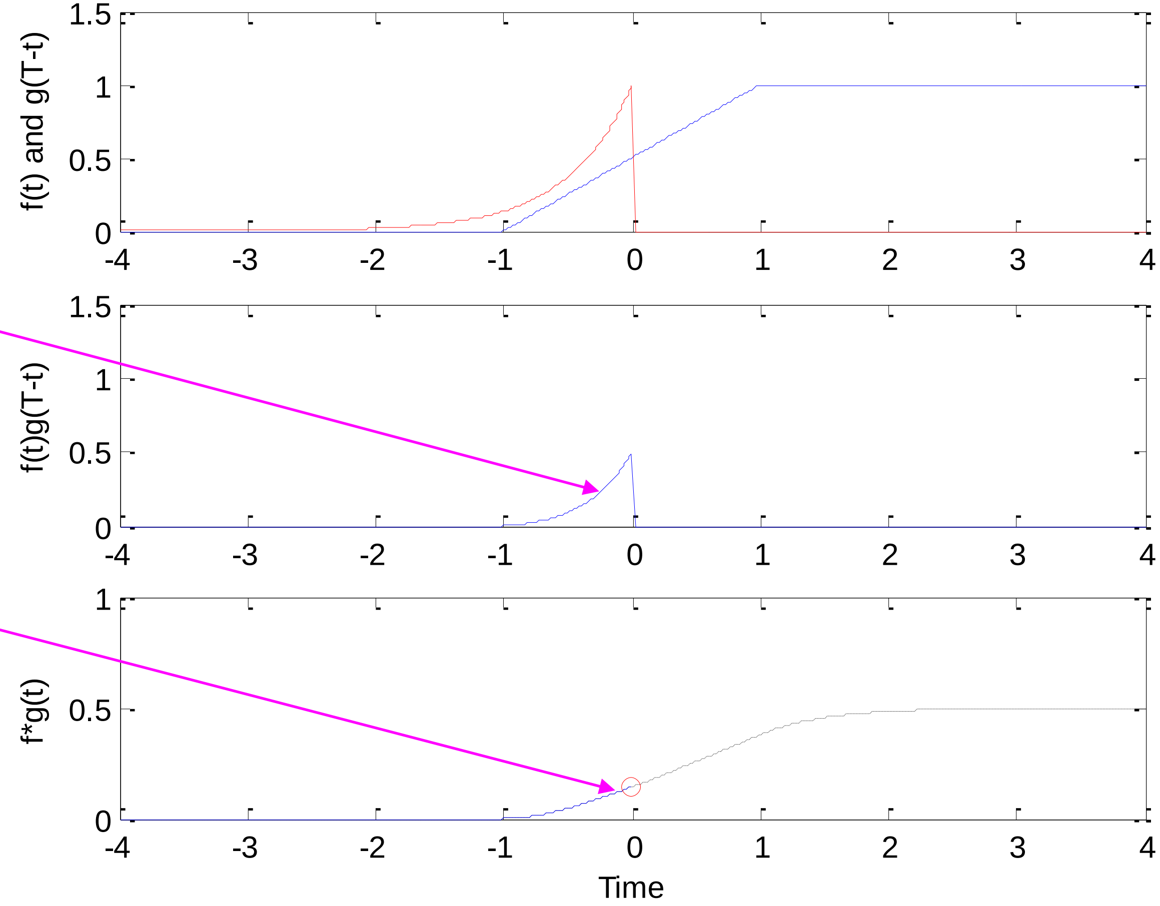

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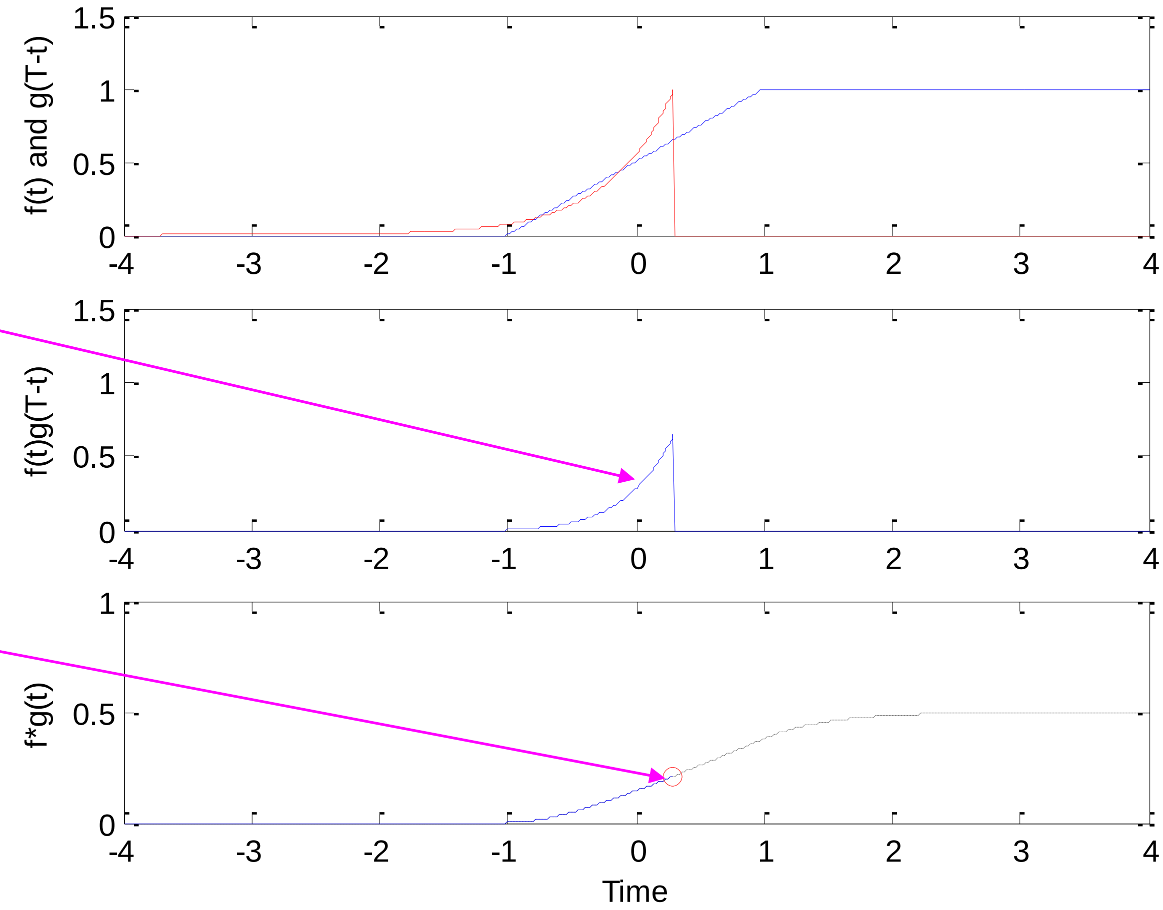

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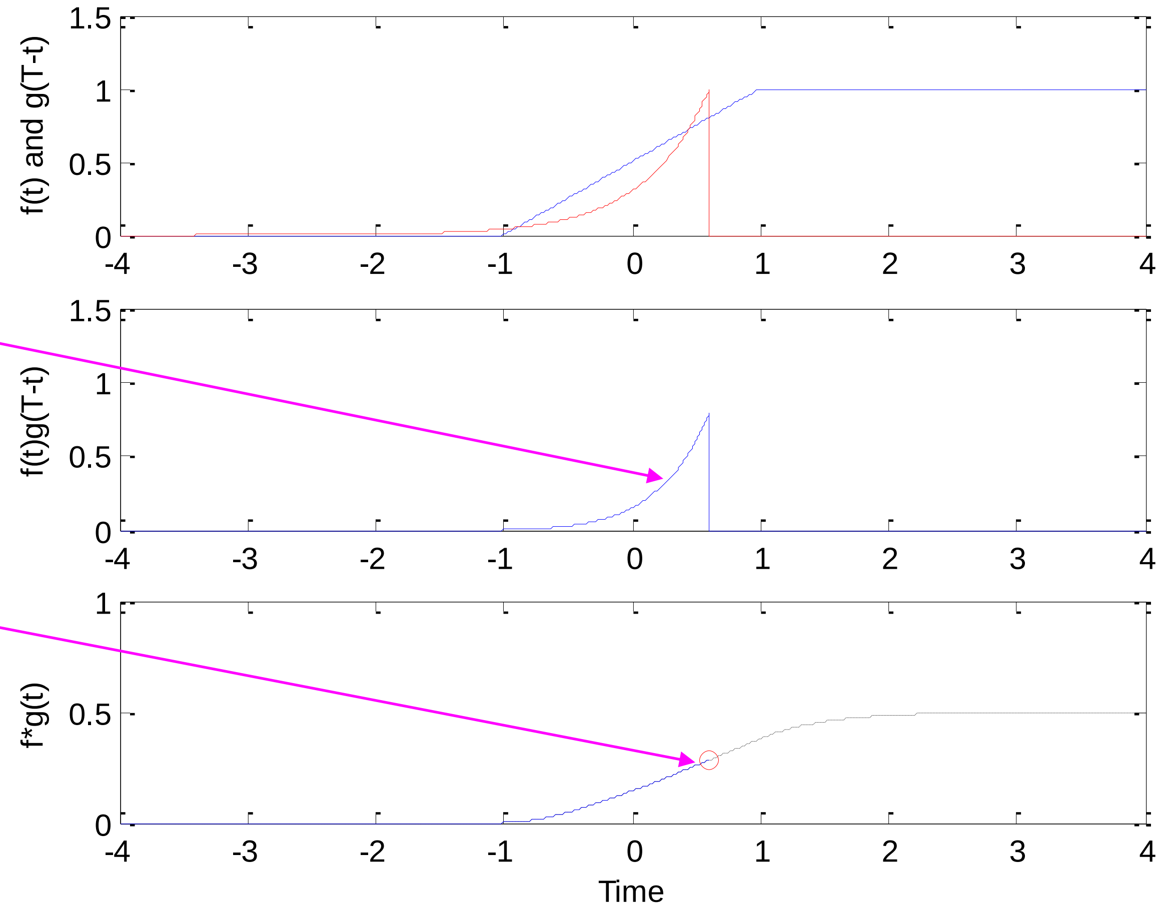

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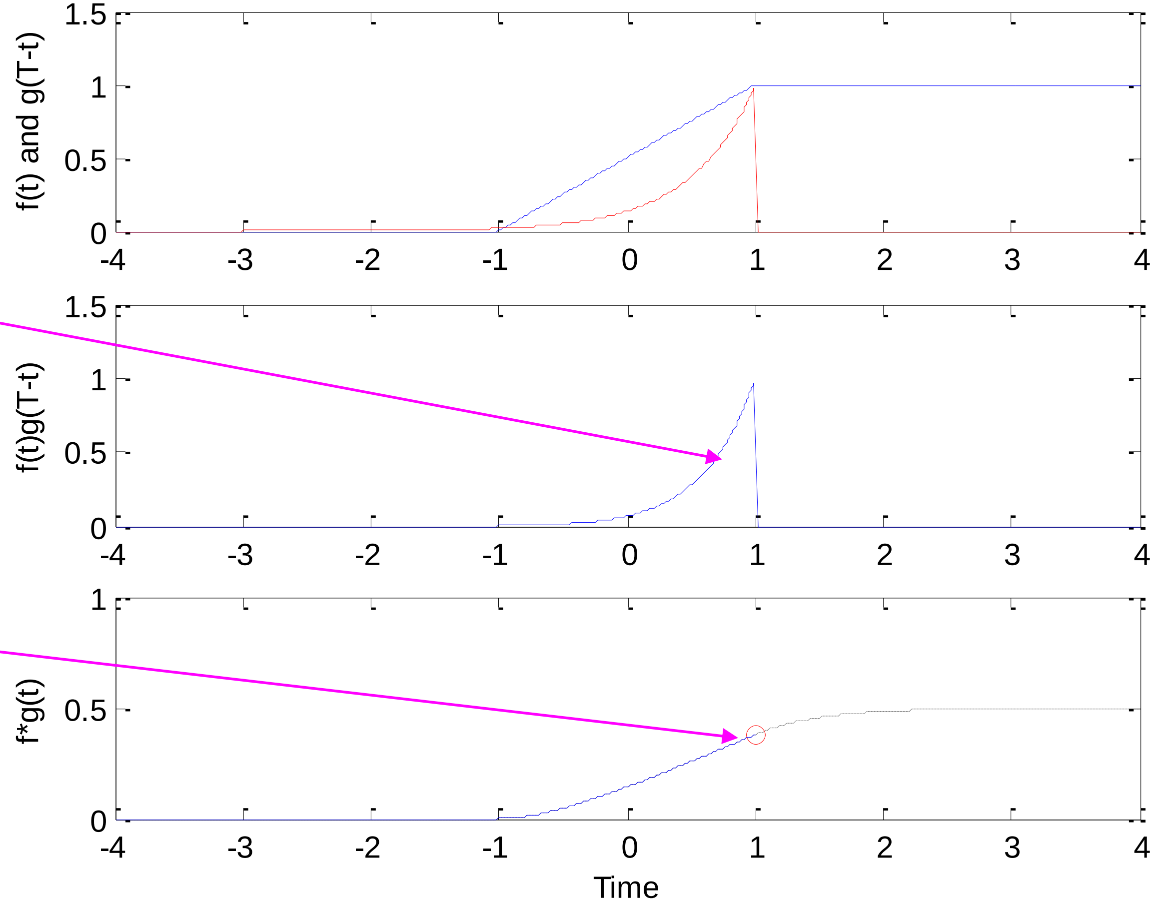

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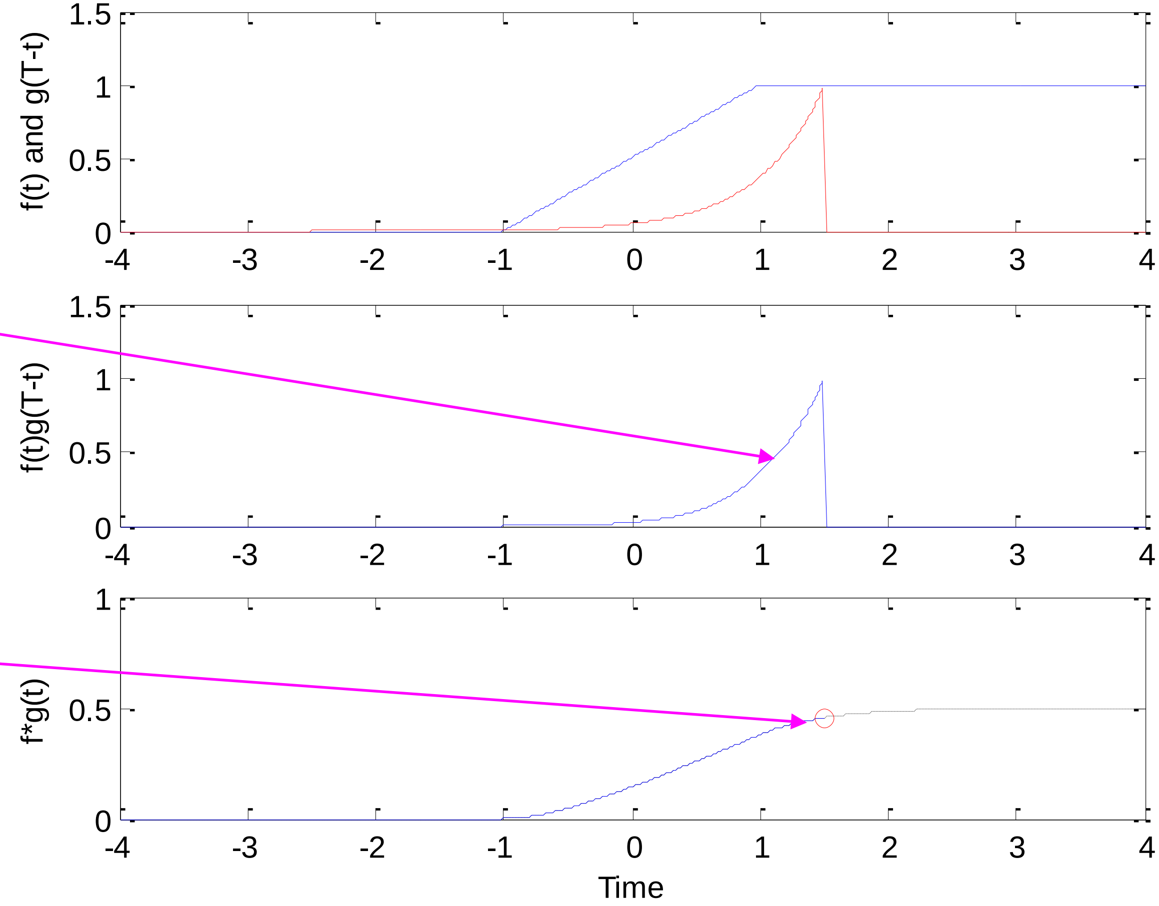

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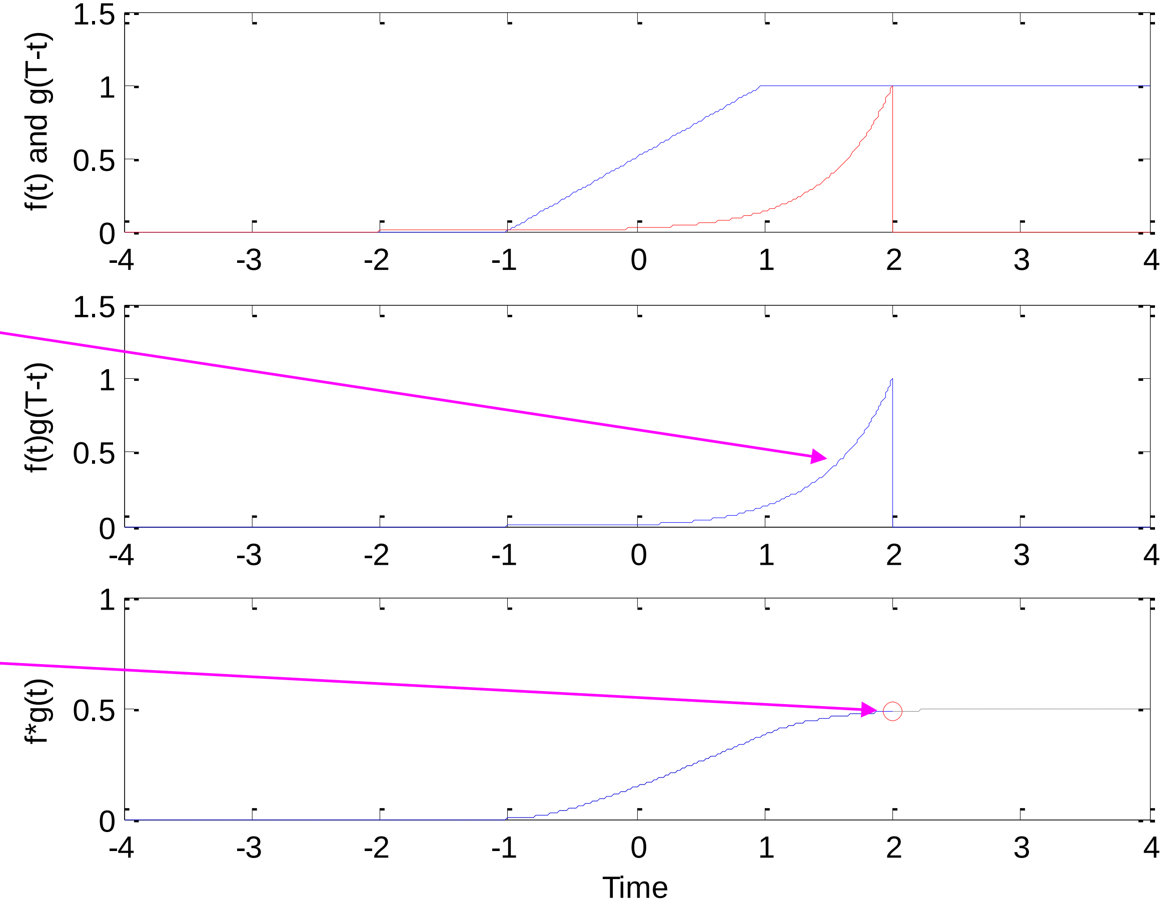

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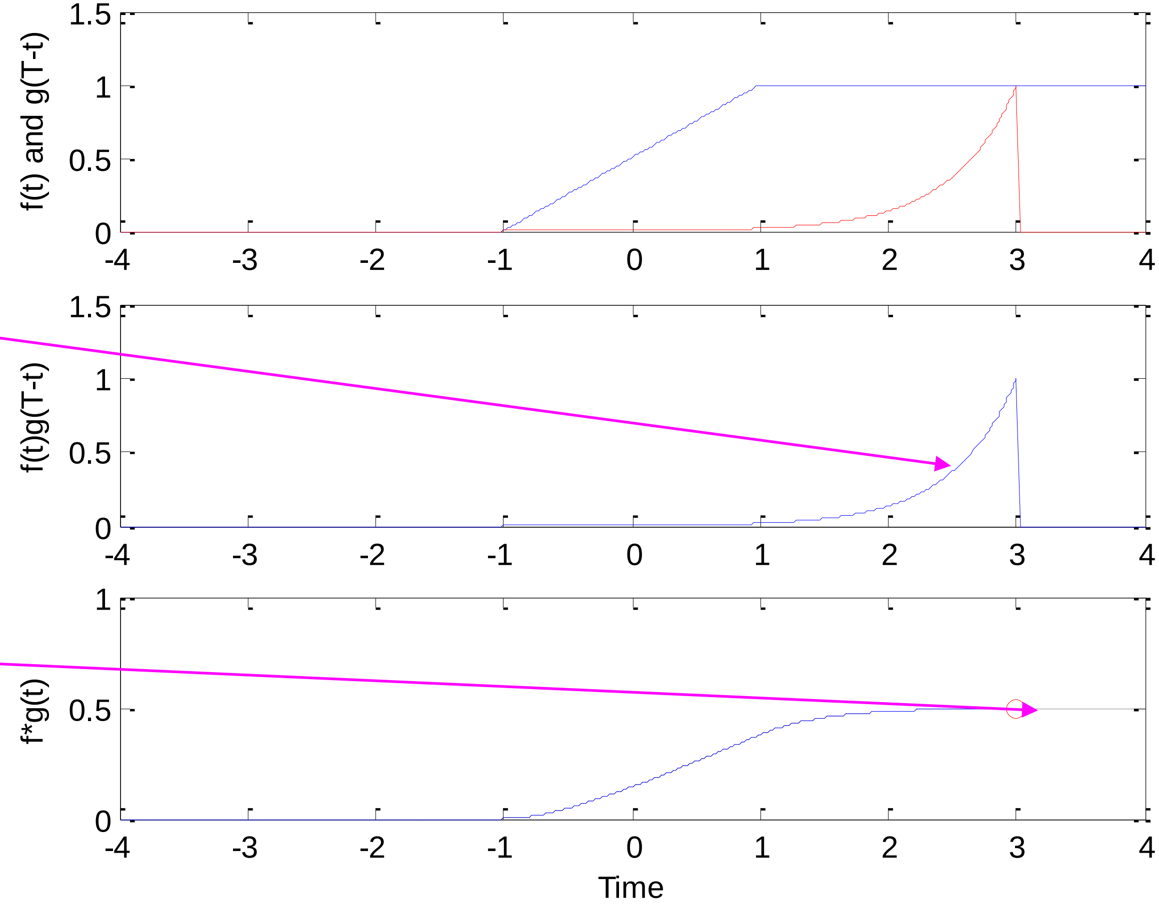

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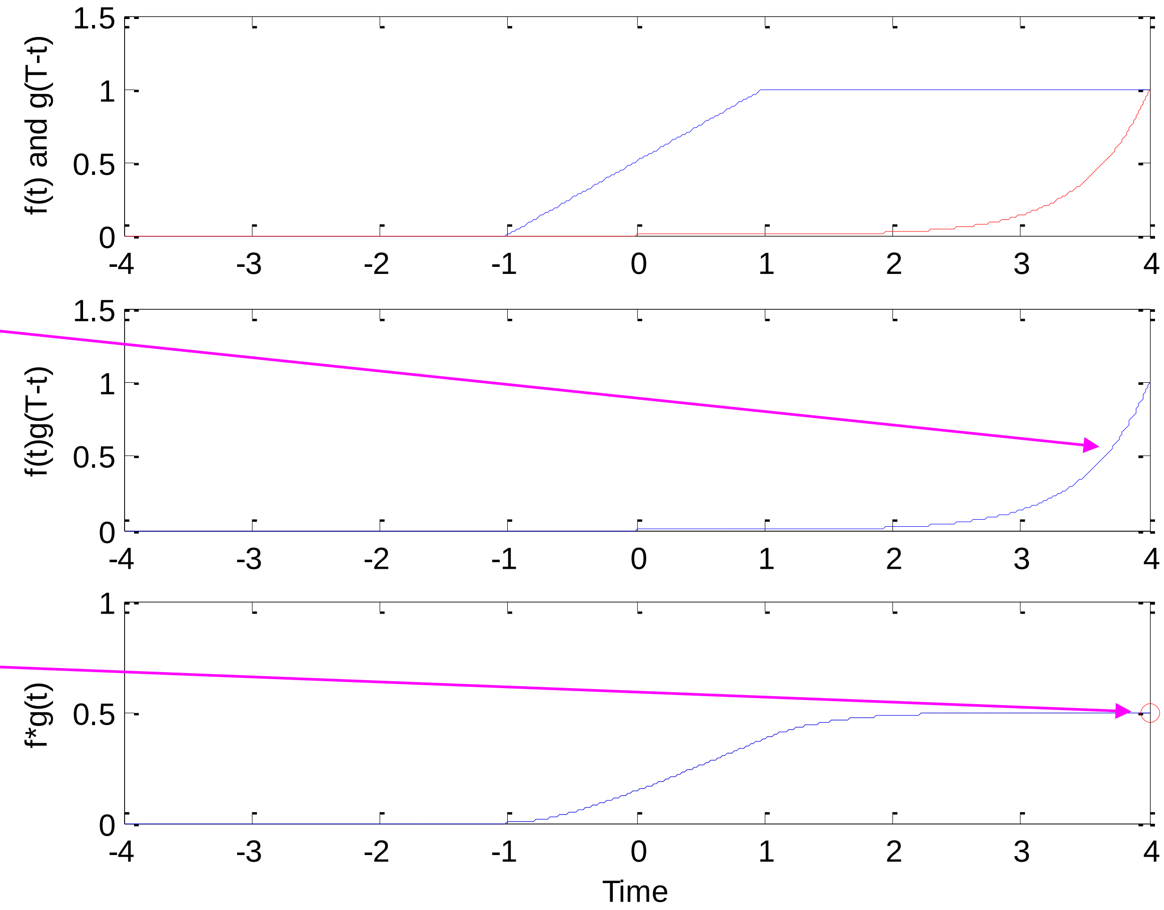
The shifted pulse and the stationary function are then multiplied to give f(t')g(t-t') as a function of t'.

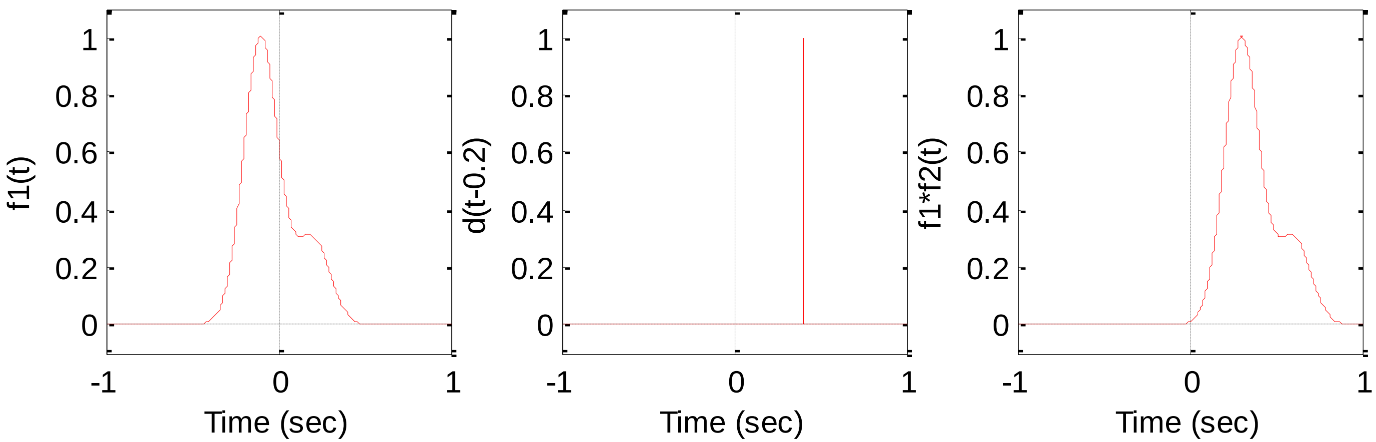
And this product function is integrated over t' to give the value of the convolution at the time t.



Analog Signals : Properties of Delta Functions

The convolution property of the delta function follows from this definition:

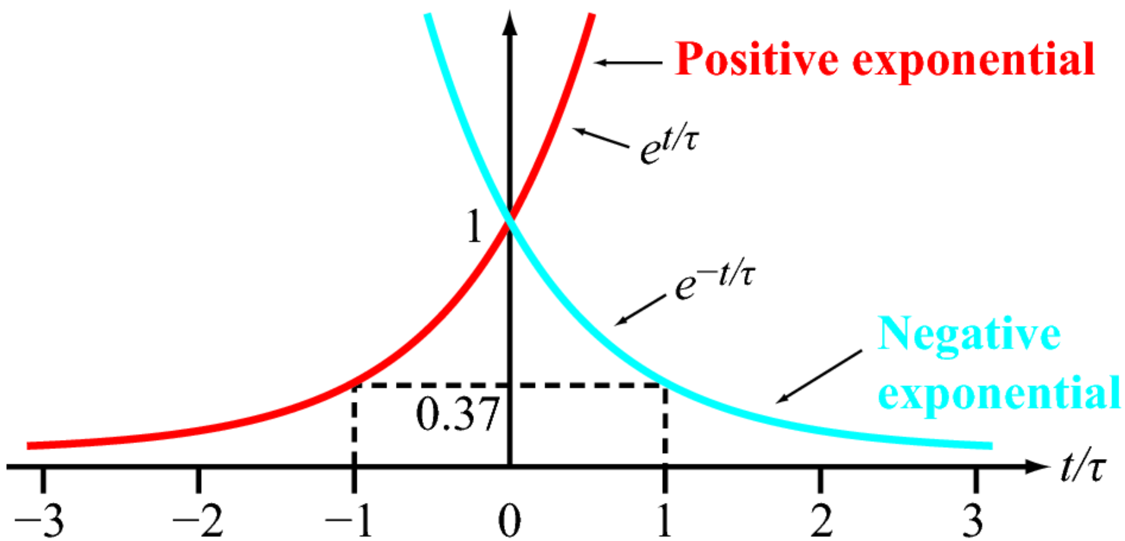
$$f(t)*\delta(t-T) = f(t-T)$$

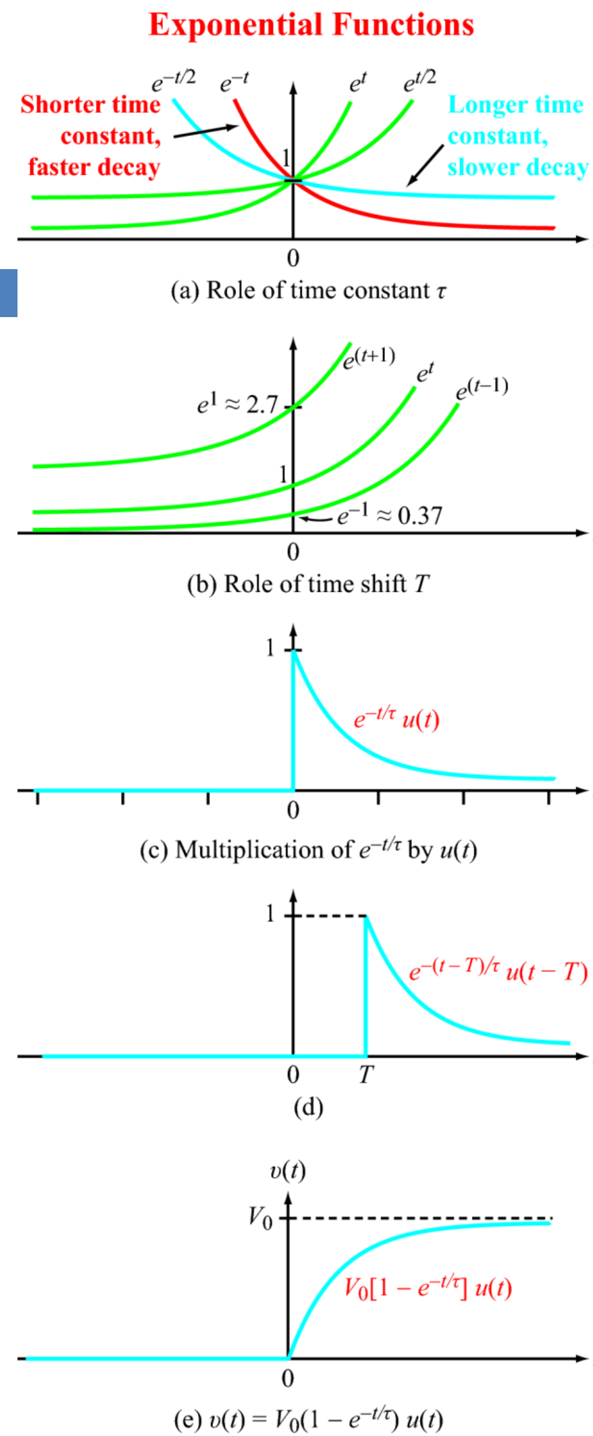


Convolving a function with a delta function simply shifts the function.

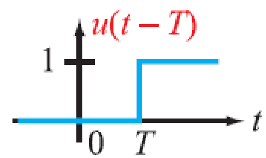
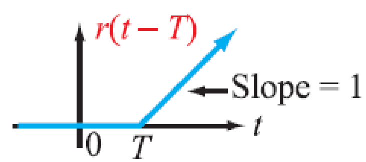
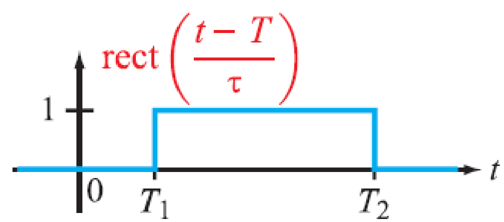
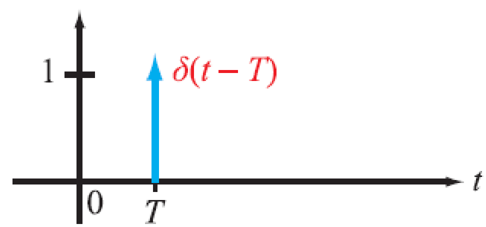
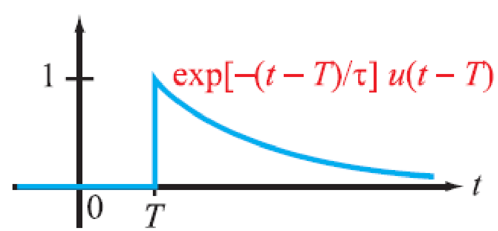
Exponential waveform

- Fast rising or decaying





Summary so far…

Function	Expression	General Shape
Step	$u(t-T) = \begin{cases} 0 & \text{for } t < T \\ 1 & \text{for } t > T \end{cases}$	
Ramp	$r(t-T) = (t-T)\, u(t-T)$	
Rectangle	$\text{rect}\left(\frac{t-T}{\tau}\right) = u(t-T_1) - u(t-T_2)$ $T_1 = T - \frac{\tau}{2}\,; \quad T_2 = T + \frac{\tau}{2}$	
Impulse	$\delta(t-T)$	
Exponential	$\exp[-(t-T)/\tau]\; u(t-T)$	

Analog Signals : Harmonic Functions

Harmonic functions are of the form :

$$f(t) = A\cos(\omega\, t + \phi)$$

which can be written in the alternative form

$$f(t) = a\cos(\omega\, t) + b\sin(\omega\, t)$$

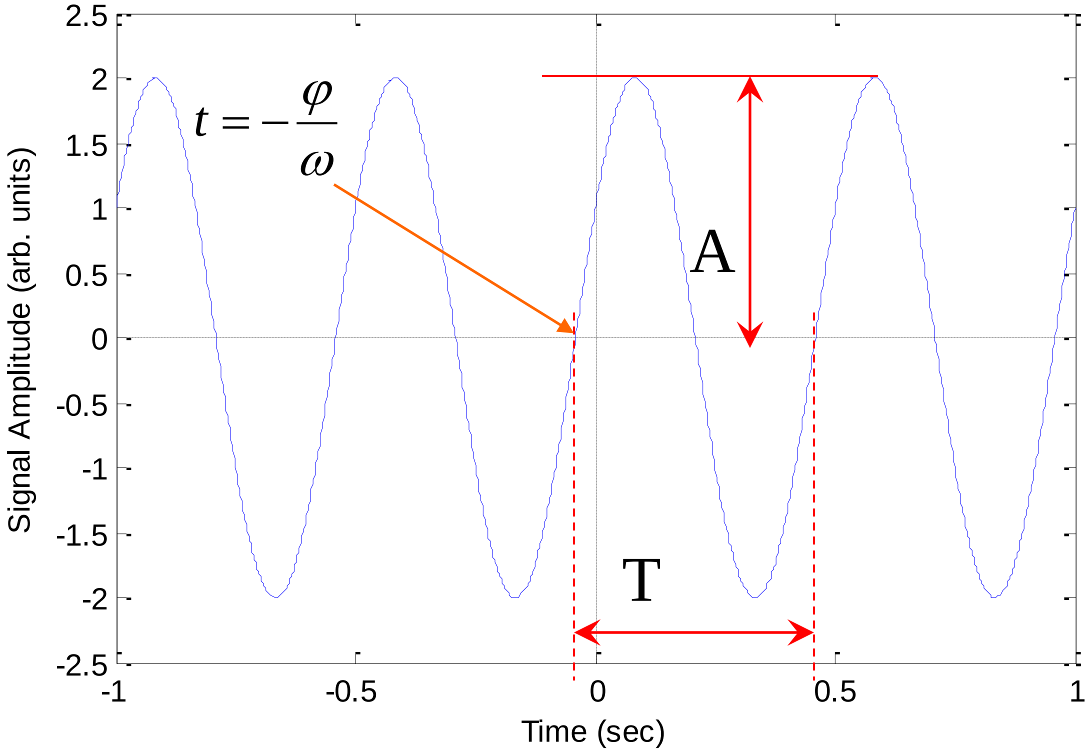
where $$a = A\cos\phi \;\text{ and }\; b = -A\sin\phi$$

The quantity $\omega = 2\pi f = 2\pi/T$ is the frequency of the signal in rad/sec (or simply s^{-1}),
f is the frequency in cycles per second or Hz,
T is the period of the signal in seconds,
ϕ is the phase constant.

Analog Signals : Harmonic Functions

The choice of the cosine is arbitrary, we could equally well have used .

$$f(t) = A\sin(\omega\, t + \varphi) \quad \text{where} \quad \varphi = \phi + \pi / 2$$

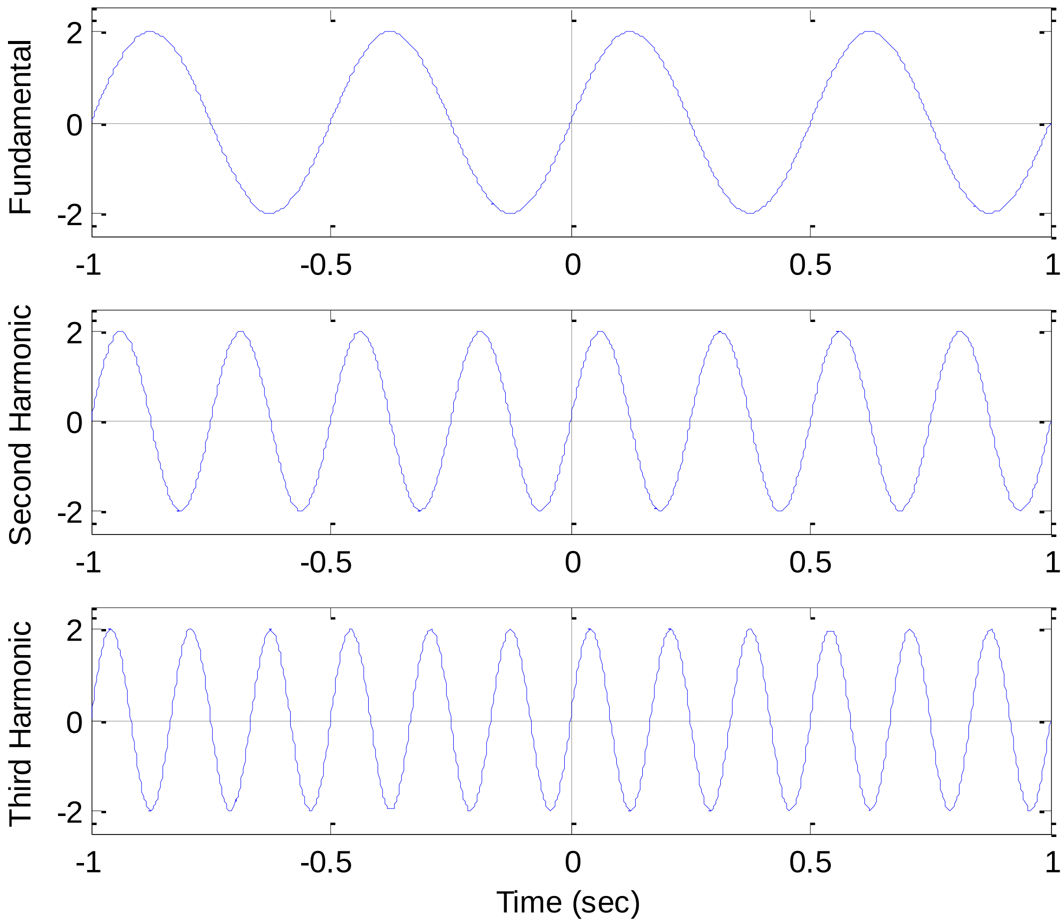


Analog Signals : Harmonic Functions

Harmonic signals at integer multiples of the frequency of a given signal are called the harmonics of that signal, and the given signal is called the fundamental.

The same terms are also used to refer to the frequencies of the signals.

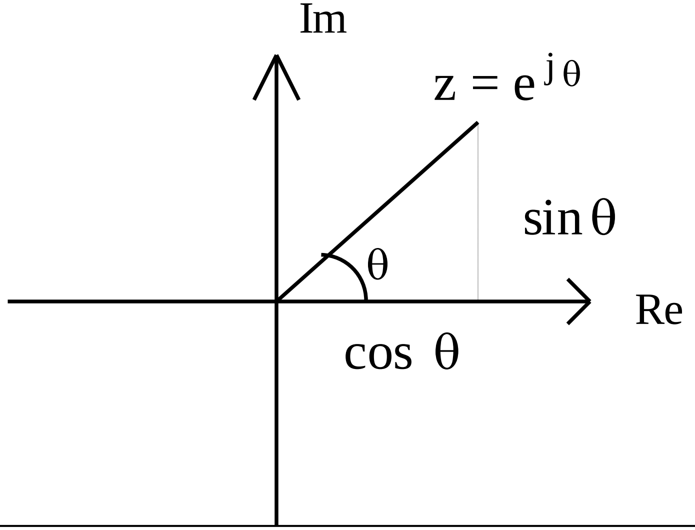
Note that the fundamental is the first harmonic.



Analog Signals : Harmonic Functions

There is a close relationship between harmonic signals and complex exponentials, which follows from Euler's formula:

$$e^{j\theta} = \cos\theta + j\sin\theta$$



Then it is easy to show that:

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

Note that for real θ

$$\left|e^{j\theta}\right| = \sqrt{(\mathrm{Re}\, e^{j\theta})^2 + (\mathrm{Im}\, e^{j\theta})^2}$$

$$= \sqrt{\cos^2\theta + \sin^2\theta}$$

$$= 1$$

CT Complex Exponential (General Form)

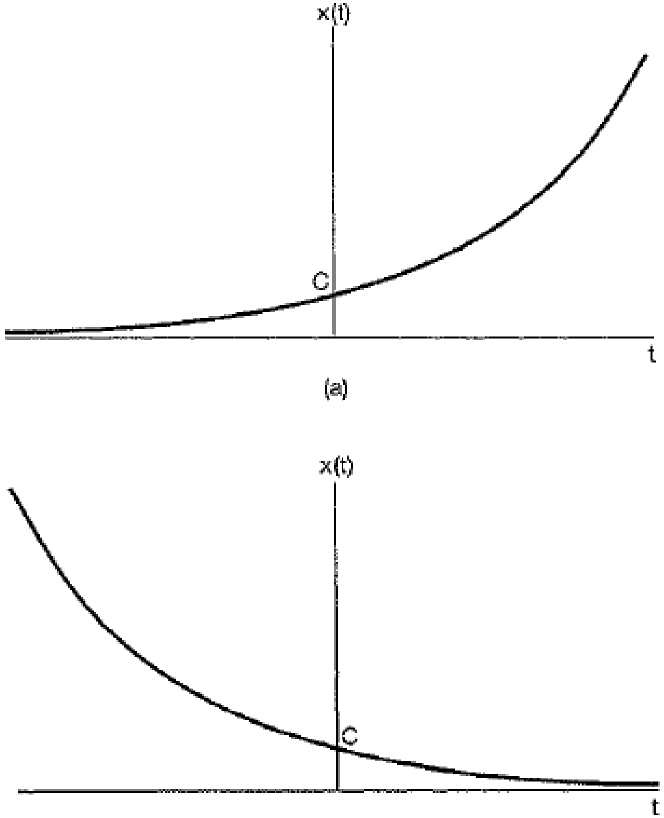
- Most general form of the Continuous Time Complex Exponential

$$x(t) = Ce^{at}$$

- C and a are typically *complex numbers*
- values of these parameters determine key characteristics...

Real Exponential

- if C and a are real
    - $a > 0$: growing exponential
    - $a < 0$: decaying exponential



(a) (b)

- $a = 0$: ???

CT Periodic Complex Exponential

- if a is purely imaginary

$$x(t) = e^{j\omega_0 t}$$

- $x(t)$ is *periodic*

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)}$$

- Let us verify
- Periodicity here implies $e^{j\omega_0 T} = 1$ for *some* value of T
  - if $\omega_0 = 0$, $e^{j\omega_0 T} = 1$ (for any T); also $x(t) = 1$, obviously periodic
  - if $\omega_0 \neq 0$: $e^{j\omega_0 T} = 1$ holds when $T = mT_0$:

$$T_0 = \frac{2\pi}{|\omega_0|}$$

  - and so $x(t)$ will be periodic with fundamental period T_0
- Note: $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same fundamental period

CT Sinusoids

- Sinusoidal signal

$$x(t) = A\cos(\omega_0 t + \phi)$$

- using Euler's relation

$$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$$

- can write

$$A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

- note: exponentials with complex amplitudes

CT Sinusoids

- Sinusoidal signal in terms of complex exponential

$$A\cos(\omega_0 t + \phi) = A\Re\left\{e^{j(\omega_0 t + \phi)}\right\}$$

$$A\sin(\omega_0 t + \phi) = A\Im\left\{e^{j(\omega_0 t + \phi)}\right\}$$

CT General Complex Exponentials

- Consider the general Ce^{at}, in polar form where

$$C = |C|e^{j\theta} \qquad a = r + j\omega_0$$

- and so

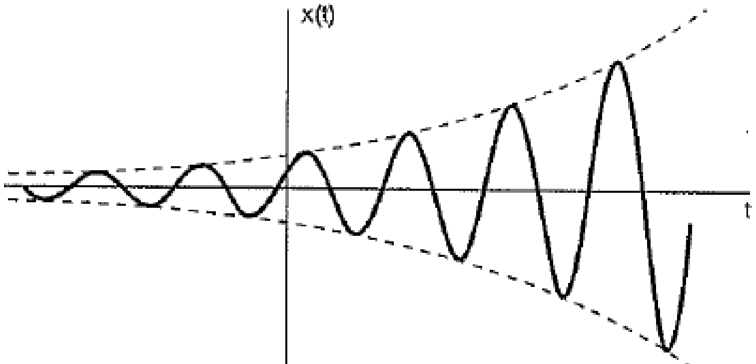
$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}$$

- using Euler

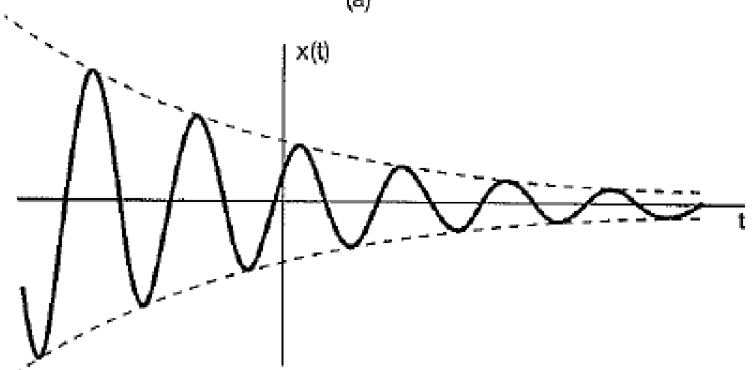
$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta)$$

CT General Complex Exponentials

$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta)$$



(a)



(b)

- $|C|e^{rt}$ is the *envelope* or amplitude of the sinusoidal
- $r > 0$ and $r < 0$ (damped sinusoid) shown above

DT Complex Exponentials (General Form)

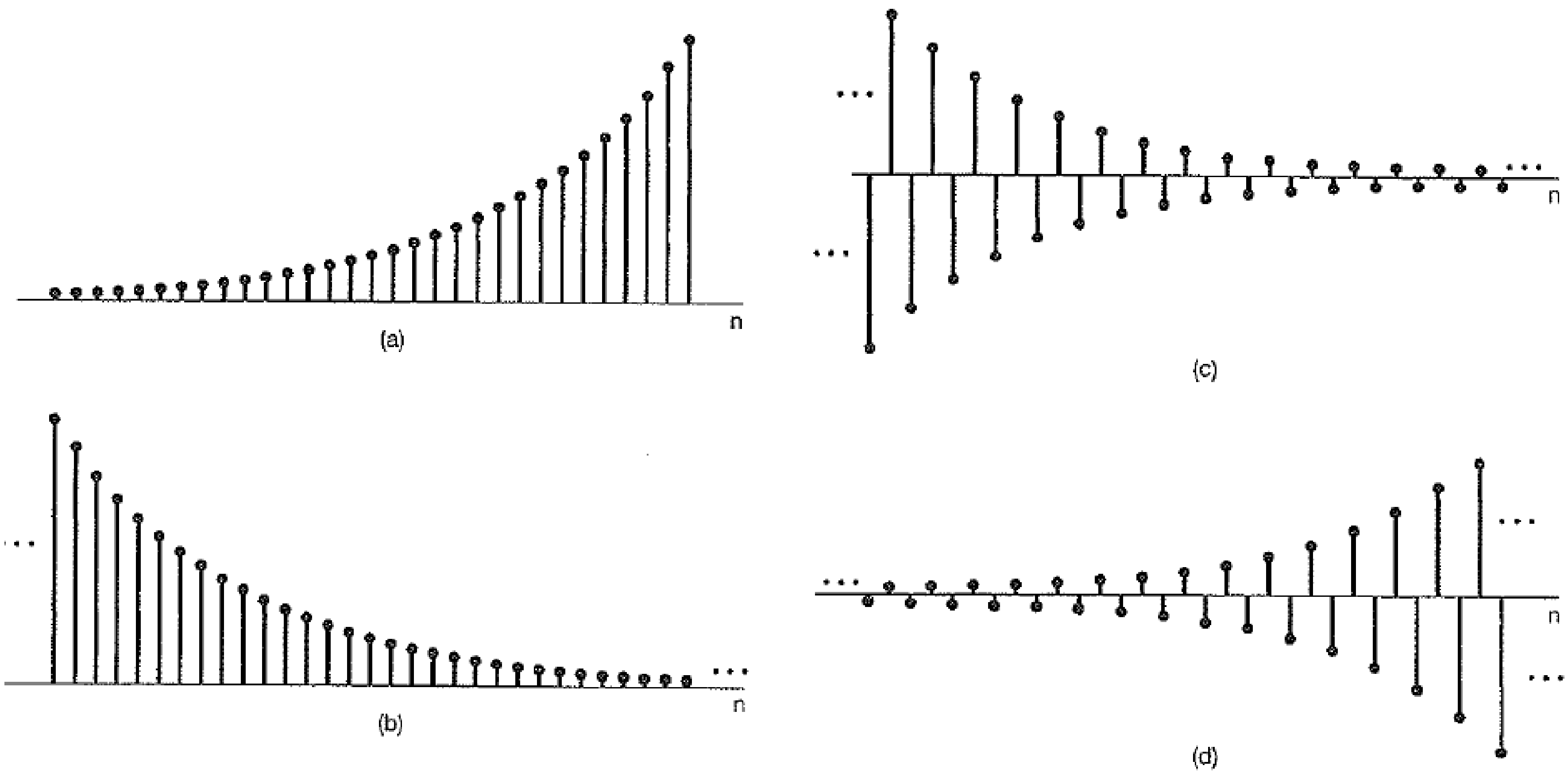
- Discrete time complex exponential *sequence*

$$x[n] = C\alpha^n$$

- C and α are typically *complex numbers*
- analogous form to continuous time exponential

$$x[n] = Ce^{\beta n}, \quad \alpha = e^{\beta}$$

DT Real Exponential

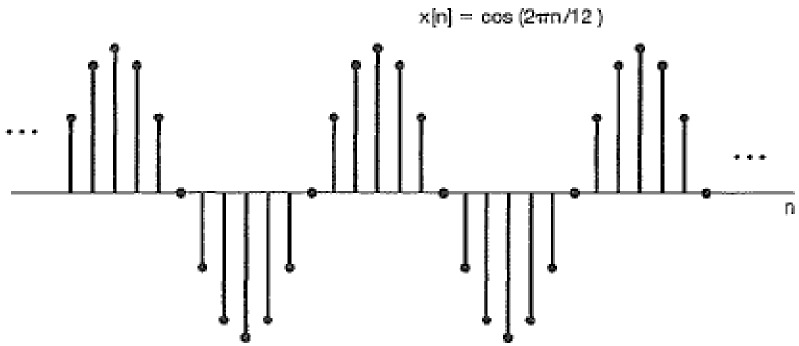


- C and α real valued
- $\alpha > 1$, $0 < \alpha < 1$, $-1 < \alpha < 0$, $\alpha < -1$

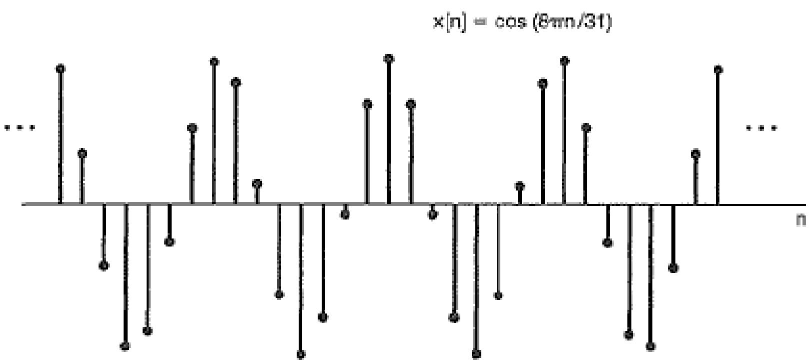
DT Complex Exponential and Sinusoids

- (Note: Unlike in the CT case, the slide title does not say 'Periodic'...)
- β purely imaginary

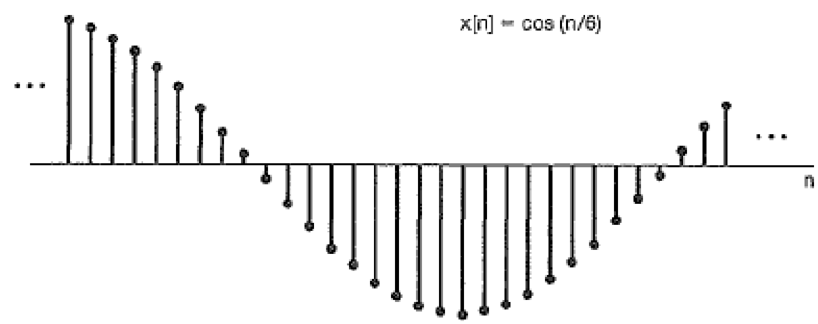
$$x[n] = Ce^{j\omega_0 n} = A\cos(\omega_0 n + \phi)$$



(a)



(b)



DT Sinusoid

- As for continuous time sinusoid, the discrete time

$$x[n] = A\cos(\omega_0 n + \phi)$$

- using Euler's relation becomes

$$e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n$$

- and

$$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

DT General Complex Exponentials

- As before, in polar form

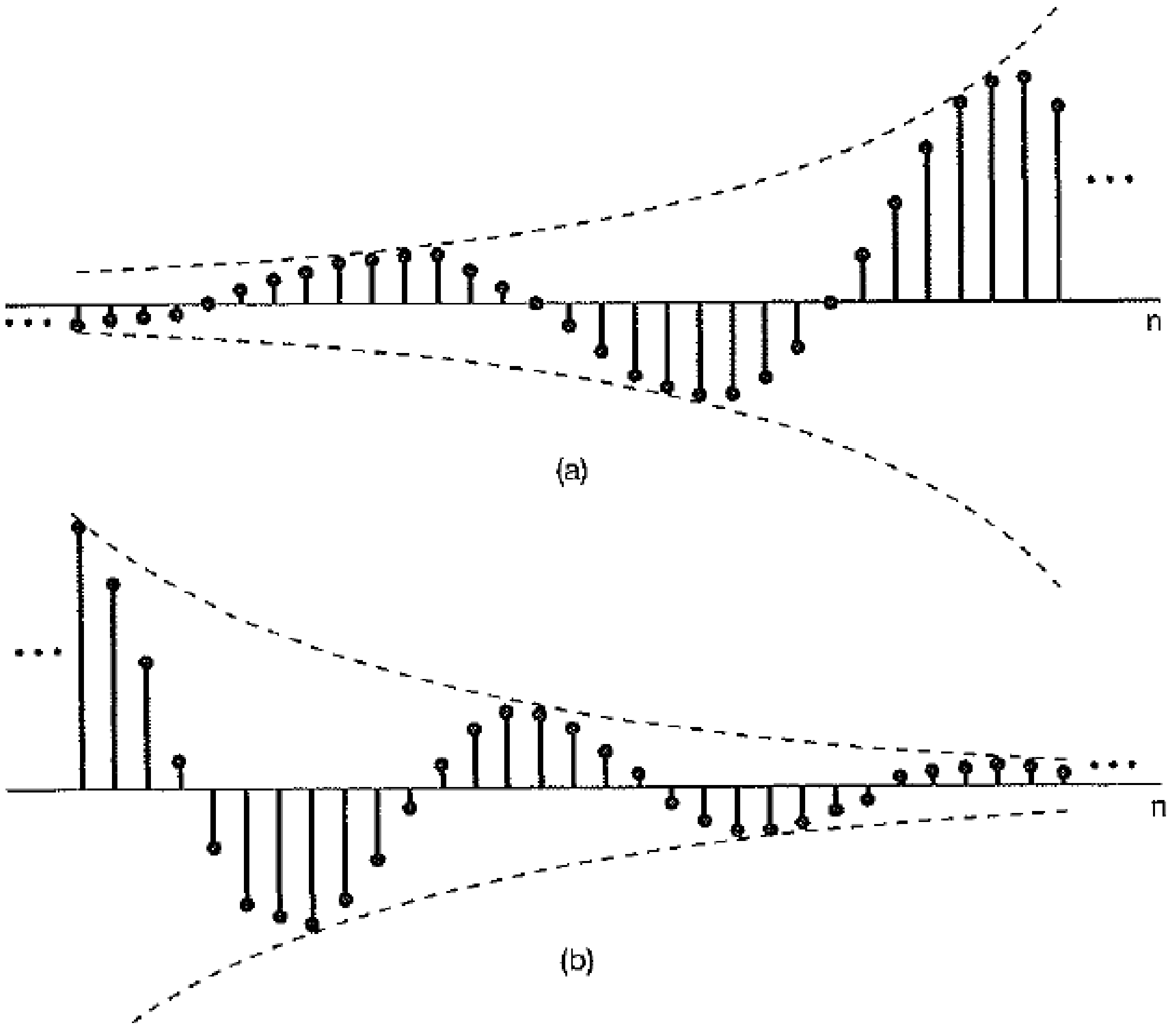
$$C = |C|e^{j\theta} \qquad \alpha = |\alpha|re^{j\omega_0}$$

- and so

$$C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta)$$

DT General Complex Exponentials

$$C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta)$$



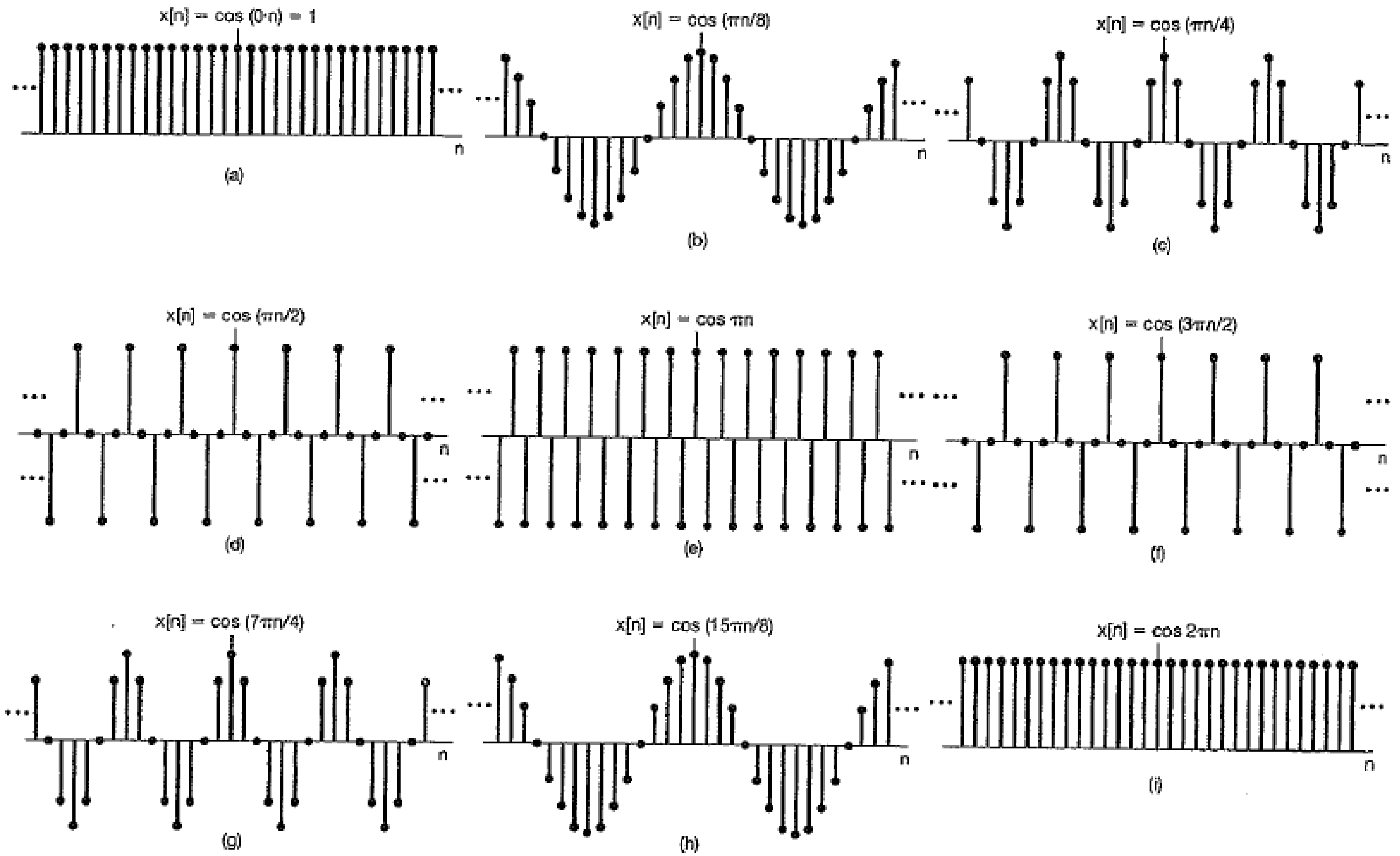
Periodicity of DT Complex Exponentials

- Two important differences between CT and DT signals: periodicity and rate of oscillation
  - $e^{j\omega_0 t}$ is periodic for any ω_0
  - Rate of oscillation of $e^{j\omega_0 t}$ increases with the magnitude of ω_0
  - Not so for DT...
- First, note that

$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}$$

- i.e., DT exponential of frequency $\omega_0 + 2\pi$ (and, for that matter, $\omega_0 + 2n\pi$) is *identical* to that at ω_0
- this is *not* the case for CT exponentials - $e^{j\omega_0 t}$ is distinct for different values of ω_0 ($e^{j(\omega_0+2\pi)t} \neq e^{j\omega_0 t}$ for all t)
- for DT, need only look at an interval of 2π (typically $0 \leq \omega_0 < 2\pi$ or $-\pi \leq \omega_0 < \pi$)

Rate of Oscillation



- Increases with ω_0 while $0 \leq \omega_0 < \pi$, then *decreases* while $\pi \leq \omega_0 < 2\pi$
- Note: $e^{j\pi n} = (-1)^n$, i.e,. oscillation for every n

Periodicity of DT Complex Exponentials

- For $e^{j\omega_0 n}$ to be periodic with period $N > 1$, need

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}$$

- i.e., need $e^{j\omega_0 N} = 1$ for some N
- which implies $\omega_0 N$ must be a multiple of 2π

$$\frac{\omega_0}{2\pi} = \frac{m}{N}$$

- which in turns means that $e^{j\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a *rational number*
- same holds for DT sinusoids $\cos \omega_0 n$
- As with CT, define fundamental frequency (DT) $\frac{2\pi}{N} = \frac{\omega_0}{m}$
- Fundamental period (DT) $N = m\left(\frac{2\pi}{\omega_0}\right)$

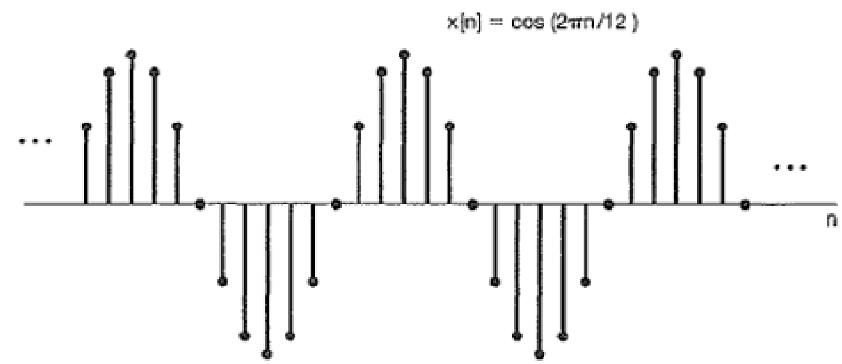
CT vs DT Complex Exponentials

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

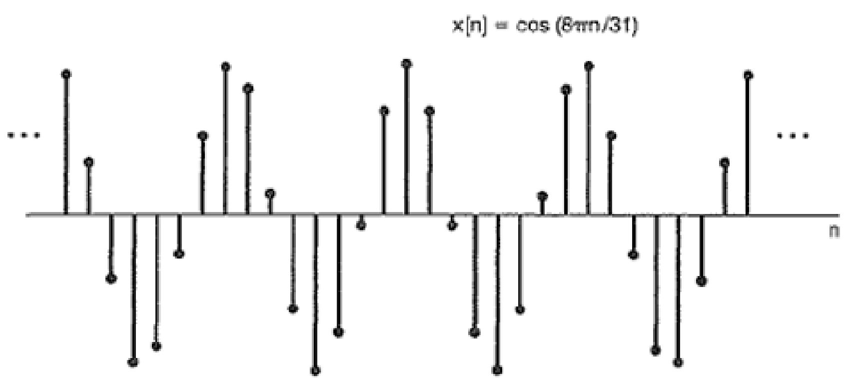
*Assumes that m and N do not have any factors in common.

Example: Periodic or not?

- Let's look at these discrete time signals again.



(a)



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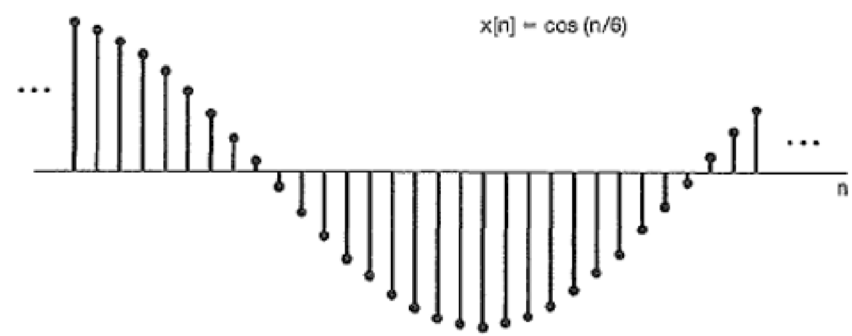
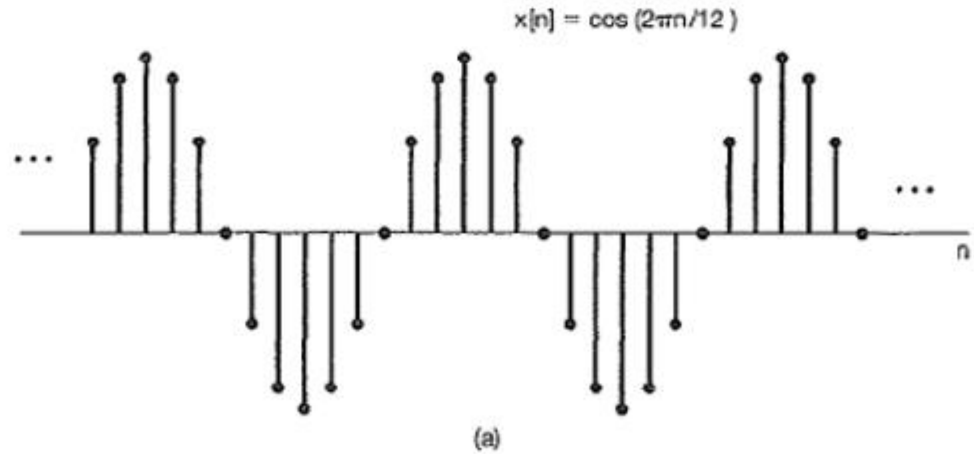
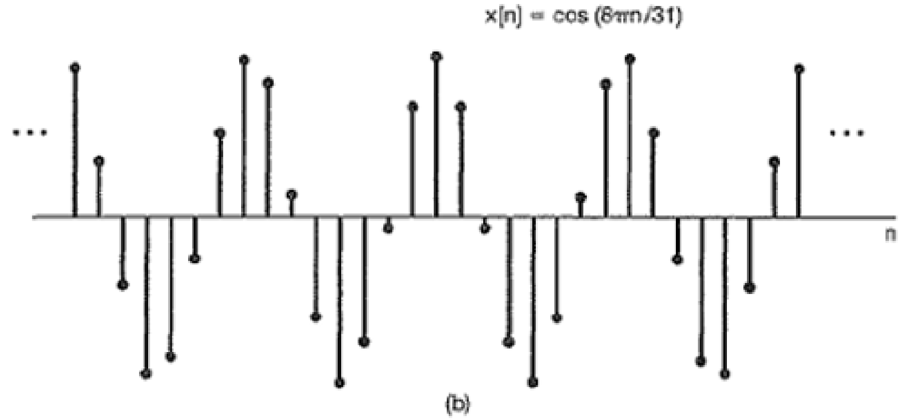


Example:



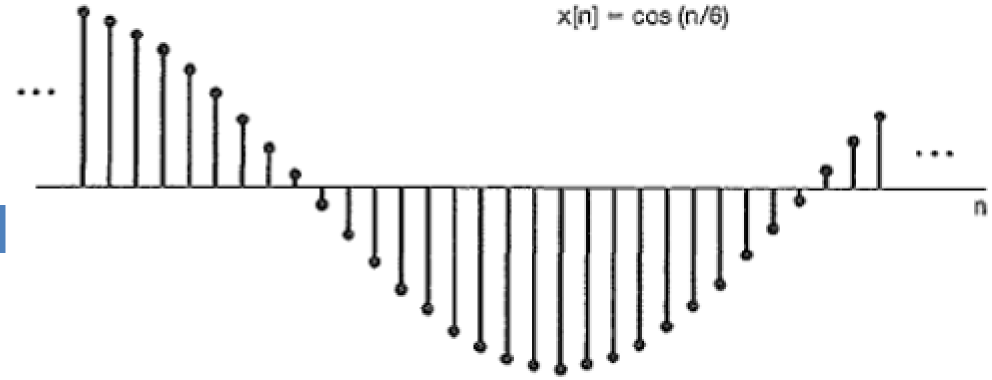
- Signal (a): sequence $x[n]=\cos(2\pi n/12)$, can be seen as the set of samples of the continuous time signal $x(t)=\cos(2\pi t/12)$ at integer time points (ie n).
- w_0 of x(t) is $2\pi/12$, meaning the frequency is 1/12.
- So, x(t) is periodic with a fundamental period of 12.
- Test for periodic? $w_0/2\pi = (2\pi/12)/\ 2\pi = 1/12$ => a rational number! So, is periodic.
- That is, the values of x[n] repeat every 12 points, exactly in step with fundamental period of x(t).

Example:



- Signal (b): sequence x[n]=cos(8 πn/31), can be seen as the set of samples of x(t)=cos(8 πt/31) at integer time points (ie n).
- In this case, x(t)is periodic with fundamental frequency 31/4.
- But, x[n] is periodic with 31. Why?
- Because discrete time signal is only defined for interger values of independent variable. So, there is no sample at time t=31/4, when x(t) completes one period (from t=0),
- There is also no samples at t=2(31/4) or 3(31/4) when x(t) completes 2 and 3 periods.
- But there is a sample when t=4(31/4)=31, when x(t) has completed 4 periods.
- We can see from graph the pattern of x[n] values repeat after 4 cycles, namely, every 31 points.

Example:



- Signal (c): sequence $x[n]=\cos(n/6)$, can be seen as the set of samples of $x(t)=\cos(t/6)$ at integer time points (ie n).
- w_0 of x(t) is 1/6, meaning the frequency is $1/12\pi$ (ie period for x(t) is 12π).
- Test for Periodic? $w_0/2\pi = (1/6)/2\pi = 1/12\pi$. => Not a rational number because π is not. So, Not periodic.
- This means, the values of x(t) at interger sample points never repeat. Why?
- Because these sample points never span an interval that is an exact multiple of the period, 12π, of x(t).
- So, x[n] is not periodic.