LECTURE BLOCK 2A: LINEAR TIME INVARIANT (LTI) SYSTEMS

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Aims of the Lecture

Learn to:

- Describe the properties of LTI systems.
- Determine the impulse and step responses of LTI systems.
- Perform convolution of two functions.
- Determine causality and stability of LTI systems.

What is a system?

- A system accepts a signal x(t) and produces an output signal y(t).
- t is a function of time; x(t) means signal x at time t.
- LTI system is very common, eg many circuits and mechanical systems.
- The system is LTI if it is linear and time invariant.
- A system is linear if it has the scaling + additivity properties.

About LTI systems

- By using an impulse as x(t), and then observing y(t), the response (input-output behavior) of a LTI system can be completely characterized. (this is known as the impulse response of the LTI system)
- Once this is known, the system's response to any other input can be computed using convolution.
- An LTI can change the amplitude and phase (but not frequency) of an input sinusoid or complex exponential signal. We use this to design a filter.
- From a desired response, we can work backward to design the system we want.

What we will learn

- Scaling, additivity and time-invariance properties of LTI systems
- Discuss impulse responses – physically and mathematically – how to compute it.
- Convolution – derivation, properties and computation.
- Causality and stability of LTI.
- Response to sinusoid or complex exponential signal.

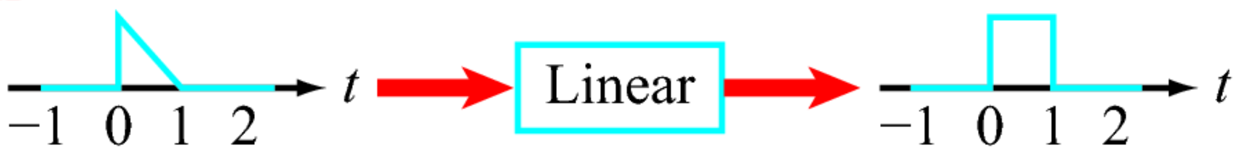
Scaling Property (homogeneity or scalability property)

Given: $x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$,

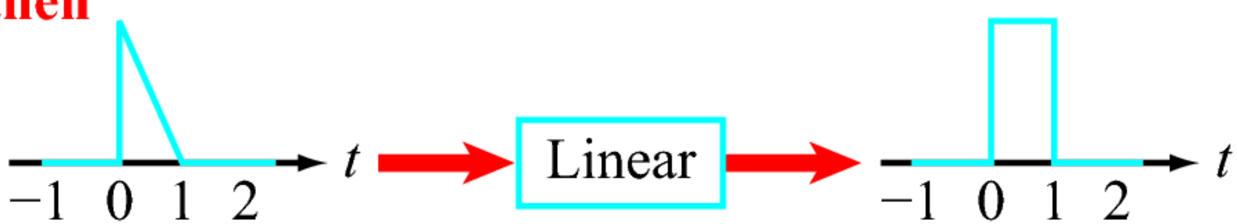
then the system is ***scalable*** (has the scaling property) if

$$c\,x(t) \longrightarrow \boxed{\text{System}} \longrightarrow c\;y(t).$$

If



then



Scaling Property (cont.)

- Example 1: System described by a 2nd Diff. Eq.

$$\frac{d^2y}{dt^2} + 2\,\frac{dy}{dt} + 3y = 4\,\frac{dx}{dt} + 5x. \qquad (2.1)$$

Upon replacing $x(t)$ with $c\,x(t)$ and $y(t)$ with $c\,y(t)$ in all terms, we end up with

$$\frac{d^2}{dt^2}\,(cy) + 2\,\frac{d}{dt}\,(cy) + 3(cy) = 4\,\frac{d}{dt}\,(cx) + 5(cx).$$

Since c is constant, we can rewrite the expression as

$$c\left[\frac{d^2y}{dt^2} + 2\,\frac{dy}{dt} + 3y\right] = c\left[4\,\frac{dx}{dt} + 5x\right], \qquad (2.2)$$

which is identical to the original equation, but multiplied by the constant c. Hence, since the response to $c\,x(t)$ is $c\,y(t)$, the system is scalable and has the scaling property.

Scaling Property (cont.)

- Example 2: System described by Diff. Eq.

$$\frac{d^2 y}{dt^2} + 2\,\frac{dy}{dt} + 3y = 4\,\frac{dx}{dt} + 5x + 6$$

Because of the constant term on the far right-hand side of the equation, this system is NOT scalable, and therefore NOT linear.

▶ If doubling the input does not double the output, the system is not linear. But if doubling the input doubles the output, the system is *probably* (but not necessarily) linear. ◀

Additivity Property

If the system responses to N inputs $x_1(t), x_2(t), \ldots, x_N(t)$ are respectively $y_1(t), y_2(t), \ldots, y_N(t)$, then the system is ***additive*** if

$$\sum_{i=1}^{N} x_i(t) \longrightarrow \boxed{\text{System}} \longrightarrow \sum_{i=1}^{N} y_i(t). \tag{2.4}$$

That is, *the response to the sum is the sum of the responses*.

► The combination of scalability and additivity is also known as the ***superposition principle***. ◄

Additivity property (cont.)

If we denote $y_1(t)$ as the response to $x_1(t)$ in Eq. (2.1), and similarly $y_2(t)$ as the response to $x_2(t)$, then

$$\frac{d^2y_1}{dt^2} + 2\,\frac{dy_1}{dt} + 3y_1 = 4\,\frac{dx_1}{dt} + 5x_1, \qquad (2.5a)$$

$$\frac{d^2y_2}{dt^2} + 2\,\frac{dy_2}{dt} + 3y_2 = 4\,\frac{dx_2}{dt} + 5x_2. \qquad (2.5b)$$

Next, if we add the two equations and denote $x_3 = x_1 + x_2$ and $y_3 = y_1 + y_2$, we get

$$\frac{d^2y_3}{dt^2} + 2\,\frac{dy_3}{dt} + 3y_3 = 4\,\frac{dx_3}{dt} + 5x_3. \qquad (2.5c)$$

Hence, the system characterized by differential equation (2.1) has the additivity property. Since it was shown earlier also to have the scaling property, the system is indeed linear.

Significance of linearity property

To appreciate the significance of the linearity property, consider the following scenario. We are given a linear system characterized by a third-order differential equation. The system is excited by a complicated input signal $x(t)$. Our goal is to obtain an analytical solution for the system's output response $y(t)$.

Fundamentally, we can pursue either of the following two approaches:

Option 1: Direct "brute-force" approach: This involves solving the third-order differential equation with the complicated signal $x(t)$. While feasible, this may be mathematically demanding and cumbersome.

Significance of linearity property

Option 2: Indirect, linear-system approach, comprised of the following steps:

(a) Synthesize $x(t)$ as the ***linear combination*** of N relatively simple signals $x_1(t)$, $x_2(t)$, ..., $x_N(t)$:

$$x(t) = c_1\, x_1(t) + c_2\, x_2(t) + \cdots + c_N\, x_N(t),$$

with signals $x_1(t)$ to $x_N(t)$ chosen so that they yield straightforward solutions of the third-order differential equation.

(b) Solve the differential equation N times, once for each of the N input signals acting alone, yielding outputs $y_1(t)$ through $y_N(t)$.

(c) Apply the linearity property to obtain $y(t)$ by summing the outputs $y_i(t)$:

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t) + \cdots + c_N\, y_N(t).$$

Significance of linearity property

- Even though the 2nd option involves solving differential equations N times, the overall system may prove significantly more tractable than the 1st option.
- This leads to the principle of Superposition.
- Eg, used in circuit analysis…

Significance of linearity property

(a) We set all but one source to zero and then analyze the (now linear) circuit using that source as the input. A voltage source set to zero becomes a short circuit, and a current source set to zero becomes an open circuit. We designate $y_1(t)$ as the circuit response to source $x_1(t)$ acting alone.

(b) We then rotate the non-zero choice of source among all of the remaining independent sources, setting all other sources to zero each time. The process generates responses $y_2(t)$ to $y_N(t)$, corresponding to sources $x_2(t)$ to $x_N(t)$, respectively.

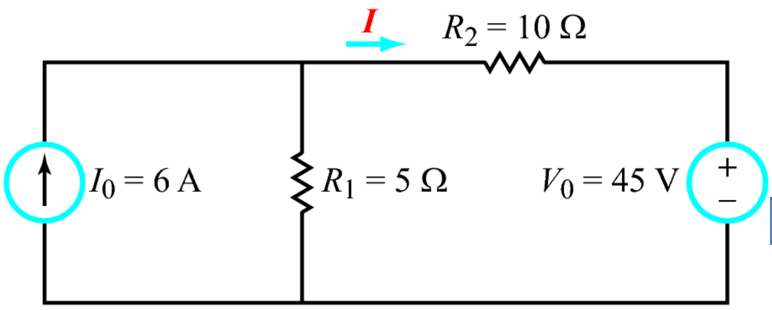
(c) We add responses $y_1(t)$ through $y_N(t)$.

Superposition Example

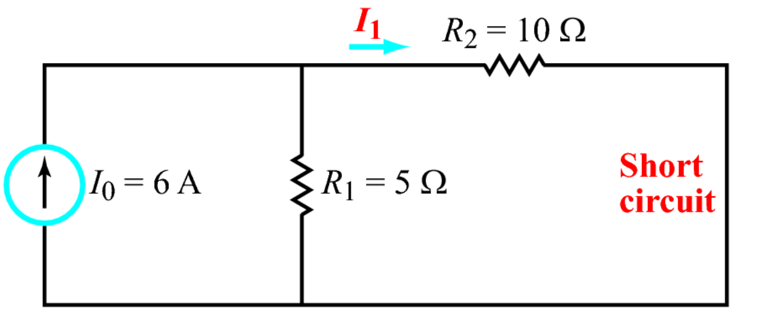
$$I_1 = \frac{I_0 R_1}{R_1 + R_2} = \frac{6 \times 5}{5 + 10} = 2\text{ A}.$$

$$I_2 = \frac{-V_0}{R_1 + R_2} = \frac{-45}{5 + 10} = -3\text{ A}.$$

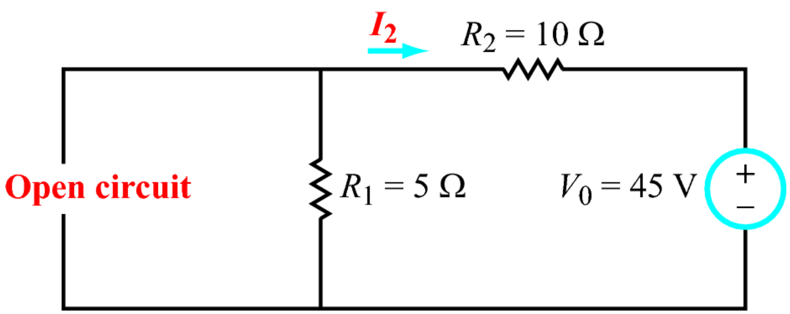
$$I = I_1 + I_2 = 2 - 3 = -1\text{ A}.$$



(a) **Original circuit**



(b) **Source I_0 alone generates I_1**



(c) **Source V_0 alone generates I_2**

Time-Invariant Systems

▶ A system is ***time-invariant*** if delaying the input signal $x(t)$ by any constant T generates the same output $y(t)$, but delayed by exactly T. ◀

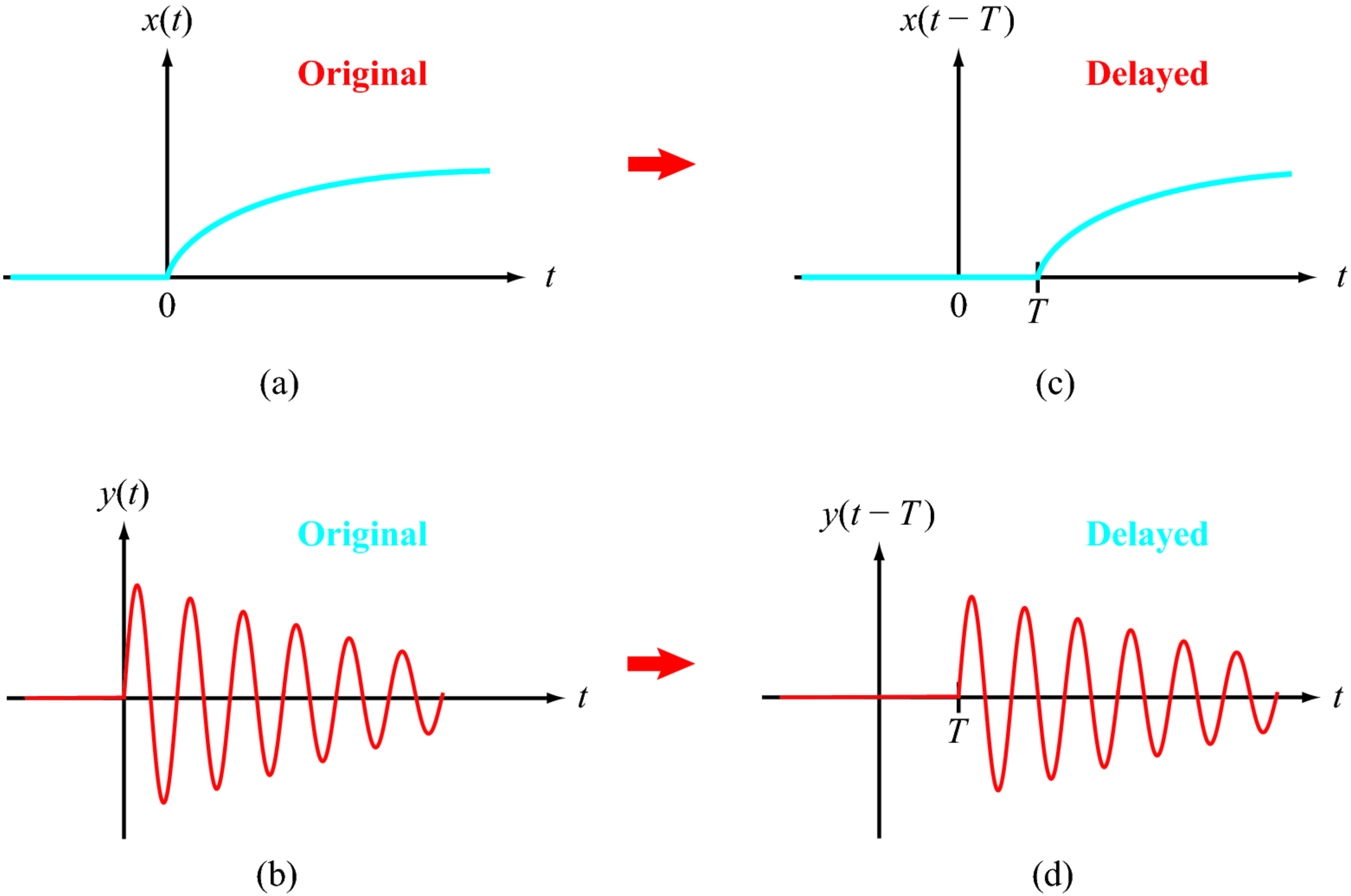
$$\text{Given: } x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t),$$

then the system is ***time-invariant*** if

$$x(t-T) \longrightarrow \boxed{\text{System}} \longrightarrow y(t-T).$$

If t appears explicitly only in the expression for $x(t)$ or in derivatives or limits of integration in the equation describing the system, the system is likely *time-invariant.*

Examples of Time-Invariant Systems



More Examples

- Time-Invariant

(a) $y_1(t) = 3\,\dfrac{d^2x}{dt^2}$,

(b) $y_2(t) = \sin[x(t)]$

(c) $y_3(t) = \dfrac{x(t+2)}{x(t-1)}$

Note1: Systems b and c are time-invariant, but not linear.

Not Time-Invariant

(d) $y_4(t) = t\;x(t)$,

(e) $y_5(t) = x(t^2)$,

(f) $y_6(t) = x(-t)$.

Note2: Systems d to f are linear, but not time-invariant.

Note3: Systems e and f are not time-invariant although they pass the rules of thumb earlier.

Impulse & Step Responses

The ***impulse response*** $h(t)$ of a system is (logically enough) the response of the system to an impulse $\delta(t)$. Similarly, the ***step response*** $y_{\text{step}}(t)$ is the response of the system to a unit step $u(t)$. In symbolic form:

$$\delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t) \tag{2.7a}$$

and

$$u(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y_{\text{step}}(t). \tag{2.7b}$$

- Once we know *h(t)*, can compute the response of any other input x(t) using convolution.
- We also want to know how the system 'track' a input step response.

Static and Dynamic Systems

- Static System (Memoryless)

► A system for which the output $y(t)$ at time t depends only on the input $x(t)$ at time t is called a ***static*** or ***memoryless*** system. ◄

An example of such a system is

$$y(t) = \frac{\sin[x(t)]}{x^2(t) + 1}.$$

- Dynamic System

  Output y(t) depends on past (or future) as well as present values of input x(t).

Computing $h(t)$ and $y_{\text{step}}(t)$ of RC Circuit

- Capacitor is initially uncharged, so $y(0^-) = 0$.
- Using Kirchoff's voltage law, we have

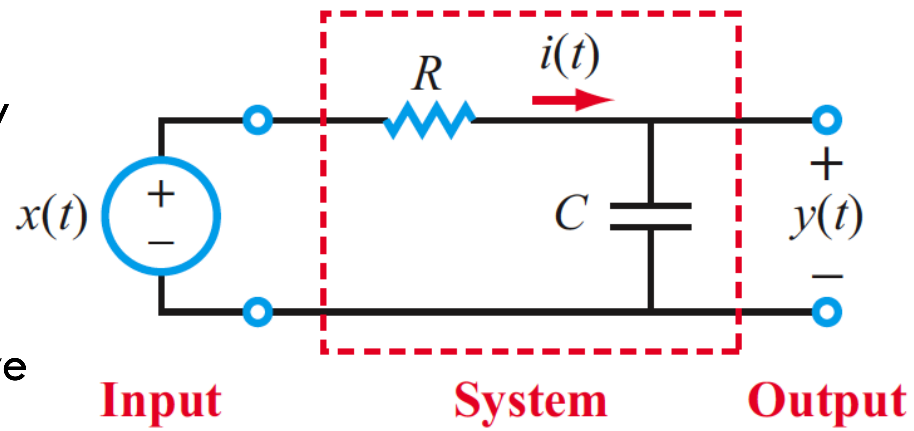
$$R\,i(t) + y(t) = x(t)$$

- For C, current is related to voltage by

$$i(t) = C\,\frac{dy}{dt}$$

- Combining the two equations, we have

$$\frac{dy}{dt} + \frac{1}{RC}\,y(t) = \frac{1}{RC}\,x(t)$$



(a) RC circuit

Impulse Response

$$\frac{dy}{dt} + \frac{1}{RC}\, y(t) = \frac{1}{RC}\, x(t)$$

Eq (1.27) $x(t)\,\delta(t) = x(0)\,\delta(t),$

To compute the ***impulse response***, we label $x(t) = \delta(t)$ and $y(t) = h(t)$ and obtain

$$\frac{dh}{dt} + \frac{1}{RC}\, h(t) = \frac{1}{RC}\, \delta(t). \qquad (2.11)$$

Next, we introduce the ***time constant*** $\tau_c = RC$ and multiply both sides of the differential equation by the ***integrating factor*** e^{t/τ_c}. The result is

$$\frac{dh}{dt}\, e^{t/\tau_c} + \frac{1}{\tau_c}\, e^{t/\tau_c}\, h(t) = \frac{1}{\tau_c}\, e^{t/\tau_c}\, \delta(t). \qquad (2.12)$$

The left side of Eq. (2.12) is recognized as

$$\frac{dh}{dt}\, e^{t/\tau_c} + \frac{1}{\tau_c}\, e^{t/\tau_c}\, h(t) = \frac{d}{dt}\, [h(t)\, e^{t/\tau_c}], \qquad (2.13a)$$

and the sampling property of the impulse function given by Eq. (1.27) reduces the right-hand side of Eq. (2.12) to

$$\frac{1}{\tau_c}\, e^{t/\tau_c}\, \delta(t) = \frac{1}{\tau_c}\, \delta(t). \qquad (2.13b)$$

Incorporating these two modifications in Eq. (2.12) leads to

$$\frac{d}{dt}\, [h(t)\, e^{t/\tau_c}] = \frac{1}{\tau_c}\, \delta(t). \qquad (2.14)$$

Integrating both sides from 0^- to t gives

$$\int_{0^-}^{t} \frac{d}{d\tau}\, [h(\tau)\, e^{\tau/\tau_c}]\, d\tau = \frac{1}{\tau_c} \int_{0^-}^{t} \delta(\tau)\, d\tau, \qquad (2.15)$$

Impulse Response of RC

where τ is a dummy variable of integration and has no connection to the time constant τ_c. We will use τ as a dummy variable throughout this chapter. By Eq. (1.24), we recognize the integral in the right-hand side of Eq. (2.15) as the step function $u(t)$. Hence,

$$h(\tau)\, e^{\tau/\tau_c}\Big|_{0^-}^{t} = \frac{1}{\tau_c}\, u(t),$$

or

$$h(t)\, e^{t/\tau_c} - h(0^-) = \frac{1}{\tau_c}\, u(t). \qquad (2.16)$$

The function $h(t)$ represents $y(t)$ for the specific case when the input is $\delta(t)$. We are told that the capacitor was initially uncharged, so $y(0^-) = 0$ for any input. Hence, $h(0^-) = 0$, and Eq. (2.16) reduces to

$$h(t) = \frac{1}{\tau_c}\, e^{-t/\tau_c}\, u(t). \qquad (2.17)$$

(impulse response of the RC circuit)

Step Response of RC

To compute the ***step response*** of the circuit, we start with the same governing equation given by Eq. (2.10), but this time we label $x(t) = u(t)$ and $y(t) = y_{\text{step}}(t)$. The procedure leads to

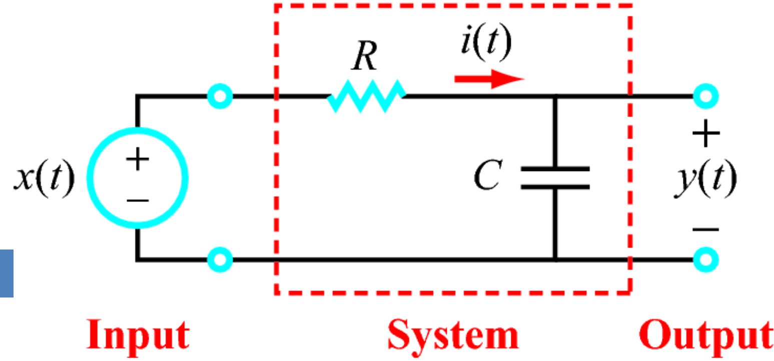
$$y_{\text{step}}(t) = [1 - e^{-t/\tau_c}]\, u(t). \tag{2.18}$$

(step response of the RC circuit)

As the initially uncharged capacitor builds up charge, the voltage across it builds up monotonically from zero at $t = 0$ to 1 V as $t \to \infty$ (**Fig. 2-5(c)**).

► Once we know the step response of an LTI system, we can apply the superposition principle to compute the response of the system to any input that can be expressed as a linear combination of scaled and delayed step functions. ◄

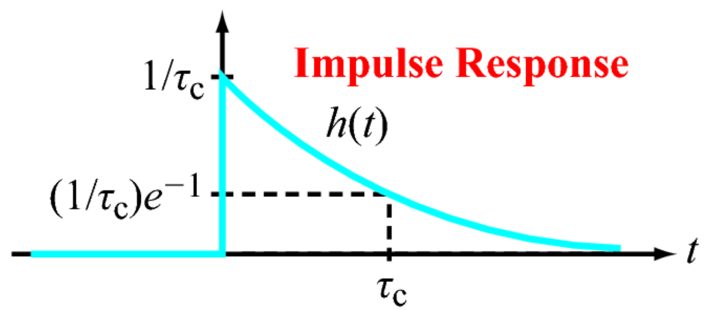
Impulse & Step Responses



(a) RC circuit

$$h(t) = \frac{1}{\tau_c} e^{-t/\tau_c} \, u(t).$$

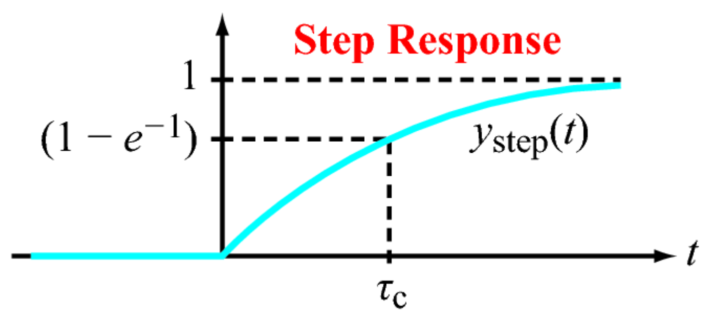
(impulse response of the RC circuit)



(b) $h(t)$

$$y_{step}(t) = [1 - e^{-t/\tau_c}] \, u(t).$$

(step response of the RC circuit)



(c) $y_{step}(t)$

Measuring Impulse Response for Circuit with RC=2s

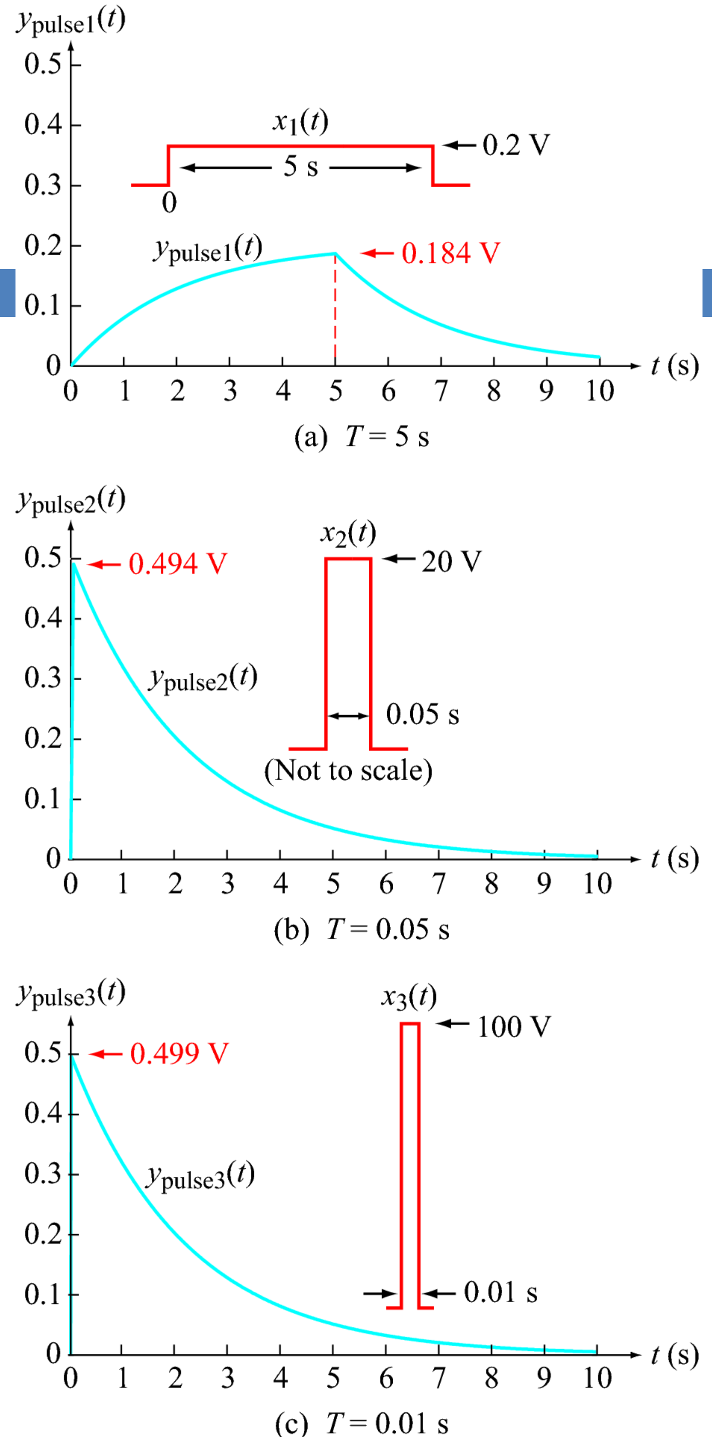
Peak value is 1/RC = ½ = 0.5V.

(a) Input pulse is 5s, longer than RC, so circuit charges and discharges.

(b) Pulse is 0.05s, shorter than RC, and Vpeak of 20V.

(c) Pulse is 0.01s, with Vpeak =100V

* We need unit area impulse at input with duration much shorter than RC.



Impulse Response From Step Response

Step 1: Physically *measure* the step response $y_{\text{step}}(t)$.

Step 2: Differentiate it to obtain

$$h(t) = \frac{dy_{\text{step}}}{dt} .$$

$$\frac{du}{dt} = \delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t) = \frac{dy_{\text{step}}}{dt}$$

Example:

$$\begin{aligned} h(t) &= \frac{d}{dt}\left[\left(1 - e^{-t/\tau_c}\right) u(t)\right] \\ &= \delta(t) - \delta(t)\, e^{-t/\tau_c} + \frac{1}{\tau_c}\, e^{-t/\tau_c}\, u(t) \\ &= \frac{1}{\tau_c}\, e^{-t/\tau_c}\, u(t), \end{aligned}$$

Ramp Response From Step Response

$$r(t) = \int_{-\infty}^{t} u(\tau)\, d\tau$$

$$r(t) = \int_{-\infty}^{t} u(\tau)\, d\tau \longrightarrow \boxed{\text{LTI}} \longrightarrow y_{\text{ramp}}(t)$$

RC Circuit Example

$$y_{\text{step}}(t) = [1 - e^{-t/\tau_c}]\, u(t).$$

(step response of the RC circuit)

$$\begin{aligned}
y_{\text{ramp}}(t) &= \int_{-\infty}^{t} y_{\text{step}}(\tau)\, d\tau \\
&= \int_{-\infty}^{t} (1 - e^{-\tau/\tau_c})\, u(\tau)\, d\tau \\
&= \int_{0}^{t} (1 - e^{-\tau/\tau_c})\, d\tau \\
&= [t - \tau_c(1 - e^{-t/\tau_c})]\, u(t).
\end{aligned}$$