LECTURE BLOCK 1C: FUNDAMENTAL FREQUENCY AND PERIODICITY OF SIGNALS

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Fundamental Frequency for CT Signals

We have learnt how to determine the period, and periodicity of an individual signals. What happens when we have 2 or more composite signals?

To identify the period T, the frequency $f = 1/T$, or the angular frequency $\omega = 2\pi f = 2\pi/T$ of a given sinusoidal or complex exponential signal, it is always helpful to write it in any of the following forms:

$$sin(\omega t) = sin(2\pi ft) = sin(2\pi t/T)$$

The fundamental frequency of a signal is the greatest common divisor (GCD) of all the frequency components contained in a signal, and, equivalently, the fundamental period is the least common multiple (LCM) of all individual periods of the components.

Example 1: Find the fundamental frequency of the following continuous signal:

$$x(t) = cos(\frac{10\pi}{3}t) + sin(\frac{5\pi}{4}t)$$

CT Signals

Example 1: Find the fundamental frequency of the following continuous signal:

$$x(t) = cos(\frac{10\pi}{3}t) + sin(\frac{5\pi}{4}t)$$

The fundamental frequency f_0 is the GCD of $f_1 = 5/3$ and $f_2 = 5/8$:

$$f_0 = GCD(\frac{5}{3}, \frac{5}{8}) = GCD(\frac{40}{24}, \frac{15}{24}) = \frac{5}{24}$$

Alternatively, the period of the fundamental T_0 is the LCM of $T_1 = 3/5$ and $T_2 = 8/5$:

$$T_0 = LCM(\frac{3}{5}, \frac{8}{5}) = \frac{24}{5}$$

CT Signals

Now we get $\omega_0 = 2\pi f_0 = 2\pi/T_0 = 5\pi/12$ and the signal can be written as

$$x(t) = cos(8\frac{5\pi}{12}t) + sin(3\frac{5\pi}{12}t) = cos(8\omega_0 t) + sin(3\omega_0 t)$$

i.e., the two terms are the 3th and 8th harmonic of the fundamental frequency ω_0, respectively.

CT Signals

Example 2:

$$x(t) = sin(\frac{5\pi}{6}t) + cos(\frac{3\pi}{4}t) + sin(\frac{\pi}{3}t)$$

The frequencies and periods of the three terms are, respectively,

$$\omega_1 = \frac{5\pi}{6},\ f_1 = \frac{5}{12},\ T_1 = \frac{12}{5},\ \ \omega_2 = \frac{3\pi}{4},\ f_2 = \frac{3}{8},\ T_2 = \frac{8}{3},\ \ \omega_3 = \frac{\pi}{3},\ f_3 = \frac{1}{6},\ T_3 = 6$$

The fundamental frequency f_0 is the GCD of f_1, f_2 and f_3:

$$f_0 = GCD(\frac{5}{12}, \frac{3}{8}, \frac{1}{6}) = GCD(\frac{10}{24}, \frac{9}{24}, \frac{4}{24}) = \frac{1}{24}$$

Alternatively, the period of the fundamental T_0 is the LCM of T_1, T_2 and T_3:

$$T_0 = LCM(\frac{12}{5}, \frac{8}{3}, 6) = LCM(\frac{36}{15}, \frac{40}{15}, \frac{90}{15})\frac{5}{24}$$

CT Signals

The signal can be written as

$$x(t) = sin(\frac{10\pi}{12}t) + cos(\frac{9\pi}{12}t) + sin(\frac{4\pi}{12}t)$$

i.e., the fundamental frequency is $\omega_0 = \pi/12$, the fundamental period is $T_0 = 2\pi/\omega) = 24$, and the three terms are the 4th, 9th and 10th harmonic of ω_0, respectively.

Example CT: Sum of two sinusoids

Example 3: Find the fundamental frequency of the following continuous signal:

$$x(t) = cos(\frac{10}{3}t) + sin(\frac{5\pi}{4}t)$$

CT Signals

Here the angular frequencies of the two terms are, respectively,

$$\omega_1 = \frac{10}{3}, \qquad \omega_2 = \frac{5\pi}{4}$$

The fundamental frequency ω_0 should be the GCD of ω_1 and ω_2:

$$\omega_0 = GCD(\frac{10}{3}, \frac{5\pi}{4})$$

which does not exist as π is an irrational number which cannot be expressed as a ratio of two integers, therefore the two frequencies can not be multiples of the same fundamental frequency. In other words, the signal as the sum of the two terms is not a periodic signal.

CT sinusoid is always periodic, but the sum of two CT sinusoids is not necessarily periodic.

Fundamental Frequency of DT Signals

For a discrete complex exponential $x[n] = e^{j\omega_1 n}$ to be periodic with period N, it has to satisfy ____________

$$e^{j\omega_1 (n+N)} = e^{j\omega_1 n}, \qquad \text{i.e.,} \qquad e^{j\omega_1 N} = 1 = e^{j2\pi k}$$

that is, $\omega_1 N$ has to be a multiple of $\underline{2\pi}$:

$$\omega_1 N = 2\pi k, \qquad \text{i.e.,} \qquad \frac{\omega_1}{2\pi} = \frac{k}{N}$$

The normalized frequency must be rational for the periodicity of discrete signals.

Distinct value of Ω_0 does not always produce periodic signals!

Is a discrete sinusoid or complex exponential periodic?
Not necessarily.

DT Signals

As k is an integer, $\omega_1/2\pi$ has to be a rational number (a ratio of two integers). In order for the period

$$N = k\frac{2\pi}{\omega_1}$$

to be the fundamental period, k has to be the smallest integer that makes N an integer, and the fundamental angular frequency is

$$\omega_0 = \frac{2\pi}{N} = \frac{\omega_1}{k}$$

The original signal can now be written as:

$$x[n] = e^{j\omega_1 n} = e^{jk\omega_0 n} = e^{jk\frac{2\pi}{N}n}$$

DT Example:

- Is sin [2k] periodic?

sin 2k is not periodic because k/Pi is not a rational number. In this case, there exists no integer k in $N = k\frac{2\pi}{\omega_1}$ such that N is an integer

- Is $\sin 0.01\pi k$ periodic?

Here $\frac{\omega_1}{2\pi}$ = 0.01Pi/2Pi = 0.005 = 5/1000 (a rational number).

Period $N = k\frac{2\pi}{\omega_1}$ = 2kPi/0.01Pi = 200k = 200 by choosing k =1.

DT Signals: Example

Consider $\sin 3.2\pi k$. It can be simplified as

$$\sin 3.2\pi k = \sin(2\pi + 1.2\pi)k = \sin 2\pi k \cos 1.2\pi k + \cos 2\pi k \sin 1.2\pi k = \sin 1.2\pi k$$

where we have used the fact that $\sin 2\pi k = 0$ and $\cos 2\pi k = 1$ for every integer k. This implies that when we are given $\sin \omega k$, we can always reduce ω to the range $[0, 2\pi)$ by subtracting or adding 2π or its multiple. Thus in the discrete-time case, we have

$$\sin 6.2\pi k = \sin 0.2\pi k \;\; and \;\; \cos(-2.4\pi k) = \cos 1.6\pi k$$

for all integer k. In the continuous-time case, $\sin 6.2\pi t$ and $\sin 0.2\pi t$ are two different functions.

DT Signals: Example

Find the fundamental period of the following discrete signal:

$$x[n] = e^{j(2\pi/3)n} + e^{j(3\pi/4)n}$$

DT Signals Example:

We first find the fundamental period for each of the two components.

- Assume the period of the first term is N_1, then it should satisfy

$$e^{j(2\pi/3)(n+N_1)} = e^{j(2\pi/3)n} \cdot 1 = e^{j(2\pi/3)n} \cdot e^{jk2\pi} = e^{j(2\pi n/3+k2\pi)}$$

where k is an integer. Equating the exponents, we have

$$\frac{2\pi}{3}n + \frac{2\pi}{3}N_1 = \frac{2\pi}{3}n + k2\pi$$

which can be solved to get $N_1 = 3k$. We find the smallest integer $k = 1$ for $N_1 = 3$ to be an integer, the fundamental period.

Example (continue)

- Assume the period of the second term is N_2, then it should satisfy

$$e^{j(3\pi/4)(n+N_2)} = e^{j(3\pi/4)n} \cdot 1 = e^{j(3\pi/4)n} \cdot e^{jk2\pi} = e^{j(3\pi n/4+k2\pi)}$$

where $\underline{k}$ is an integer. Equating the exponents, we have

$$\frac{3\pi}{4}n + \frac{3\pi}{4}N_2 = \frac{3\pi}{4}n + k2\pi$$

which can be solved to get $N_2 = 8k/3$. We find the smallest integer $\underline{k = 3}$ for $N_2 = 8$ to be an integer, the fundamental period. Now the second term can be written as

$$e^{j\frac{3\pi}{4}n} = e^{j2\frac{3\pi}{8}n}$$

Continue…

Given the fundamental periods $N_1 = 3$ and $N_2 = 8$ of the two terms, the fundamental period N_0 of their sum is easily found to be their least common multiple

$$N_0 = lcm(3, 8) = 24$$

and the fundamental frequency is

$$\omega_0 = \frac{2\pi}{24} = \frac{\pi}{12}$$

Now the original signal can be written as

$$x[n] = e^{8\frac{2\pi}{24}n} + e^{9\frac{2\pi}{24}n}$$

i.e., the two terms are the 8th and 9th harmonic of the fundamental frequency $\omega_0 = \pi/12$.

Similar to the continuous case, to find the fundamental frequecy of a signal containing multiple terms all expressed as a fraction multiplied by $\frac{\pi}{}$, we can rewrite these fractions in terms of the least common multiple of all the denominators.

1 more example

$$x[n] = sin(\frac{5\pi}{6}n) + cos(\frac{3\pi}{4}n) + sin(\frac{\pi}{3}n)$$

answer

The least common multiple of the denominators is 12, therefore

$$x[n] = sin(\frac{10\pi}{12}n) + cos(\frac{9\pi}{12}n) + sin(\frac{4\pi}{12}n)$$

i.e., the fundamental frequency is $\omega_0 = \pi/12$, the fundamental period is $T = 2\pi/\omega_0 = 24$ and the three terms are the 4th, 9th and 10th harmonic of ω_0, respectively.

Product of periodic Signals

$X(t) = x_a(t) * x_b(t) =$

$= 2\sin[t(7\pi/24)] * \cos[t(\pi/24)]$;

find the period of x(t)

- We know: $= 2\sin[t(7\pi/12)/2] * \cos[t(\pi/12)/2]$;
  - Using Trig. Itentity:
- $x(t) = \sin(t\pi/3)+\sin(t\pi/4)$
  - Expressed as $\sin(t2\pi/6)+\sin(t2\pi/8)$
  - f =(1/6, 1/8), so T = (6, 8)
  - LCM (6, 8) = 24
  - So, the combined period is 24!
  - And frequency = 1/24

- **Sum-to-Product Formulas**

$$\boxed{\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)}$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$