LECTURE BLOCK 2B: LINEAR TIME INVARIANT (LTI) SYSTEMS

Dr. Yau Hee Kho

yauhee.kho@vuw.ac.nz

Previously, we have learnt…

- Scaling, additivity and time-invariance properties of LTI systems
- Discuss impulse responses – physically and mathematically – how to compute it.

Today, we will look at:

- Convolution – derivation, properties and computation.
- Causality and stability of LTI.
- Response to sinusoid or complex exponential signal.

Convolution

► The response $y(t)$ of an LTI system with impulse response $h(t)$ to *any* input $x(t)$ can be computed *explicitly* using the ***convolution integral***

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = h(t) * x(t).$$

(convolution integral) (2.30)

All initial conditions must be zero. ◄

Convolution

(a) τ is a dummy variable of integration.

(b) $h(t-\tau)$ is obtained from the impulse response $h(t)$ by (1) replacing t with the variable of integration τ, (2) reversing $h(\tau)$ along the τ axis to obtain $h(-\tau)$, and (3) delaying $h(-\tau)$ by time t to obtain $h(t-\tau) = h(-(\tau - t))$.

(c) As will be demonstrated later in this section, convolution is commutative; we can interchange x and h in Eq. (2.30).

According to the convolution property represented by Eq. (2.30), once $h(t)$ of an LTI system has been determined, the system's response can be readily evaluated for any specified input excitation $x(t)$ by performing the convolution integration. The integration may be carried out analytically, graphically, or numerically—all depending on the formats of $x(t)$ and $h(t)$. Multiple examples are provided in the latter part of this section.

Derivation of Convolution (5 steps)

LTI System with Zero Initial Conditions

a)	$\delta(t)$	$\rightarrow$	LTI	$\rightarrow$	$y(t) = h(t)$
b)	$\delta(t-\tau)$	$\rightarrow$	LTI	$\rightarrow$	$y(t) = h(t-\tau)$
c)	$x(\tau)\,\delta(t-\tau)$	$\rightarrow$	LTI	$\rightarrow$	$y(t) = x(\tau)\,h(t-\tau)$
d)	$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$	$\rightarrow$	LTI	$\rightarrow$	$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$
e)	$x(t)$	$\rightarrow$	LTI	$\rightarrow$	$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$

5 steps of Convolution

Make use of definition of impulse response and the properties of LTI systems:

a) Here we use the definition of impulse response

$$\delta(t) \rightarrow \boxed{\text{LTI}} \rightarrow y(t) = h(t)$$

b) Make use of time invariance property of LTI system: delay of input and output

$$\delta(t-\tau) \rightarrow \boxed{\text{LTI}} \rightarrow y(t) = h(t-\tau)$$

5 Steps of Convolution

c) Make use of scaling property of LTI systems: input and output scaled by an amplitude

$$x(\tau)\,\delta(t-\tau) \rightarrow \boxed{\text{LTI}} \rightarrow y(t) = x(\tau)\,h(t-\tau)$$

d) An impulsive excitation at τ_1 for input will generate corresponding output.

$$x(\tau_1)\,\delta(t-\tau_1) \rightarrow \boxed{\text{LTI}} \rightarrow x(\tau_1)\,h(t-\tau_1).$$

Similarly, if τ_2 will give

$$x(\tau_2)\,\delta(t-\tau_2) \rightarrow \boxed{\text{LTI}} \rightarrow x(\tau_2)\,h(t-\tau_2).$$

5 Steps of Convolution

If the LTI system is excited by both τ_1 and τ_2 at the same time, the additivity property of LTI assures that the output will be equal the sum of both outputs.

For a continuous-time input extending over $-\infty < \tau < \infty$, the input and output become definite integrals:

$$\int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

$$\downarrow$$

$$\boxed{\text{LTI}}$$

$$\downarrow$$

$$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

5 Steps of Convolution

e) From the sampling property of impulses, we have

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t).$$

Therefore, $x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = \displaystyle\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau.$

As a shorthand notation, the convolution integral is represented by an asterisk *, so output $y(t)$ is expressed as

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau.$$

- Convolution is Commutative:

$$x(t) * h(t) = h(t) * x(t)$$

- For Causal Signals and Systems

$$y(t) = u(t) \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

$$= u(t) \int_{0}^{t} x(t-\tau)\, h(\tau)\, d\tau.$$

(causal signals and systems)

Computing Convolutional Integrals

(a) analytically, by performing the integration to obtain an expression for y(t), and then evaluating y(t) as a function of time,

(b) Graphically, by simulating the integration using plots of the waveforms of x(t) and h(t),

(c) Analytically with simplification, by taking advantage of the expedient conclusion properties discussed later, to simplify the integration algebra,

(d) Numerically on a digital computer.

Method (a) is straightforward but algebra is tedious.

Method (b) allows visualisation but is slow.

Example: Response of Circuit with RC=1s to Triangular Pulse

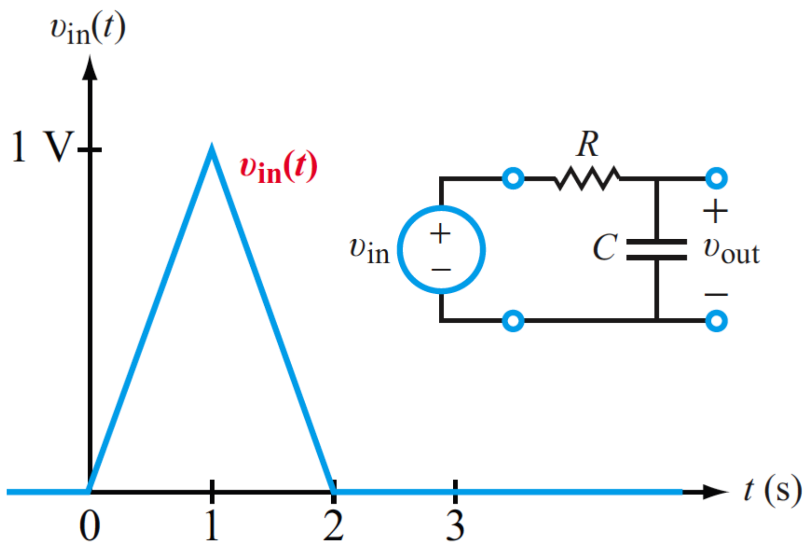
Solution:

The input signal, measured in volts, is given by

$$\upsilon_{\text{in}}(t) = \begin{cases} 0 & \text{for } t \le 0, \\ t & \text{for } 0 \le t \le 1 \text{ s}, \\ 2-t & \text{for } 1 \le t \le 2 \text{ s}, \\ 0 & \text{for } t \ge 2 \text{ s}, \end{cases}$$

and according to Eq. (2.17), the impulse response for $\tau_c = 1$ is

$$h(t) = \frac{1}{\tau_c}\, e^{-t/\tau_c}\, u(t) = e^{-t}\, u(t).$$



(a) Triangular pulse

$$\upsilon_{\text{out}}(t) = \upsilon_{\text{in}}(t) * h(t) = \int_0^t \upsilon_{\text{in}}(\tau)\, h(t-\tau)\, d\tau$$

with

$$h(t-\tau) = e^{-(t-\tau)}\, u(t-\tau) = \begin{cases} 0 & \text{for } t < \tau, \\ e^{-(t-\tau)} & \text{for } t > \tau. \end{cases}$$

(1) $t < 0$:

The lowest value that the integration variable τ can assume is zero. Therefore, when $t < 0$, $t < \tau$ and $h(t-\tau) = 0$. Consequently,

$$v_{out}(t) = 0 \qquad \text{for } t < 0.$$

(2) $0 \leq t \leq 1$ s:

$$h(t-\tau) = e^{-(t-\tau)}, \qquad v_{in}(\tau) = \tau,$$

and

$$v_{out}(t) = \int_0^t \tau e^{-(t-\tau)}\, d\tau$$
$$= e^{-t} + t - 1, \qquad \text{for } 0 \leq t \leq 1 \text{ s}.$$

(3) $1 \text{ s} \leq t \leq 2$ s:

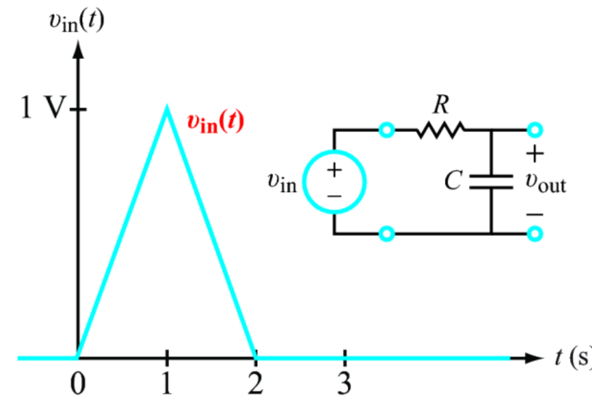
$$v_{in}(\tau) = \begin{cases} \tau & \text{for } 0 \leq \tau \leq 1 \text{ s}, \\ 2-\tau & \text{for } 1 \text{ s} \leq \tau \leq 2 \text{ s}, \end{cases}$$

and

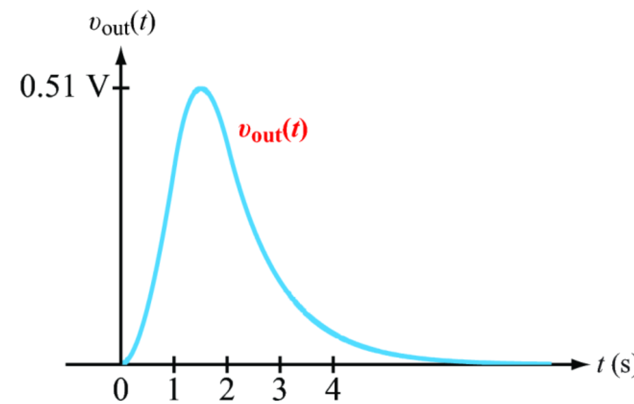
$$v_{out}(t) = \int_0^1 \tau e^{-(t-\tau)}\, d\tau + \int_1^t (2-\tau) e^{-(t-\tau)}\, d\tau$$
$$= (1-2e)e^{-t} - t + 3, \qquad \text{for } 1 \text{ s} \leq t \leq 2 \text{ s}.$$

(4) $t \geq 2$ s:

$$v_{out}(t) = \int_0^1 \tau e^{-(t-\tau)}\, d\tau + \int_1^2 (2-\tau) e^{-(t-\tau)}\, d\tau$$
$$= (1-2e+e^2)e^{-t} \qquad \text{for } t \geq 2 \text{ s}.$$



(a) Triangular pulse



(b) Output response

Graphical Convolution Technique

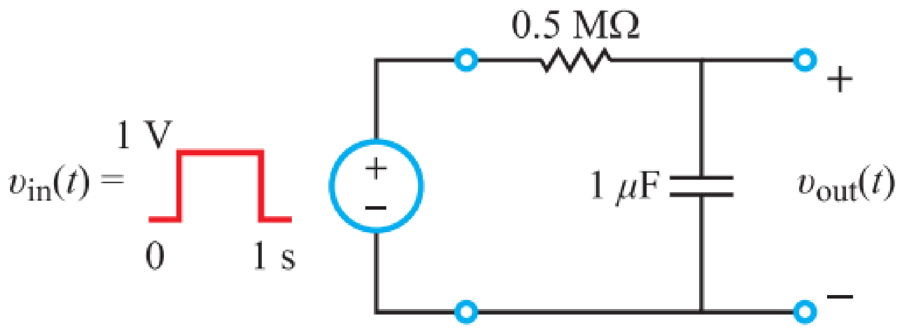
Step 1: On the τ-axis, display $x(\tau)$ and $h(-\tau)$ with the latter being a folded image of $h(\tau)$ about the vertical axis.

Step 2: Shift $h(-\tau)$ to the right by a small increment t to obtain $h(t - \tau) = h(-(\tau - t))$.

Step 3: Determine the product of $x(\tau)$ and $h(t - \tau)$ and integrate it over the τ-domain from $\tau = 0$ to $\tau = t$ to get $y(t)$. The integration is equal to the area overlapped by the two functions.

Step 4: Repeat steps 2 and 3 for each of many successive values of t to generate the complete response $y(t)$.

Example: RC circuit with rectangular pulse

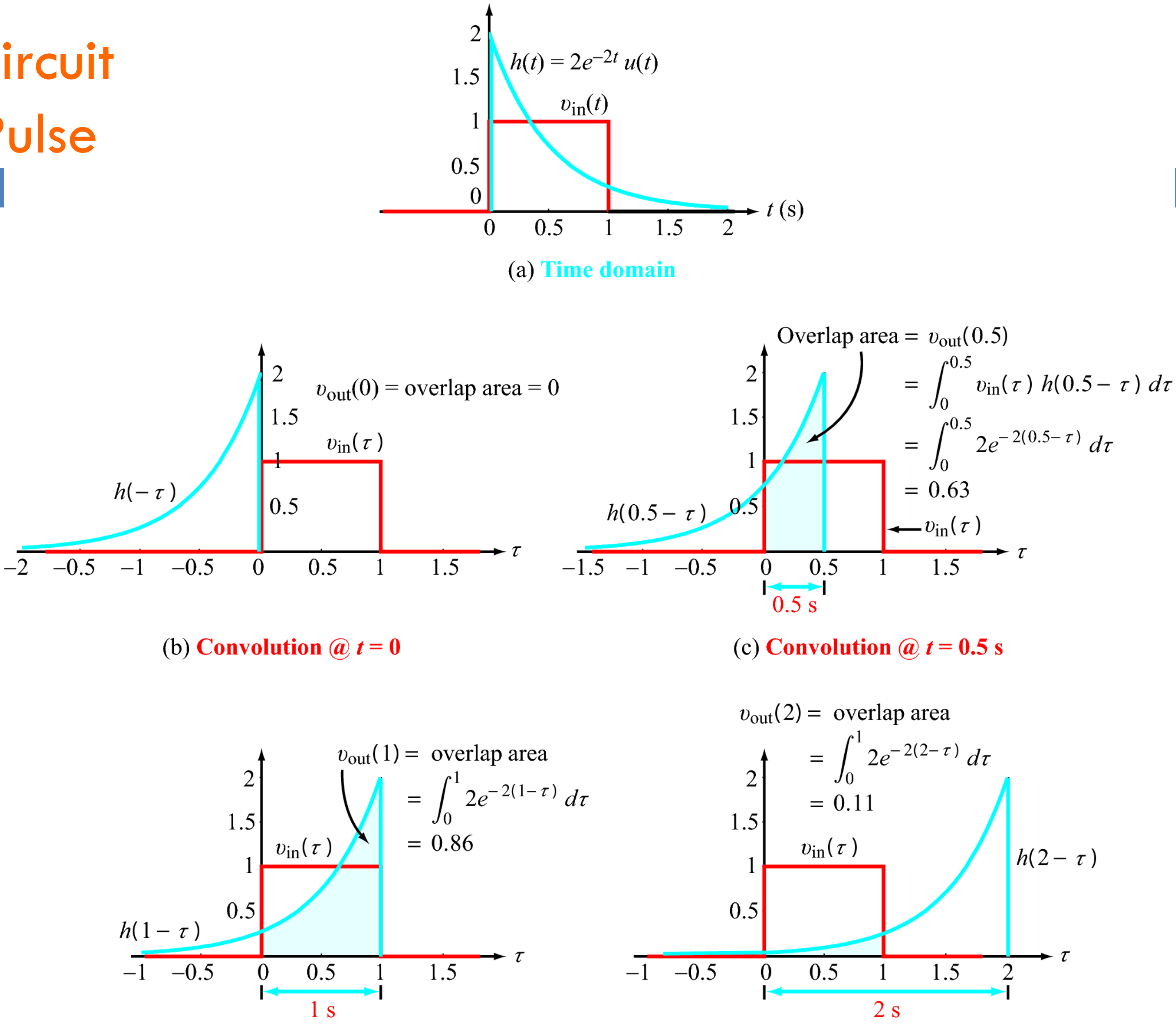


(a) RC lowpass filter

Solution: The time constant of the RC circuit is $\tau_c = RC = (0.5 \times 10^6) \times 10^{-6} = 0.5$ s. In view of Eq. (2.17), the impulse response of the circuit is

$$h(t) = \frac{1}{\tau_c} e^{-t/\tau_c} u(t) = 2e^{-2t} u(t). \qquad (2.53)$$

Example: RC Circuit Excited by a Pulse



(a) Time domain

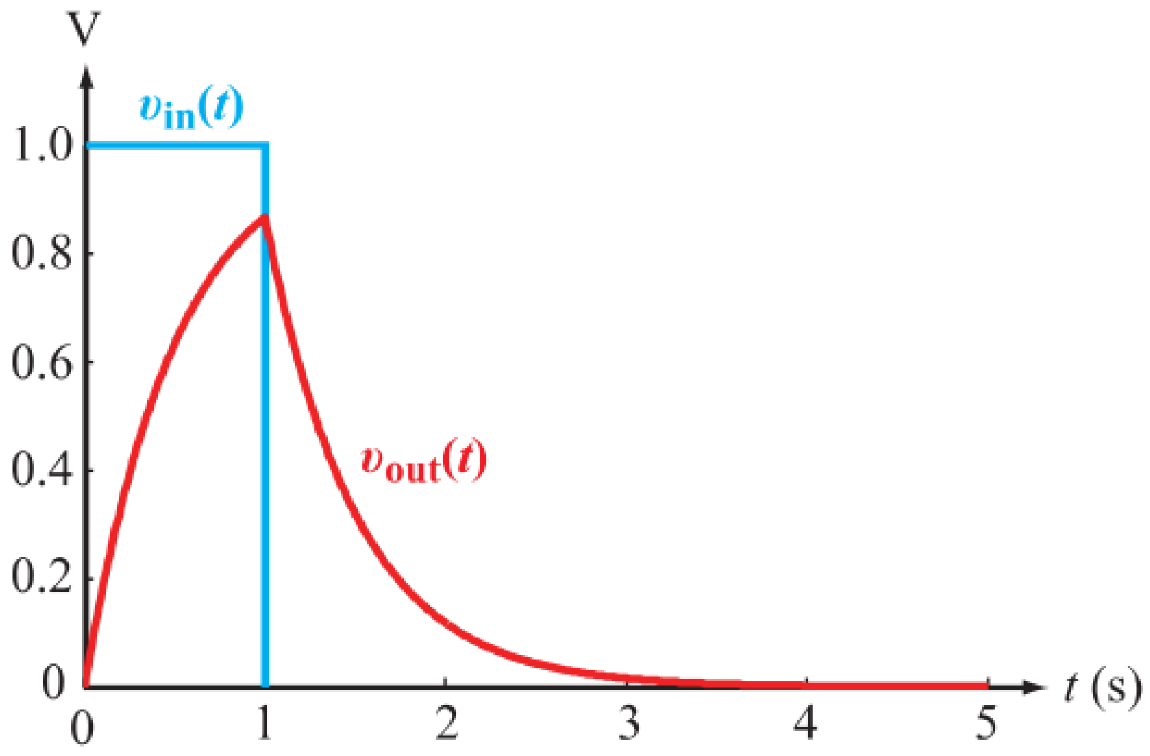
(b) Convolution @ $t = 0$

(c) Convolution @ $t = 0.5$ s

(d) Convolution @ $t = 1$ s

(e) Convolution @ $t = 2$ s

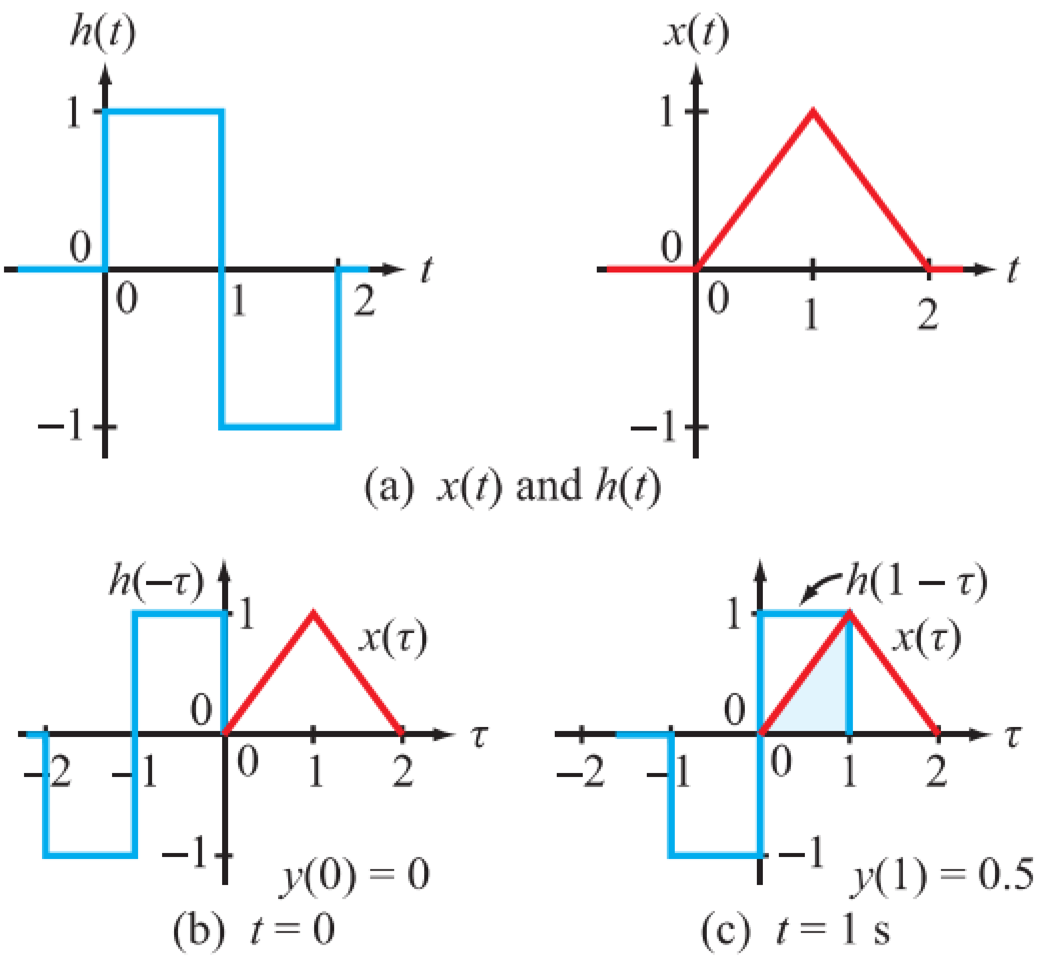
Example: RC Circuit Excited by a Pulse



(b) Output response

Example:

- Convolution of 1 cycle of square wave with Triangle

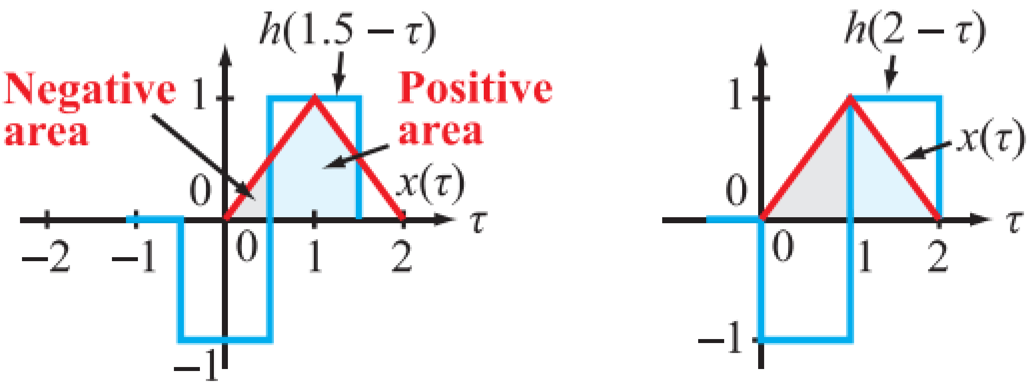


(a) $x(t)$ and $h(t)$

(b) $t = 0$

(c) $t = 1$ s

Example:



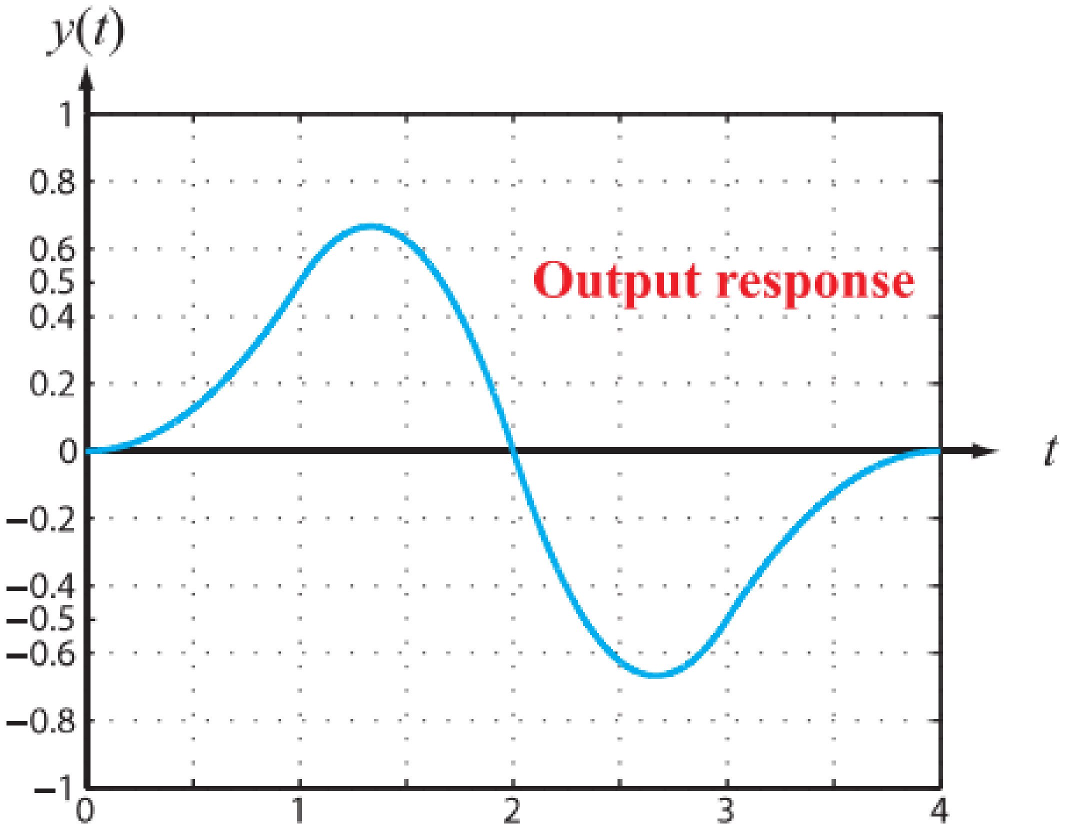
$y(1.5) = \frac{6}{8} - \frac{1}{8} = \frac{5}{8}$

(d) $t = 1.5$ s

$y(2) = 0$

(e) $t = 2$ s

Example:



(f) $y(t)$

Convolution Properties

- **Commutative**

$$y(t) = h(t) * x(t) = x(t) * h(t).$$

Its validity can be demonstrated by changing variables from
to $\tau' = t - \tau$ in the convolution integral:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{\infty}^{-\infty} h(t-\tau')\, x(\tau')\, (-d\tau')$$

$$= \int_{-\infty}^{\infty} x(\tau')\, h(t-\tau')\, d\tau' = x(t) * h(t).$$

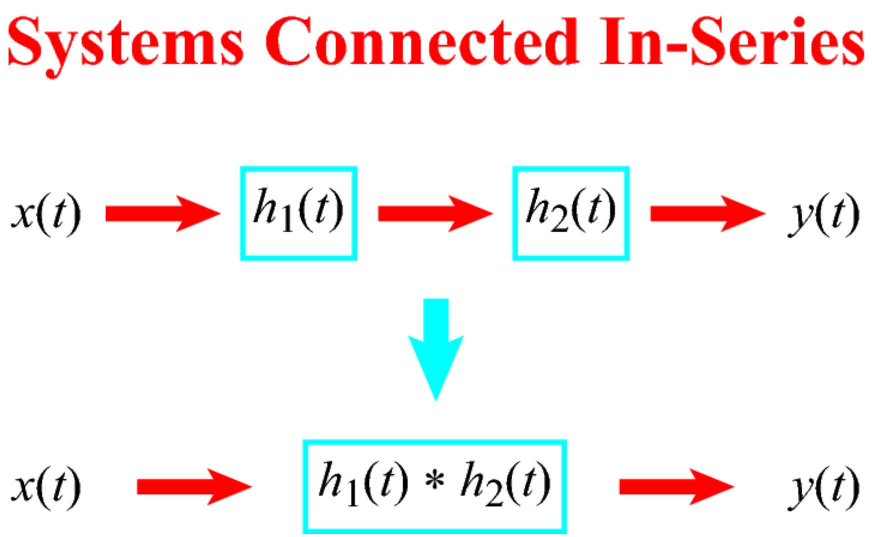
If $\quad x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t),$

then $\quad h(t) \longrightarrow \boxed{x(t)} \longrightarrow y(t).$

Convolution Properties

- Associative

$$y(t) = [g(t) * h(t)] * x(t) = g(t) * [h(t) * x(t)].$$

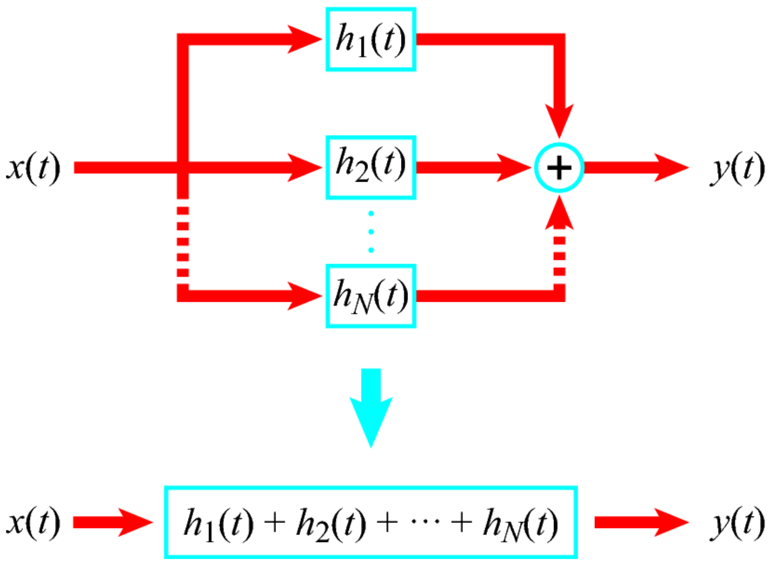


Convolution Properties

- Distributive

Systems Connected In-Parallel

$$h(t) * [x_1(t) + x_2(t) + \cdots + x_N(t)] =$$
$$h(t) * x_1(t) + h(t) * x_2(t) + \cdots + h(t) * x_N(t).$$



In combination, the associative and distributive properties state that for LTI systems:

(1) Systems in series: Impulse responses convolved.
(2) Systems in parallel: Impulse responses added.

Convolution Properties

- Time Shift property

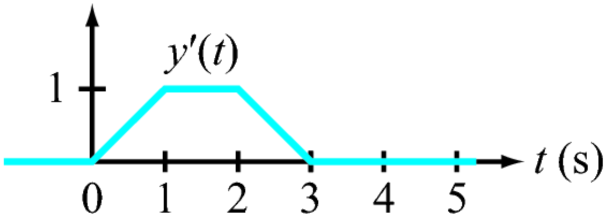
$$h(t - T_1) * x(t - T_2) = y(t - T_1 - T_2)$$

(time-shift property)

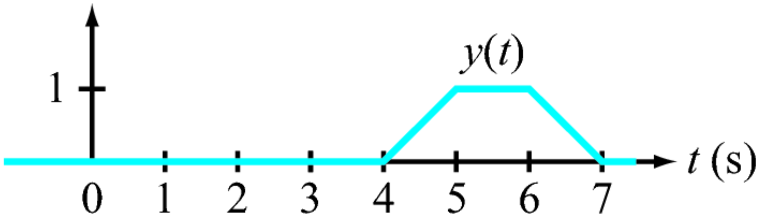
Example:

(a) **Original pulses**

(b) **Pulses shifted to start at $t = 0$**



(c) **Convolution of $x'_1(t)$ with $x'_2(t)$**



(d) **$y'(t)$ delayed by 4 s gives $y(t)$**

Convolution Properties

- Convolution with an Impulse

Convolution of a signal $x(t)$ with an impulse function $\delta(t-T)$ simply delays $x(t)$ by T:

$$x(t) * \delta(t-T) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-T-\tau)\, d\tau$$
$$= x(t-T). \tag{2.74}$$

The result follows from the sampling property of the impulse function given by Eq. (1.29), namely that $\delta(t-T-\tau)$ is zero except at $\tau = t-T$.

Convolution Properties

- Width Property

> ▶ When a function of time width T_1 is convolved with another of time width T_2, the width of the resultant function is $(T_1 + T_2)$, regardless of the shapes of the two functions. ◀

(see example of time-shift property)

Convolution Properties

- Area property

► The area under the convolution $y(t)$ of two functions $x(t)$ and $h(t)$ is the product of the areas under the individual functions. ◄

Convolution Properties

- Convolution with a Step Function

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\, d\tau,$$

(ideal integrator)

Convolution Properties

- Differentiation and Integration Properties

$$\left(\frac{d^m x}{dt^m}\right) * \left(\frac{d^n h}{dt^n}\right) = \frac{d^{m+n} y}{dt^{m+n}}.$$

$$\int_{-\infty}^{t} y(\tau)\, d\tau = x(t) * \left[\int_{-\infty}^{t} h(\tau)\, d\tau\right]$$

$$= \left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] * h(t).$$

Table 2-1: Convolution properties.

Convolution Integral	$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$
• **Causal Systems:**	Replace lower limit with 0
Property	**Description**
1. Commutative	$x(t) * h(t) = h(t) * x(t)$
2. Associative	$[g(t) * h(t)] * x(t) = g(t) * [h(t) * x(t)]$
3. Distributive	$x(t) * [h_1(t) + \cdots + h_N(t)] = x(t) * h_1(t) + \cdots + x(t) * h_N(t)$
4. Causal $*$ Causal $=$ Causal	$y(t) = u(t) \int_{0}^{t} h(\tau)\, x(t-\tau)\, d\tau$
5. Time-shift	$h(t-T_1) * x(t-T_2) = y(t-T_1-T_2)$
6. Convolution with Impulse	$x(t) * \delta(t-T) = x(t-T)$
7. Width	Width of $y(t)$ = width of $x(t)$ + width of $h(t)$
8. Area	Area of $y(t)$ = area of $x(t)$ $\times$ area of $h(t)$
9. Convolution with $u(t)$	$y(t) = x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ (Ideal integrator)
10a. Differentiation	$\left(\frac{d^m x}{dt^m}\right) * \left(\frac{d^n h}{dt^n}\right) = \frac{d^{m+n} y}{dt^{m+n}}$
10b. Integration	$\int_{-\infty}^{t} y(\tau)\, d\tau = x(t) * \left[\int_{-\infty}^{t} h(\tau)\, d\tau\right] = \left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] * h(t)$

2-6.1 Causality

We define a ***causal system*** as a system for which the present value of the output $y(t)$ can only depend on present and past values of the input $\{x(\tau), \quad \tau \leq t\}$. For a noncausal system, the present output could depend on future inputs. Noncausal systems are also called ***anticipatory*** systems, since they anticipate the future.

A physical system must be causal, because a noncausal system must have the ability to see into the future! For example, the noncausal system $y(t) = x(t+2)$ must know the input two seconds into the future to deliver its output at the present time. This is clearly impossible in the real world.

► An LTI system is causal ***if and only if*** its impulse response is a causal function: $h(t) = 0$ for $t < 0$. ◄

Stability

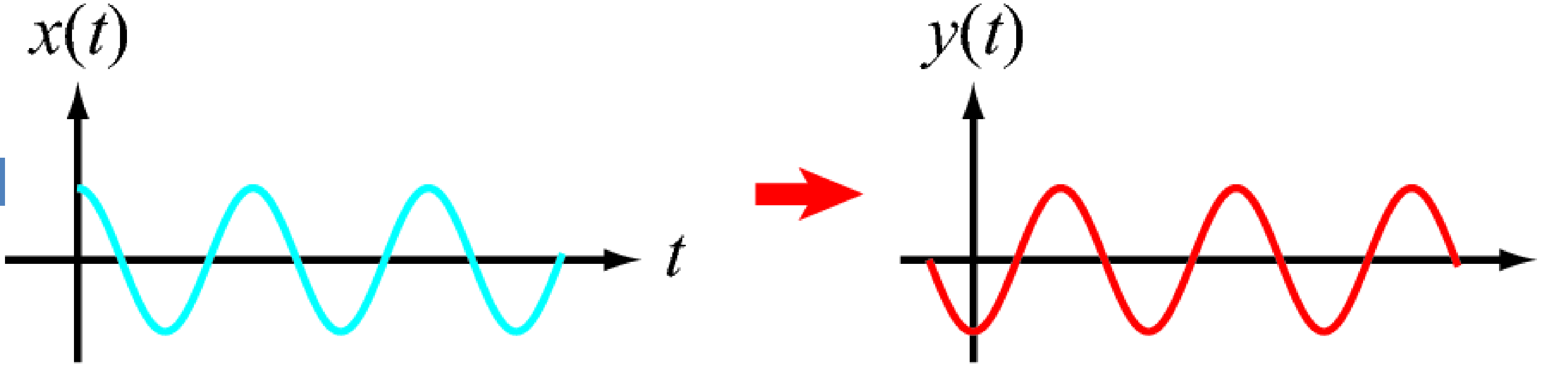
A signal $x(t)$ is ***bounded*** if there is a constant C so that $|x(t)| \le C$ for all t. Examples of bounded signals include:

- $\cos(3t)$, $7e^{-2t}\,u(t)$, and $e^{2t}\,u(1-t)$.

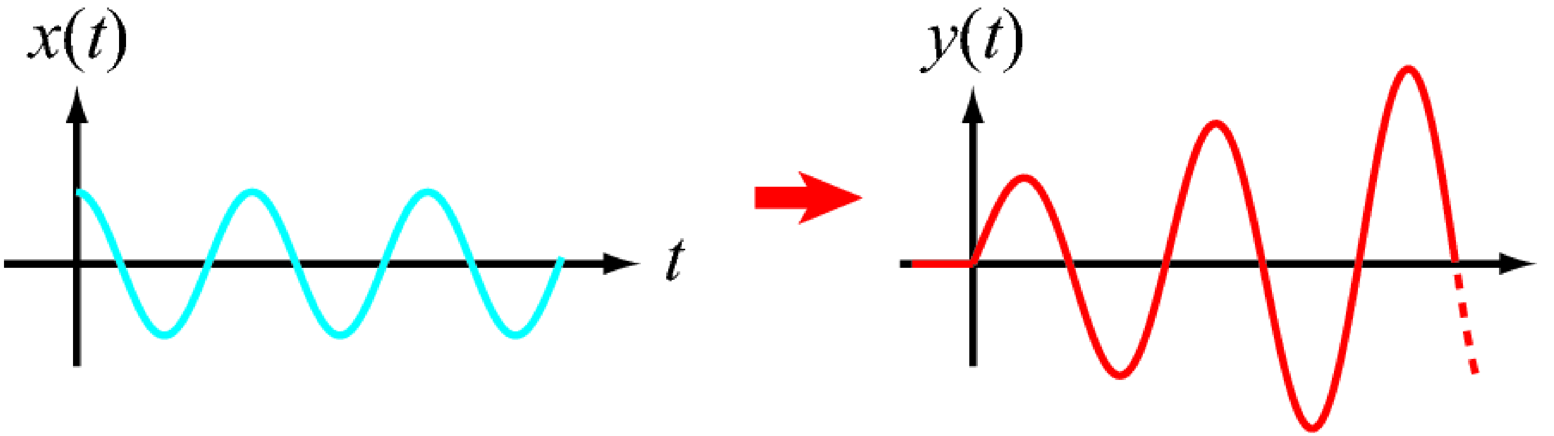
Examples of unbounded signals include:

- t^2, $e^{2t}\,u(t)$, e^{-t}, and $1/t$.

A system is ***BIBO (bounded input/bounded output) stable*** every bounded input $x(t)$ results in a bounded output $y(t)$,



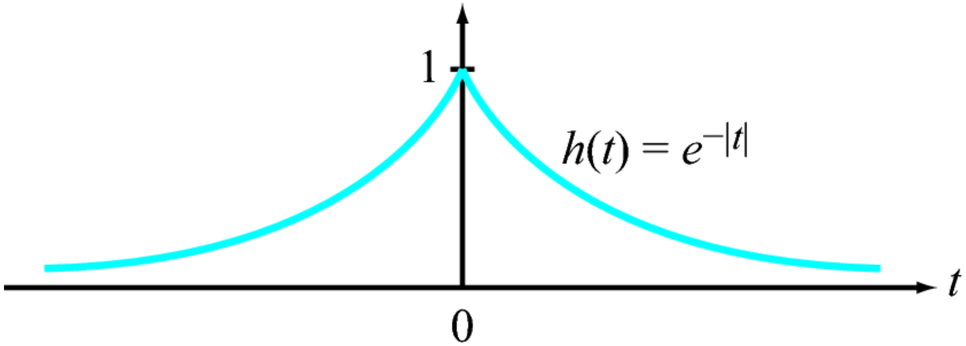
(a) BIBO-stable system



(b) Non-BIBO-stable system

▶ An LTI system is BIBO stable *if and only if* its impulse response $h(t)$ is ***absolutely integrable*** (i.e., if $\int_{-\infty}^{\infty} |h(t)|\, dt$ is finite). ◀

Example



$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} e^{-|t|}\, dt = 2\int_{0}^{\infty} e^{-t}\, dt = 2.$$

Hence, the system is BIBO stable.

2-6.4 BIBO Stability of System with Decaying Exponentials

Consider a causal system with an impulse response

$$h(t) = Ce^{\gamma t}\, u(t), \tag{2.93}$$

where C is a finite constant and γ is, in general, a finite complex coefficient given by

$$\gamma = \alpha + j\beta, \quad \alpha = \mathfrak{Re}[\gamma], \quad \text{and} \quad \beta = \mathfrak{Im}[\gamma]. \tag{2.94}$$

Such a system is BIBO stable if and only if $\alpha < 0$ (i.e., $h(t)$ is a one-sided exponential with an exponential coefficient whose real part is negative). To verify the validity of this statement, we test to see if $h(t)$ is absolutely integrable. Since $|e^{j\beta t}| = 1$ and $e^{\alpha t} > 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} |h(t)|\, dt &= \int_{0}^{\infty} |Ce^{\alpha t} e^{j\beta t}|\, dt \\ &= |C| \int_{0}^{\infty} e^{\alpha t}\, dt.\end{aligned}$$

(a) $\boldsymbol{\alpha} < 0$

If $\alpha < 0$, we can rewrite it as $\alpha = -|\alpha|$ in the exponential, which leads to

$$\int_{-\infty}^{\infty} |h(t)|\, dt = |C| \int_{0}^{\infty} e^{-|\alpha| t}\, dt = \frac{|C|}{|\alpha|} < \infty. \tag{2.96}$$

Hence, $h(t)$ is absolutely integrable and the system is BIBO stable.

(b) $\boldsymbol{\alpha} \geq 0$

If $\alpha \geq 0$, Eq. (2.95) becomes

$$\int_{-\infty}^{\infty} |h(t)|\, dt = |C| \int_{0}^{\infty} e^{\alpha t}\, dt \to \infty,$$

thereby proving that the system is not BIBO stable when $\alpha \geq 0$.

► By extension, for any positive integer N, an impulse response composed of a linear combination of N exponential signals

$$h(t) = \sum_{i=1}^{N} C_i e^{\gamma_i t} \, u(t) \tag{2.97}$$

is absolutely integrable, and its LTI system is BIBO stable, if and only if all of the exponential coefficients γ_i *have negative real parts.* This is a fundamental attribute of LTI system theory. ◄

LTI Sinusoidal Response

▶ The response of an LTI system to a complex exponential input signal $x(t) = Ae^{j\omega t}$ is another complex exponential signal $y(t) = \mathbf{H}(\omega)\, Ae^{j\omega t}$, where $\mathbf{H}(\omega)$ is a complex coefficient that depends on ω. A similar statement applies to sinusoidal inputs and outputs. ◀

Later we will learn that a complicated periodic signal can be expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t}$$

with constant coefficients $\mathbf{x}_n$.

▶ A similar process applies to nonperiodic signals, wherein $\sum_{n=-\infty}^{\infty}$ becomes an integral $\int_{-\infty}^{\infty}$. ◀

LTI Sinusoidal Response

According to the sinusoidal-response property, the corresponding output $y(t)$ will assume the form

$$y(t) = \sum_{n=-\infty}^{\infty} \mathbf{H}_n \mathbf{x}_n e^{jn\omega_0 t} \tag{2.99}$$

with constant coefficients $\mathbf{H}_n$.

This is how filtering of signals is performed by LTI systems to remove noise, unwanted frequency components etc.

Now, we will learn how:

an LTI system excited by an input signal $x(t) = e^{j\omega t}$ generates an output signal $y(t) = \mathbf{H}(\omega)\, e^{j\omega t}$, and we will learn how to compute $\mathbf{H}(\omega)$ from the system's impulse response $h(t)$.

2-7.1 LTI System Response to Complex Exponential Signals

Let us examine the response of an LTI system to a complex exponential $x(t) = Ae^{j\omega t}$. Since the system is LTI, we may (without loss of generality) set $A = 1$. Hence,

$$
\begin{aligned}
y(t) = h(t) * x(t) = h(t) * e^{j\omega t} &= \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)}\, d\tau \\
&= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau \\
&= \mathbf{H}(\omega)\, e^{j\omega t}, \qquad (2.100)
\end{aligned}
$$

where the ***frequency response function*** $\mathbf{H}(\omega)$ is defined as

$$
\mathbf{H}(\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau. \qquad (2.101)
$$

That is: $e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow \mathbf{H}(\omega)\, e^{j\omega t}.$ and $e^{-j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow \mathbf{H}(-\omega)\, e^{-j\omega t}.$

Properties of H(w)

The frequency response function is characterized by the following properties:

(a) $\mathbf{H}(\omega)$ is independent of t.

(b) $\mathbf{H}(\omega)$ is a function of $j\omega$, not just ω.

(c) $\mathbf{H}(\omega)$ is defined for an input signal $e^{j\omega t}$ that exists for all time, $-\infty < t < \infty$; i.e., $x(t) = e^{j\omega t}$, not $e^{j\omega t}\, u(t)$.

If $h(t)$ is real, $\mathbf{H}(\omega)$ is related to $\mathbf{H}(-\omega)$ by

$$\mathbf{H}(-\omega) = \mathbf{H}^*(\omega) \qquad \text{(for } h(t) \text{ real).} \tag{2.107}$$

Properties of H(w)

The condition defined by Eq. (2.107) is known as ***conjugate symmetry***. It follows from the definition of $\mathbf{H}(\omega)$ given by Eq. (2.101), namely,

$$\mathbf{H}^*(\omega) = \int_{-\infty}^{\infty} h^*(\tau)\,(e^{-j\omega\tau})^*\,d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\,e^{j\omega\tau}\,d\tau = \mathbf{H}(-\omega). \qquad (2.108)$$

▶ Since $h(t)$ is real, it follows from Eq. (2.107) that the magnitude $|\mathbf{H}(\omega)|$ is an even function and the phase $\underline{/\mathbf{H}(\omega)}$ is an odd function:

- $$|\mathbf{H}(\omega)| = |\mathbf{H}(-\omega)| \qquad (2.109a)$$

and

- $$\underline{/\mathbf{H}(-\omega)} = -\underline{/\mathbf{H}(\omega)}\,. \qquad (2.109b)$$

◀

H(w) of linear, constant-coefficient, differential equation (LCCDE)

- If an LTI is a LCCDE, we can determine H(w) without having to know h(t). Consider

$$\frac{d^2y}{dt^2} + 2\,\frac{dy}{dt} + 3y = 4\,\frac{dx}{dt} + 5x. \qquad (2.110)$$

According to Eq. (2.100), $y(t) = \mathbf{H}(\omega)\, e^{j\omega t}$ when $x(t) = e^{j\omega t}$. Substituting these expressions for $x(t)$ and $y(t)$ in Eq. (2.110) leads to the ***frequency domain equivalent*** of Eq. (2.110):

$$(j\omega)^2\, \mathbf{H}(\omega)\, e^{j\omega t} + 2(j\omega)\, \mathbf{H}(\omega)\, e^{j\omega t} + 3\mathbf{H}(\omega)\, e^{j\omega t}$$

$$= 4(j\omega)e^{j\omega t} + 5e^{j\omega t}. \qquad (2.111)$$

After canceling $e^{j\omega t}$ (which is never zero) from all terms and then solving for $\mathbf{H}(\omega)$, we get

$$\mathbf{H}(\omega) = \frac{5 + j4\omega}{-\omega^2 + 2j\omega + 3} = \frac{5 + j4\omega}{(3 - \omega^2) + j2\omega} . \qquad (2.112)$$

► We note that every d/dt operation in the time domain translates into a multiplication by $j\omega$ in the frequency domain. ◄

LTI Response to Sinusoid

- Consider a cosine signal, BY Euler's identity, we have:

$$x(t) = \cos \omega t = \frac{1}{2}[e^{j\omega t} + e^{-j\omega t}]. \qquad (2.115)$$

Using the additivity property of LTI system:

$$\begin{array}{ccccc} \frac{1}{2}\, e^{j\omega t} & \longrightarrow & \boxed{h(t)} & \longrightarrow & \frac{1}{2}\, \mathbf{H}(\omega)\, e^{j\omega t} \\ + & & & & + \\ \frac{1}{2}\, e^{-j\omega t} & \longrightarrow & \boxed{h(t)} & \longrightarrow & \frac{1}{2}\, \mathbf{H}(-\omega)\, e^{-j\omega t}. \end{array} \qquad (2.116)$$

Since $h(t)$ is real, Eq. (2.107) stipulates that

$$\mathbf{H}(-\omega) = \mathbf{H}^*(\omega).$$

If we define

$$\mathbf{H}(\omega) = |\mathbf{H}(\omega)|e^{j\theta},$$
$$\mathbf{H}(-\omega) = |\mathbf{H}(\omega)|e^{-j\theta},$$

where $\theta = \underline{/\mathbf{H}(\omega)}$, the sum of the two outputs in becomes

$$\begin{aligned} y(t) &= \frac{1}{2}\left[\mathbf{H}(\omega)\, e^{j\omega t} + \mathbf{H}(-\omega)\, e^{-j\omega t}\right] \\ &= \frac{1}{2}|\mathbf{H}(\omega)|\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right] \\ &= |\mathbf{H}(\omega)|\cos(\omega t + \theta). \end{aligned}$$

In summary,

$$y(t) = |\mathbf{H}(\omega)|\cos(\omega t + \theta)$$
$$(\text{for } x(t) = \cos\omega t).$$

In symbolic form:

$$A\cos(\omega t + \phi)$$
$$\downarrow$$
$$\boxed{h(t)}$$
$$\downarrow$$
$$A|\mathbf{H}(\omega)|\cos(\omega t + \theta + \phi).$$

Example 2-13: Response of an LCCDE to a Sinusoidal Input

$$5\cos(4t+30^\circ) \longrightarrow \boxed{\frac{dy}{dt}+3y=3\frac{dx}{dt}+5x} \longrightarrow ?$$

Solution: Inserting $x(t)=e^{j\omega t}$ and $y(t)=\mathbf{H}(\omega)\,e^{j\omega t}$ in the specified LCCDE gives

$$j\omega\,\mathbf{H}(\omega)\,e^{j\omega t}+3\mathbf{H}(\omega)\,e^{j\omega t}=3\,j\omega e^{j\omega t}+5e^{j\omega t}.$$

Solving for $\mathbf{H}(\omega)$ gives

$$\mathbf{H}(\omega)=\frac{5+j3\omega}{3+j\omega}.$$

The angular frequency ω of the input signal is 4 rad/s. Hence,

$$\mathbf{H}(4)=\frac{5+j12}{3+j4}=\frac{13e^{j67.38^\circ}}{5e^{j53.13^\circ}}=\frac{13}{5}\,e^{j14.25^\circ}.$$

Application of Eq. (2.120) with $A=5$ and $\phi=30^\circ$ gives

$$y(t)=5\times\frac{13}{5}\cos(4t+14.25^\circ+30^\circ)=13\cos(4t+44.25^\circ).$$