# LECTURE BLOCK 2C: LINEAR TIME INVARIANT (LTI) SYSTEMS

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# Aim of lecture

Today we will study the discrete-time equivalent of signals, LTI system and convolution.

There are similarities between continuous-time and discrete-time signals and systems.

This means they are similar but not exactly the same, for example, periodicity etc.

# Discrete-Time Signals: Finite Length

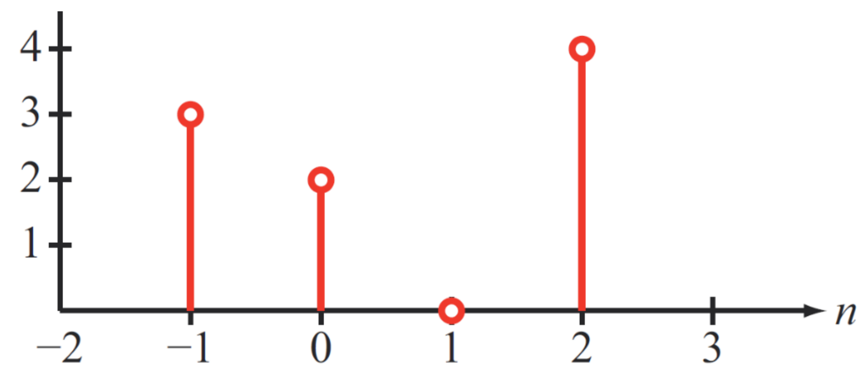
Discrete-time signals x[n] are defined only at integer times n.
Continuous-time signals x(t) are defined for all real numbers t.
Discrete-time signals can be represented in 3 different ways:

1. Enclosing all non-zero values in brackets: $x[n]=\{3,\underline{2},0,4\}$.
   The underline indicates x[n] at time n=0. Here, x[0]=2.

2. Listing all of its non-zero values:

$$x[n] = \begin{cases} 3 & \text{for } n = -1, \\ 2 & \text{for } n = 0, \\ 4 & \text{for } n = 2, \\ 0 & \text{for all other } n. \end{cases}$$

3. Using a stem plot:

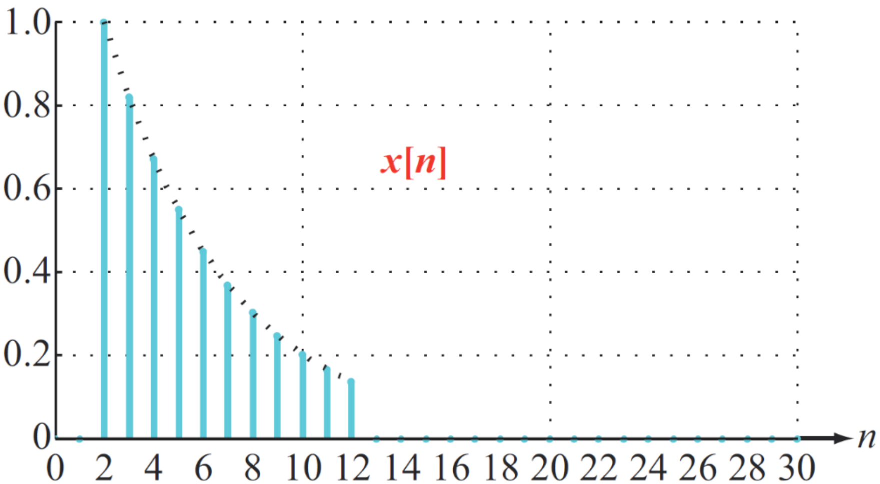


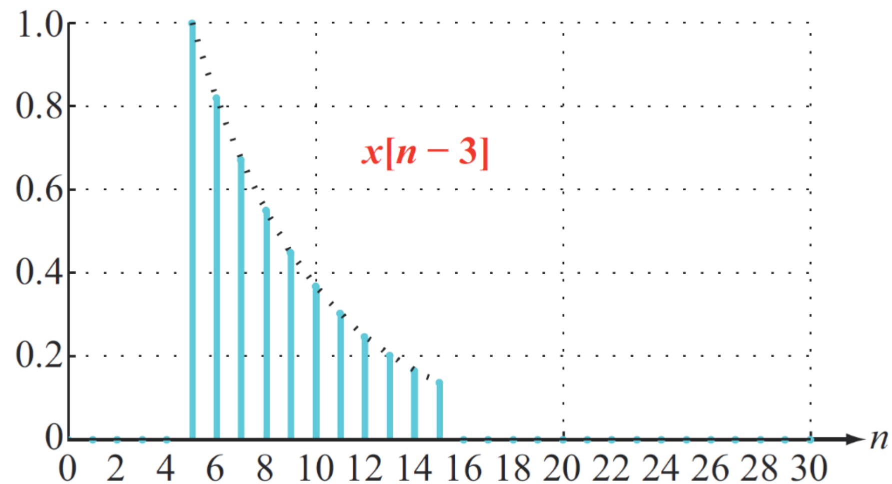
# Discrete-Time Signals: Finite Length

Let x[n]=0 for all n<a and all n>b. The length or duration of x[n] is b-a+1, not b-a.

Example: Let $x[n]=\{3,\underline{2},0,4\}=0$ for n<-1 and n>2. The length of x[n] is 2-(-1)+1=4.

Time shifts work the same way in discrete time as time delays work in continuous time.

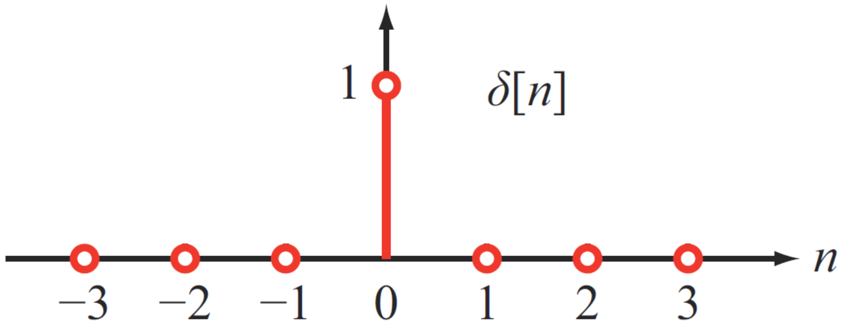


# Discrete-Time Impulse and Step Functions

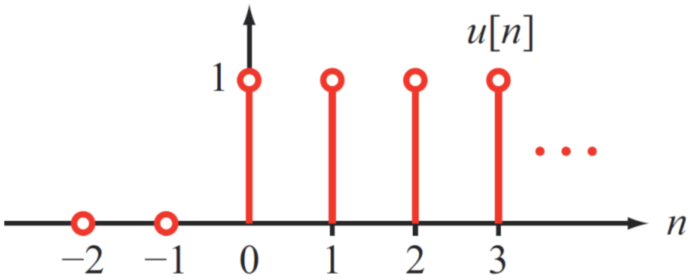
## Discrete-time Impulse:

$$\delta[n] = \{\underline{1}\} = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases}$$




## Discrete-time Step:

$$u[n] = \{\underline{1}, 1, 1, \ldots\} = \begin{cases} 1 & \text{for } n \geq 0, \\ 0 & \text{for } n < 0, \end{cases}$$



# Discrete vs. Cont. Time: Impulses and Steps

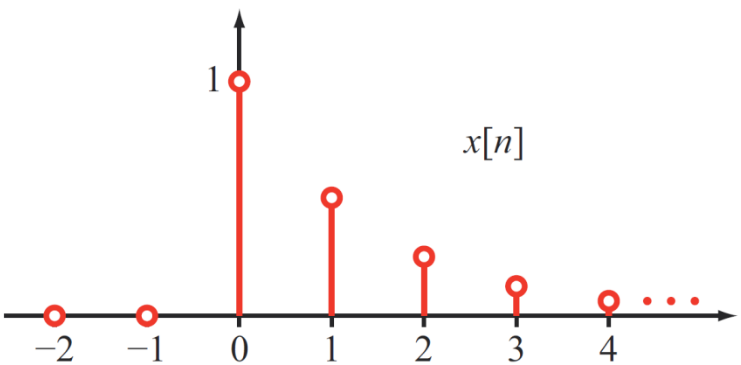
| Property: | Discrete Time: | Continuous Time: |
|---|---|---|
| Sampling: | $x[n] = \sum_{i=-\infty}^{\infty} x[i]\,\delta[i-n]$ | $x(T) = \int_{-\infty}^{\infty} x(t)\,\delta(t-T)\,dt$ |
| Relation between step and impulse | $u[n] = \sum_{i=-\infty}^{n} \delta[i]$ | $u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$ |
| Value at time=0 | u[0]=1 and $\delta$(0)=1. | u(0) undefined<br>$\delta(0) \rightarrow \infty$ |

# Discrete vs. Cont. Time: Exponentials

## Discrete Time:

Geometric Signals

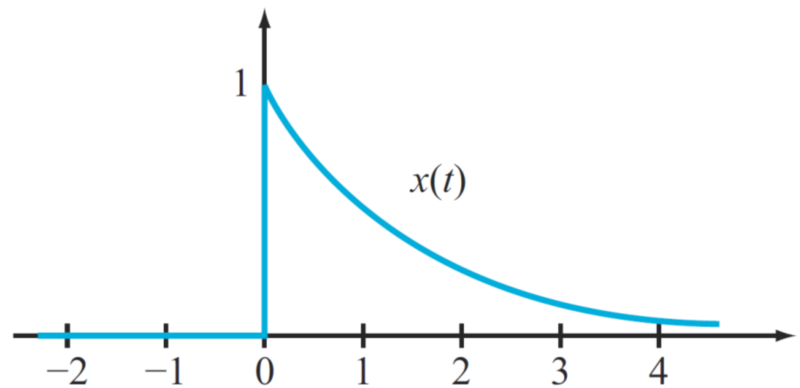
$$x[n] = \mathbf{p}^n \; u[n]$$




## Continuous Time:

Exponential Signals

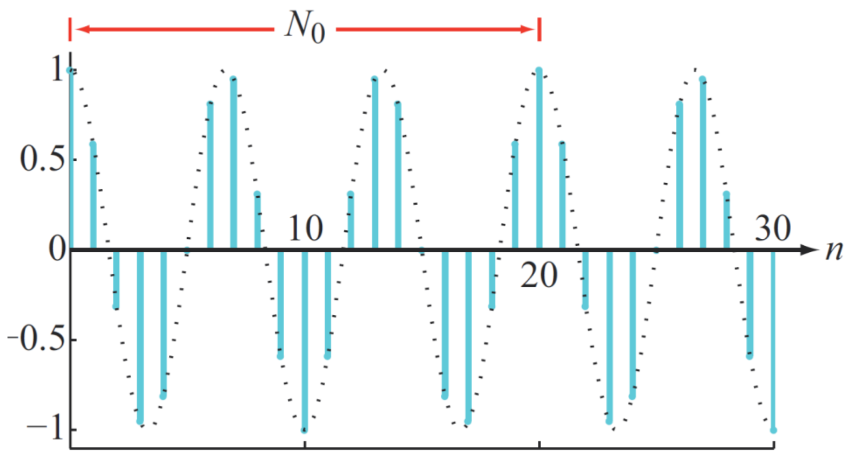
$$x_1(t) = e^{-at} \; u(t)$$



# Discrete vs. Cont. Time: Sinusoids

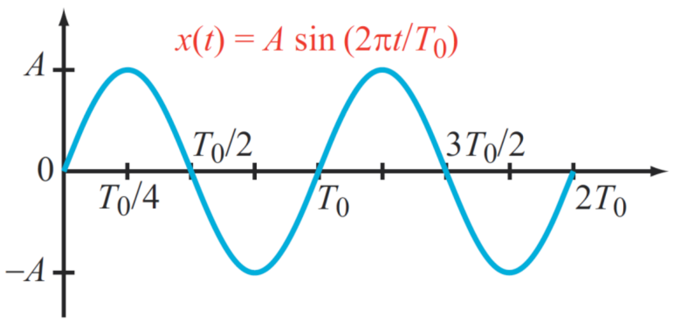
## Discrete Time:

$$x[n] = A\cos(\Omega n + \theta)$$




## Continuous Time:

$$x(t) = A\cos(\omega_0 t + \theta)$$




Period: $T_0 = 2\pi/\omega_0$

Period: $N_0 = \dfrac{2\pi k}{\Omega}$ if k exists that makes this an integer

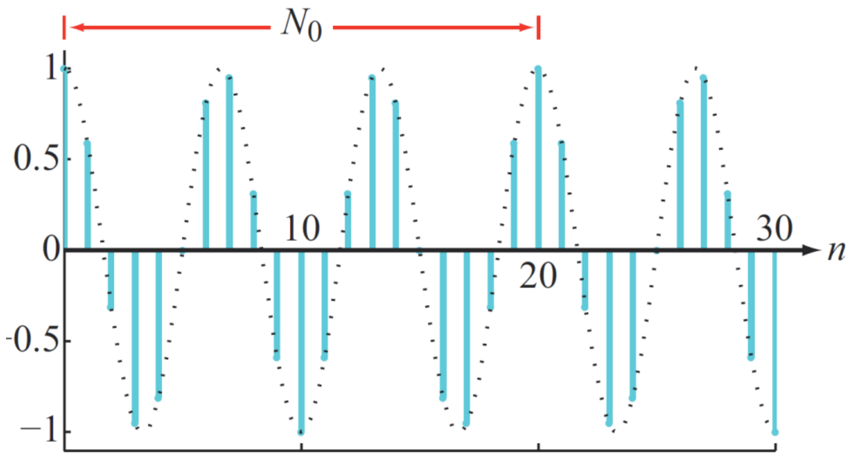
# Example: Period of Discrete-Time Sinusoid

**Goal:** Determine the period of $x[n] = \cos(0.3\pi n)$

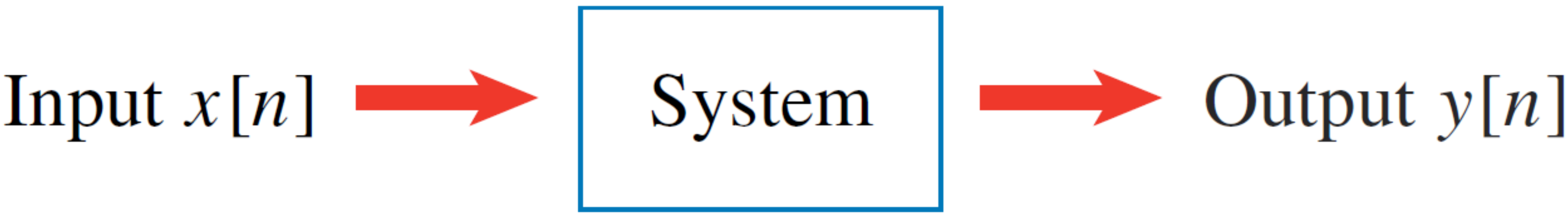
**Solution:** $N_0 = \dfrac{2\pi k}{\Omega} = \dfrac{2\pi k}{0.3\pi} = \dfrac{20k}{3}$ k=3→period=20.

x[n+20]=cos(0.3π(n+20))=cos(0.3πn+6π)=cos(0.3πn).

**Note:** Most discrete-time sinusoids are not periodic!

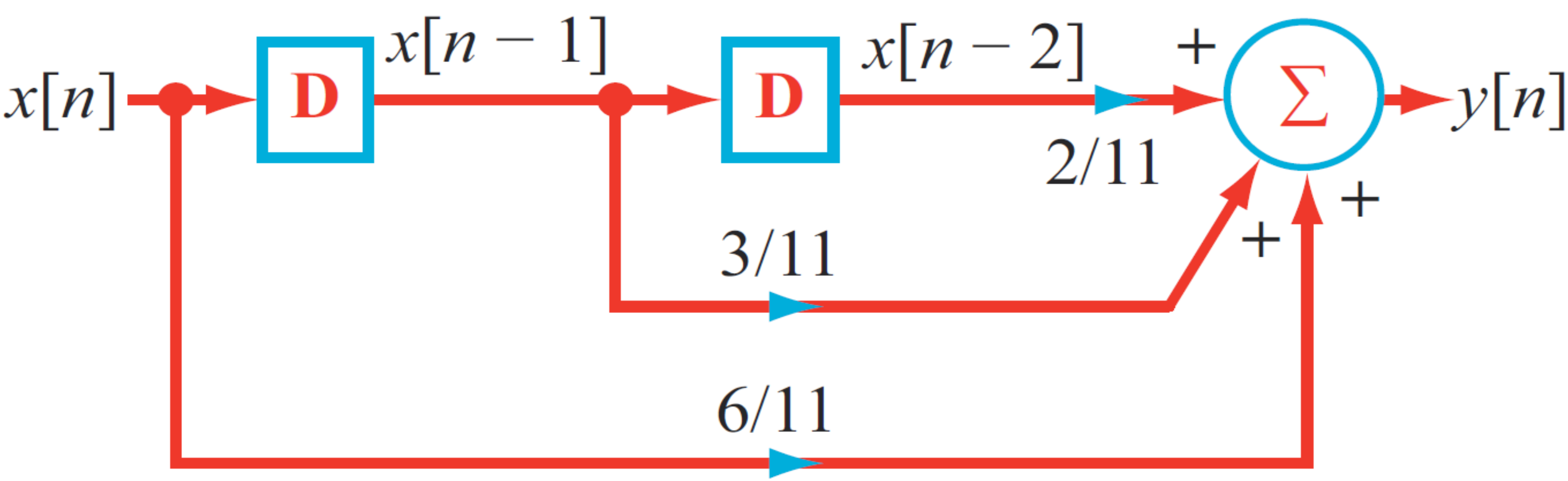


# Discrete-Time Systems




Example: Weighted Moving-Average (MA) of 3 most recent inputs:

$$y[n] = \frac{6}{11}\, x[n] + \frac{3}{11}\, x[n-1] + \frac{2}{11}\, x[n-2].$$



# Discrete vs. Cont. Time: System Eqns.

## Discrete Time:

Difference Equation

$$\sum_{i=0}^{N} a_i \; y[n-i] = \sum_{i=0}^{M} b_i \; x[n-i].$$

## Continuous Time:

Differential Equation

$$\sum_{i=0}^{N} c_{N-i} \, \frac{d^i y}{dt^i} = \sum_{i=0}^{M} d_{M-i} \, \frac{d^i x}{dt^i}$$

An ARMA difference equation has two parts:

$$\sum_{i=0}^{N} a_i \; y[n-i] = x[n] \qquad \textbf{(autoregressive)}.$$

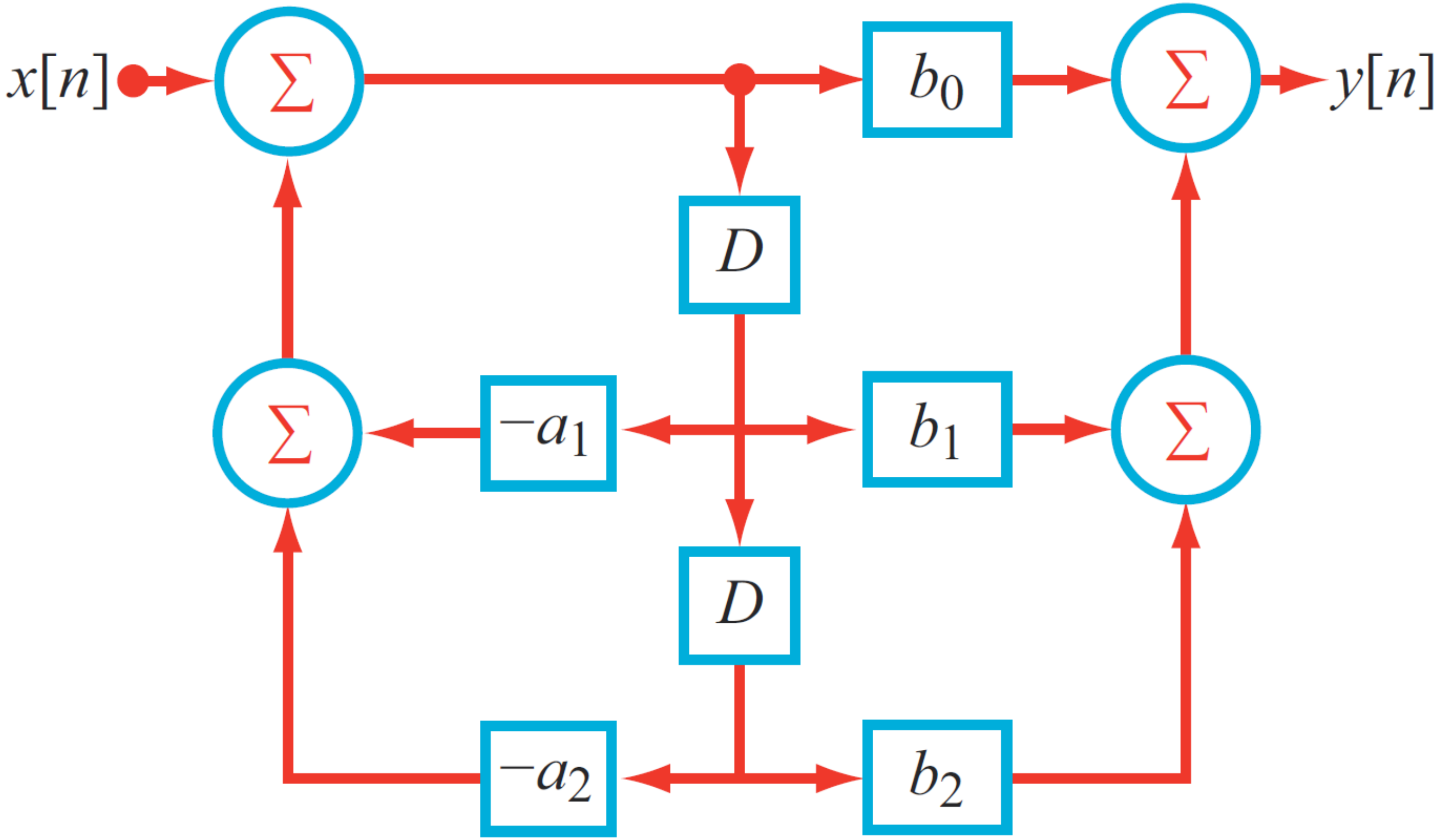
(left side of equation)

$$y[n] = \sum_{i=0}^{M} b_i \; x[n-i] \qquad \textbf{(moving average)}.$$

(right side of equation)

# Realization of ARMA Difference Equation

$$y[n] + a_1\, y[n-1] + a_2\, y[n-2] = b_0\, x[n] + b_1\, x[n-1] + b_2\, x[n-2].$$



# Discrete-Time LTI Systems

Definition of an LTI (Linear Time-Invariant) system is the same in discrete time as in continuous time:

A system is linear if it has the scaling and additivity properties.

A system is time-invariant if delaying the input delays the output.

A system is LTI if it is both linear and time-invariant.

ARMA difference equations with constant coefficients are LTI.

$$\sum_{i=0}^{N} a_i \; y[n-i] = \sum_{i=0}^{M} b_i \; x[n-i].$$

This describes an LTI system.

# Impulse Response of an MA System

The impulse response h[n] of a system is its response to an impulse:

$$\delta[n] \longrightarrow \boxed{\text{System}} \longrightarrow h[n].$$

We can read off the impulse response of an MA difference equation:

$$y[n] = \sum_{i=0}^{M} b_i \; x[n-i] \longleftrightarrow h[n] = \{\underline{b_0}, b_1, \ldots, b_M\}.$$

Example: Impulse response of the system described by the MA difference equation

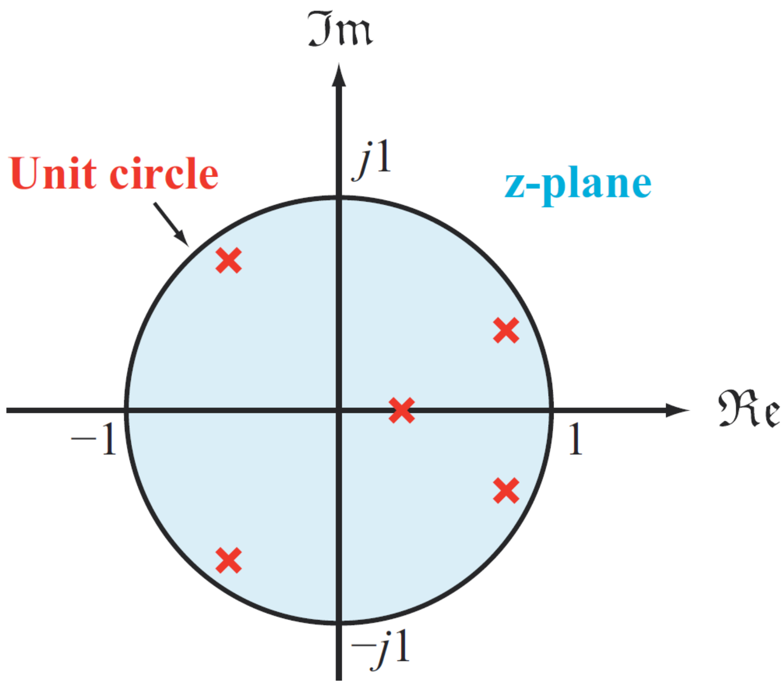
$y[n] = 2x[n] - 3x[n-1] - 4x[n-2]$. is $h[n] = \{\underline{2}, -3, -4\}$

# Discrete vs. Cont. Time: BIBO Stability

| Property: | Discrete Time: | Continuous Time: |
|---|---|---|
| Causality: | $h[n] = 0$ for all $n < 0$ | $h(t) = 0$ for $t < 0$ |
| Stability: | $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert$ is finite | $\int_{-\infty}^{\infty} \lvert h(t)\rvert \, dt$ is finite |
| Impulse response has the form | $h[n] = \sum_{i=1}^{N} \mathbf{C}_i \mathbf{p}_i^n \, u[n].$ | $h(t) = \sum_{i=1}^{N} C_i e^{\gamma_i t} \, u(t)$ |
| Stability: | $\lvert\mathbf{p}_i\rvert < 1$ | Real[$\gamma_i$]<0. |

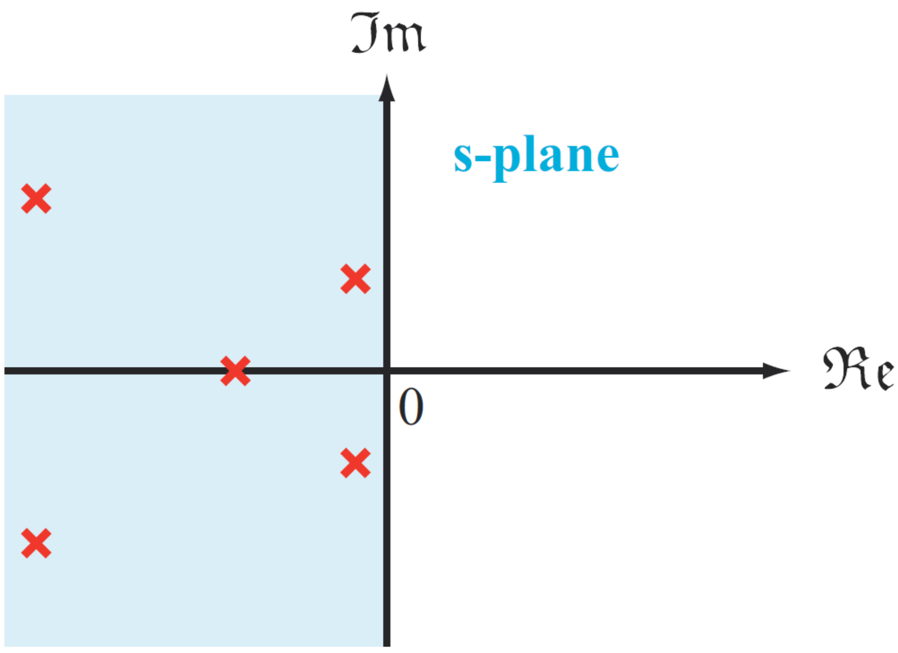
# Discrete vs. Cont. Time: Locations of Poles of BIBO-Stable LTI Systems

## Discrete Time:



(b) Poles inside unit circle

## Continuous Time:



(a) Poles in open left half-plane

# Discrete vs. Cont. Time: Convolution

## Discrete Time:

$$y[n] = x[n] * h[n] = \sum_{i=-\infty}^{\infty} x[i]\, h[n-i]$$

$$x[n] \rightarrow \boxed{\text{LTI}} \rightarrow \sum_{i=-\infty}^{\infty} x[i]\, h[n-i]$$

## Continuous Time:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$x(t) \rightarrow \boxed{\text{LTI}} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

# Discrete vs. Cont. Time: Convolution

**Table 7-4:** Comparison of convolution properties for continuous-time and discrete-time signals.

| Property | Continuous Time | Discrete Time |
|---|---|---|
| Definition | $y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$ | $y[n] = h[n] * x[n] = \sum_{i=-\infty}^{\infty} h[i]\, x[n-i]$ |
| 1. Commutative | $x(t) * h(t) = h(t) * x(t)$ | $x[n] * h[n] = h[n] * x[n]$ |
| 2. Associative | $[g(t) * h(t)] * x(t) = g(t) * [h(t) * x(t)]$ | $[g[n] * h[n]] * x[n] = g[n] * [h[n] * x[n]]$ |
| 3. Distributive | $x(t) * [h_1(t) + \cdots + h_N(t)] =$ $x(t) * h_1(t) + \cdots + x(t) * h_N(t)$ | $x[n] * [h_1[n] + \cdots + h_N[n]] =$ $x[n] * h_1[n] + \cdots + x[n] * h_N[n]$ |
| 4. Causal $*$ Causal $=$ Causal | $y(t) = u(t) \int_{0}^{t} h(\tau)\, x(t-\tau)\, d\tau$ | $y[n] = u[n] \sum_{i=0}^{n} h[i]\, x[n-i]$ |
| 5. Time-shift | $h(t-T_1) * x(t-T_2) = y(t-T_1-T_2)$ | $h[n-a] * x[n-b] = y[n-a-b]$ |
| 6. Convolution with Impulse | $x(t) * \delta(t-T) = x(t-T)$ | $x[n] * \delta[n-a] = x[n-a]$ |
| 7. Width | width $y(t) =$ width $x(t) +$ width $h(t)$ | width $y[n] =$ width $x[n] +$ width $h[n] - 1$ |
| 8. Area | area of $y(t) =$ area of $x(t) \times$ area of $h(t)$ | $\sum_{n=-\infty}^{\infty} y[n] = \left(\sum_{n=-\infty}^{\infty} h[n]\right)\left(\sum_{n=-\infty}^{\infty} x[n]\right)$ |
| 9. Convolution with step | $y(t) = x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ | $x[n] * u[n] = \sum_{i=-\infty}^{n} x[i]$ |

# Example: Computing Convolution

Compute the convolution {<u>2</u>,3,4}*{<u>5</u>,6,7} using $y[n] = u[n] \sum_{i=0}^{n} h[i]\, x[n-i]$

Solution: The length of the convolution is 3+3-1=5: {y[0]…y[4]}.
Compute each of y[0],y[1],y[2],y[3],y[4] in succession as follows:

$$y[0] = \sum_{i=0}^{0} x[i]\, h[0-i] = x[0]\, h[0] = 2 \times 5 = 10,$$

$$\begin{aligned} y[1] &= \sum_{i=0}^{1} x[i]\, h[1-i] \\ &= x[0]\, h[1] + x[1]\, h[0] = 2 \times 6 + 3 \times 5 = 27, \end{aligned}$$

$$\begin{aligned} y[2] &= \sum_{i=0}^{2} x[i]\, h[2-i] \\ &= x[0]\, h[2] + x[1]\, h[1] + x[2]\, h[0] \\ &= 2 \times 7 + 3 \times 6 + 4 \times 5 = 52, \end{aligned}$$

$$\begin{aligned} y[3] &= \sum_{i=1}^{2} x[i]\, h[3-i] \\ &= x[1]\, h[2] + x[2]\, h[1] = 3 \times 7 + 4 \times 6 = 45, \end{aligned}$$

$$y[4] = \sum_{i=2}^{2} x[i]\, h[4-i] = x[2]\, h[2] = 4 \times 7 = 28,$$

$$y[n] = 0, \ \text{otherwise.}$$

Hence,

$$y[n] = \{\underline{10}, 27, 52, 45, 28\}.$$

# Example: Computing Convolution

Compute the convolution $\{\underline{2},3,4\}*\{\underline{5},6,7\}$ graphically.

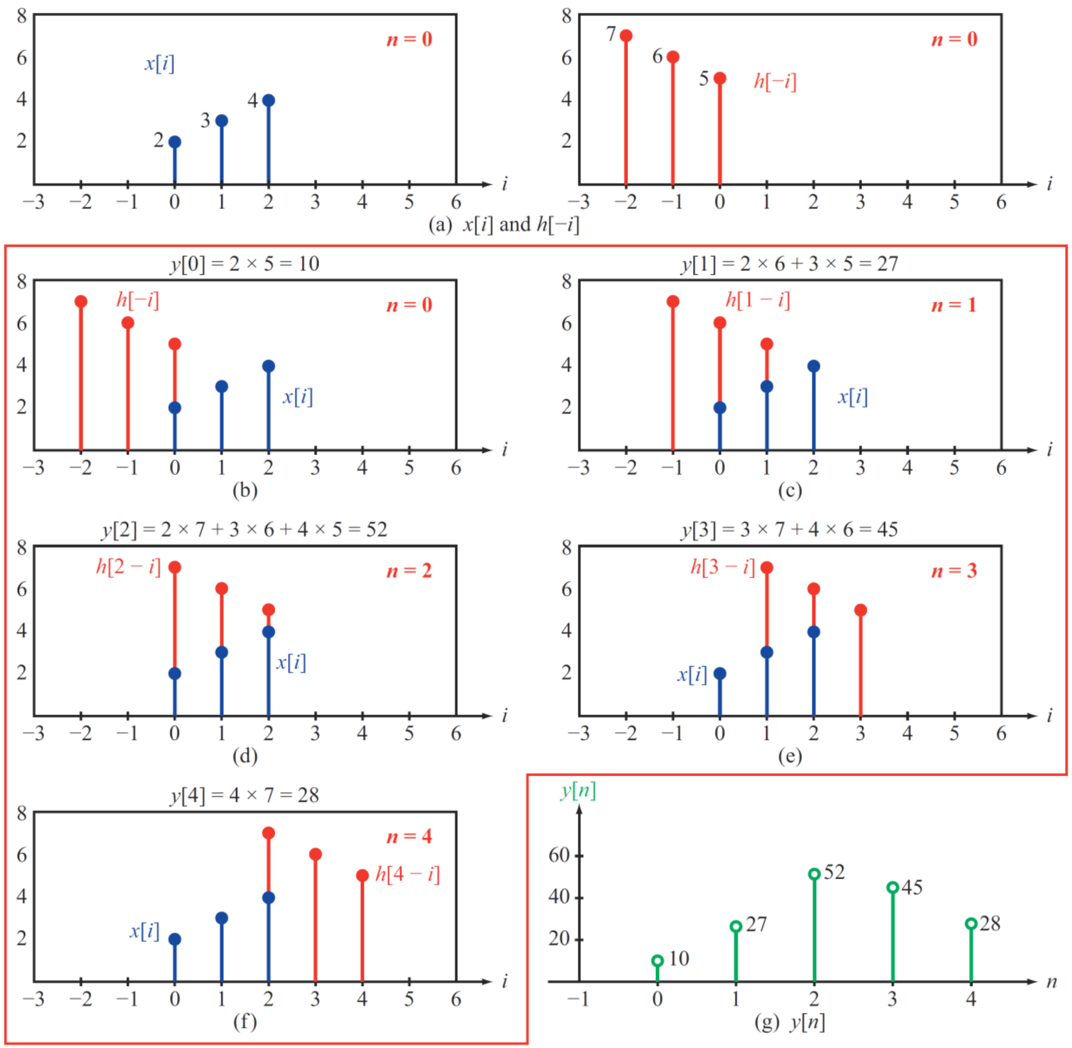
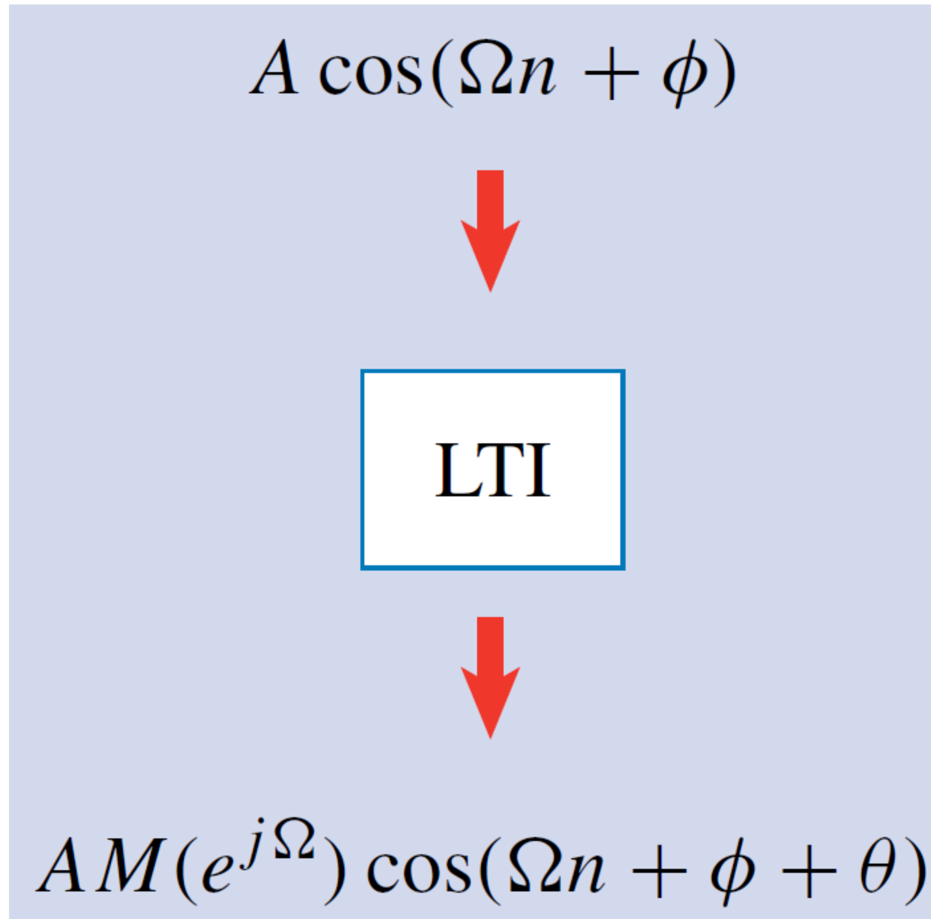



**Figure 7-12:** Graphical computation of convolution sum.

# Response to Sinusoidal Input

Response of LTI system to an input sinusoid is an output sinusoid at the same frequency: "sine-in-sine-out."

The amplitude and phase are altered by frequency response function of LTI system (see the next slide).



# Discrete vs. Cont. Time: Frequency Response Function for Sinusoidal Inputs

## Discrete Time:

$$e^{j\Omega n} \longrightarrow \boxed{\text{LTI}} \longrightarrow \mathbf{H}(e^{j\Omega})\, e^{j\Omega n}$$

$$\mathbf{H}(e^{j\Omega}) = \sum_{i=0}^{\infty} h[i]\, e^{-ji\Omega}.$$

$$\underbrace{\mathbf{H}(e^{j\Omega})}_{\text{frequency response}} = \underbrace{\mathbf{H}(\mathbf{z})}_{\text{transfer function}}\Big|_{\mathbf{z}=e^{j\Omega}}$$

## Continuous Time:

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow \mathbf{H}(\omega)\, e^{j\omega t}$$

$$\mathbf{H}(\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau.$$

$$\mathbf{H}(\omega) = \mathbf{H}(\mathbf{s})\Big|_{\mathbf{s}=j\omega}$$

# Example: Response to a Sinusoid

The impulse response of an LTI system is $h[n] = \left(\frac{1}{2}\right)^n u[n]$.

(a) Determine and plot the frequency response $\mathbf{H}(e^{j\Omega})$ and (b) compute the system response to input $x[n] = \cos\left(\frac{\pi}{3}\, n\right)$.

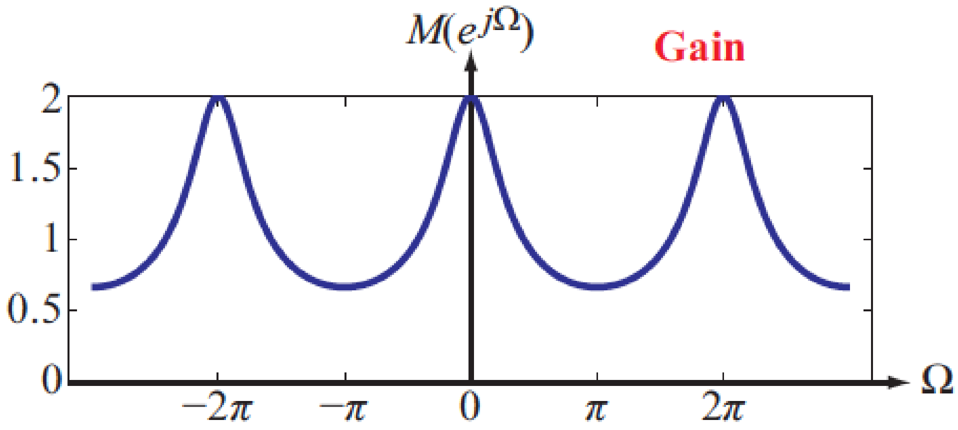
**Solution:** $\mathbf{H}(e^{j\Omega}) = \sum_{i=0}^{\infty} h[i]\, e^{-ji\Omega} = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i e^{-ji\Omega} = \dfrac{1}{1 - \frac{1}{2}\, e^{-j\Omega}}$

Magnitude and phase are plotted on the next slide.

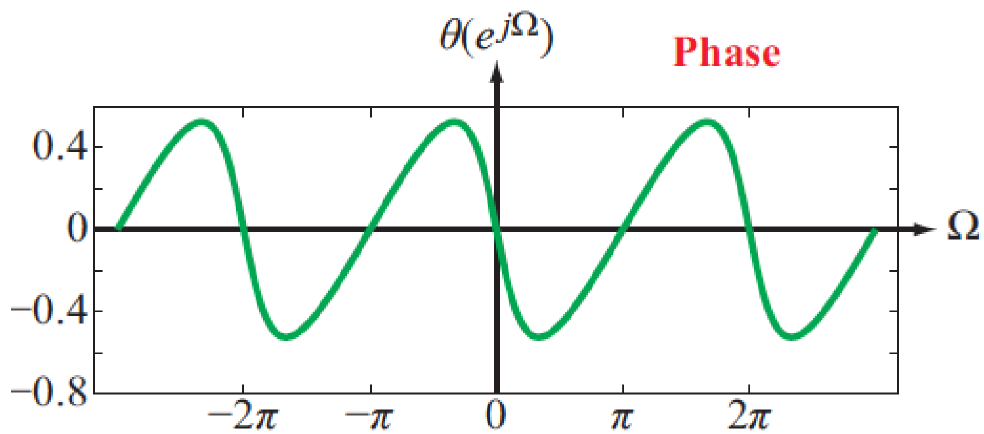
**(b)** $\mathbf{H}(e^{j\pi/3}) = \dfrac{1}{1 - \frac{1}{2}\, e^{-j\pi/3}} = 1.155 e^{-j\pi/6}$.

$$y[n] = |\mathbf{H}(e^{j\pi/3})| \cos\left(\frac{\pi}{3}\, n + \theta\right) = 1.155 \cos\left(\frac{\pi}{3}\, n - \frac{\pi}{6}\right).$$

# Example: Response to a Sinusoid




(a) $M(e^{j\Omega}) = |\mathbf{H}(e^{j\Omega})|$




(b) Phase $\theta(e^{j\Omega}) = \angle \mathbf{H}(e^{j\Omega})$