LECTURE BLOCK 3A: FOURIER SERIES OF PERIODIC SIGNALS

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Fourier Analysis Techniques

We will learn:

- Express periodic signals in term of Fourier Series.
- Use Fourier Series to analyse system driven by continuous periodic signals.
- Apply Parseval's theorem to compute the power or energy contained in a signal.
- Study Fourier Series' examples, and properties,
- Look at Discrete Time Fourier Series.

Earlier, on LTI systems…

- Convolutional sum (and integral) is based on representing signals as linear combination of shifted impulses.
- This means, we can represent signals as linear combinations of a set of basic signals, for example by using a set of complex exponentials.
- Superposition: the response of an LTI system to any input consisting of a linear combination of basic signals is the same linear combination of the individual responses of each of the basic signals.
- Lecture 2B: studied the Response of LTI to a complex exponentials; another way to analyse LTI systems.
- So now, we will represent signals as a linear combination of complex exponentials.

Fourier Series

In 1807 Jean-Baptiste Joseph Fourier[†] (1768-1830) completed a paper on the theory of heat conduction (eventually published in 1822) in which he introduced the idea of expanding a function as a sum of trigonometric functions.

This idea was also proposed in a different context by Daniel Bernoulli in 1748.

At the time the idea was controversial, but has since gained acceptance as one of the fundamental tools in the study of differential equations.

[†]For a short biography see : http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html

Fourier Series

Awareness of the physical and technical (as opposed to mathematical) importance of superposed oscillations has existed in physics and musical theory since antiquity.

This was sharpened by the systematic investigations involved in the development of tuned bells in the Netherlands in the 17-th century and the work on the harmonic structure of musical tones in the early 18th century.

In 1863 Hermann von Helmholtz[†] published *On the Sensations of Tone as a Physiological Basis for the Theory of Music*, in which he advocated the use of Fourier Series.

[†]For a short biography see : http://www-gap.dcs.st-and.ac.uk/history/Mathematicians/Helmholtz.html

Fourier Series

Since the mid 19-th century the importance of the *spectral representation* of signals has continued to grow, to the extent that it is now forms a basic part of the language of most technical disciplines.

This is especially true of signal processing, where (perhaps in one of its more advanced forms) it is arguably *the* basic idea.

In this lecture we shall briefly revise the basic mathematics of Fourier Series.

Fourier Series Representation of Periodic Signals

- Let x(t) be periodic: $x(t) = x(t + nT_0)$
- Then x(t) can be expressed as a linear combination of sinusoids at harmonic frequencies:

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \qquad (5.26a)$$

(sine/cosine representation)

$$= c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n) \qquad (5.26b)$$

(amplitude/phase representation)

Fourier Series Expansions of Periodic Signals

- Let x(t) be periodic: $x(t) = x(t + nT_0)$
- Now x(t) need not be real-valued, ie it can be complex
- Then x(t) can be expressed as a linear combination of complex exponentials at harmonic frequencies:

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t},$$

(exponential representation)

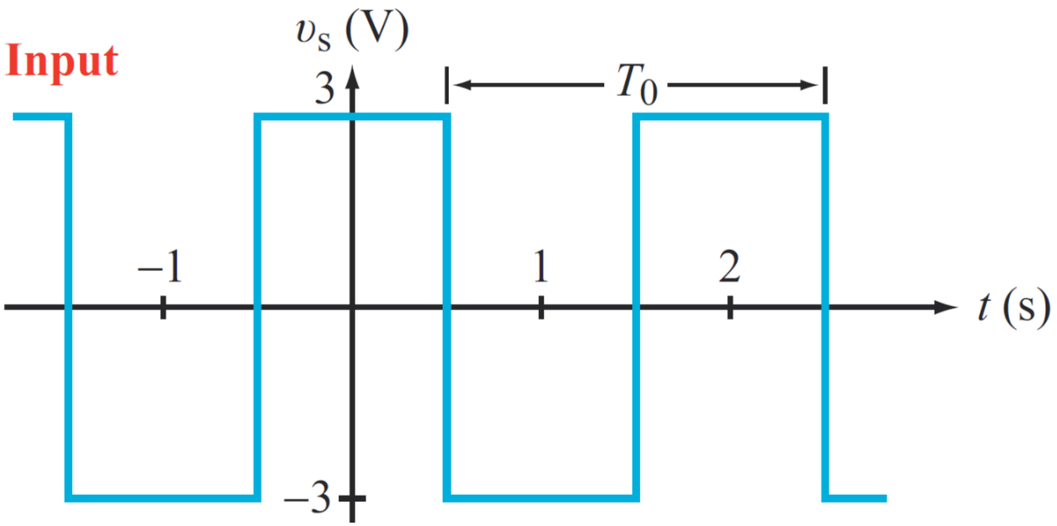
Remember that: sinusoidal is related to complex exponential by Euler's identity

Fourier Series Expansions of Periodic Signals

- Fundamental angular frequency: $\omega_0 = 2\pi / T_0$
- dc or average term: a_0 or c_0
- Fundamental term: $c_1 \cos(\omega_0 t + \phi_1)$
- Harmonics: $c_n \cos(n\omega_0 t + \phi_n)$
- Fundamental has same period as x(t)
- In music: harmonics are known as overtones
- $\{a_n, b_n, c_n, \mathbf{x}_n\}$ are Fourier coefficients

Fourier Series Example

- The Fourier series expansion of the periodic signal

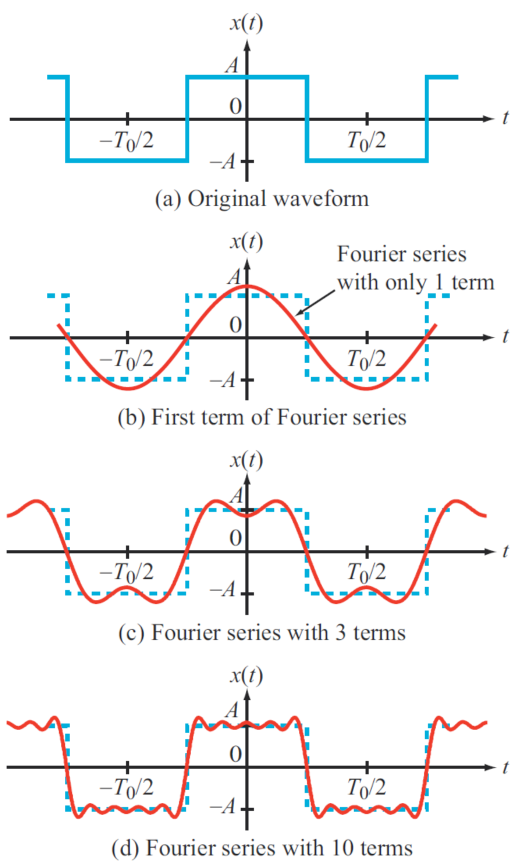


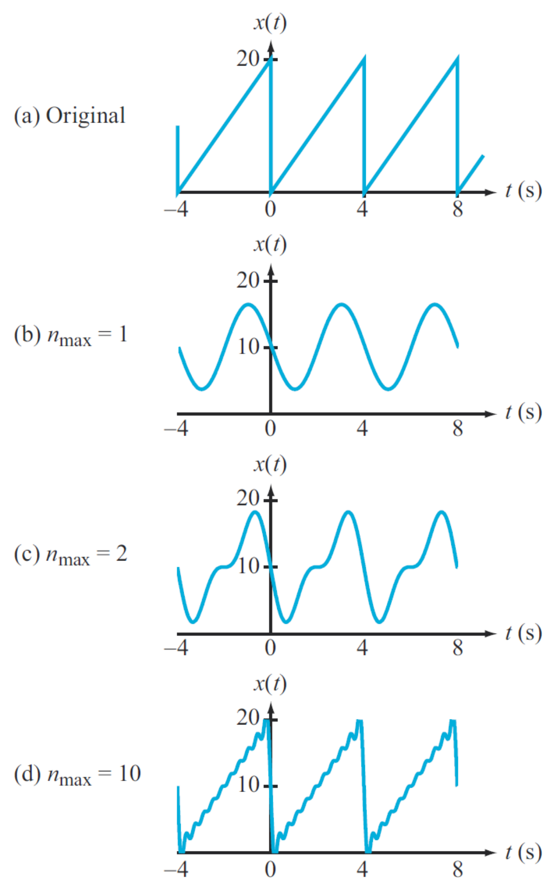
- is the infinite series

$$\upsilon_s(t) = \frac{12}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \cdots\right)$$

Fourier Series Expansions of Periodic Signals

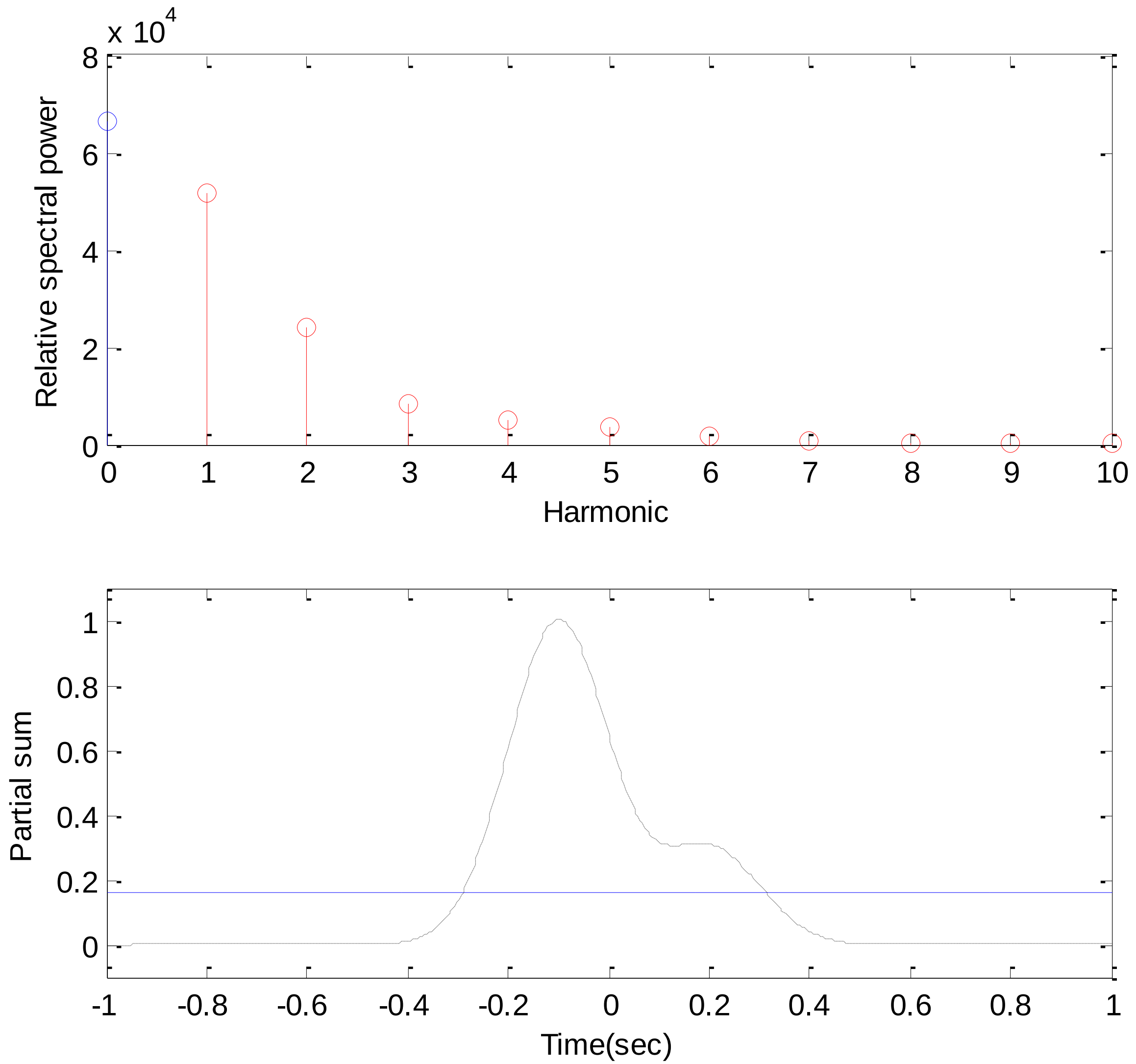
Adding more terms makes the Fourier series resemble more closely the original signal:





Fourier Series : Fourier Synthesis

Periodic signals (which in addition satisfy some not-very-restrictive extra conditions) can be approximated arbitrarily closely by a sum of the harmonics of a fundamental that has the same frequency as the signal.



DC

Fourier Series : Fourier Synthesis

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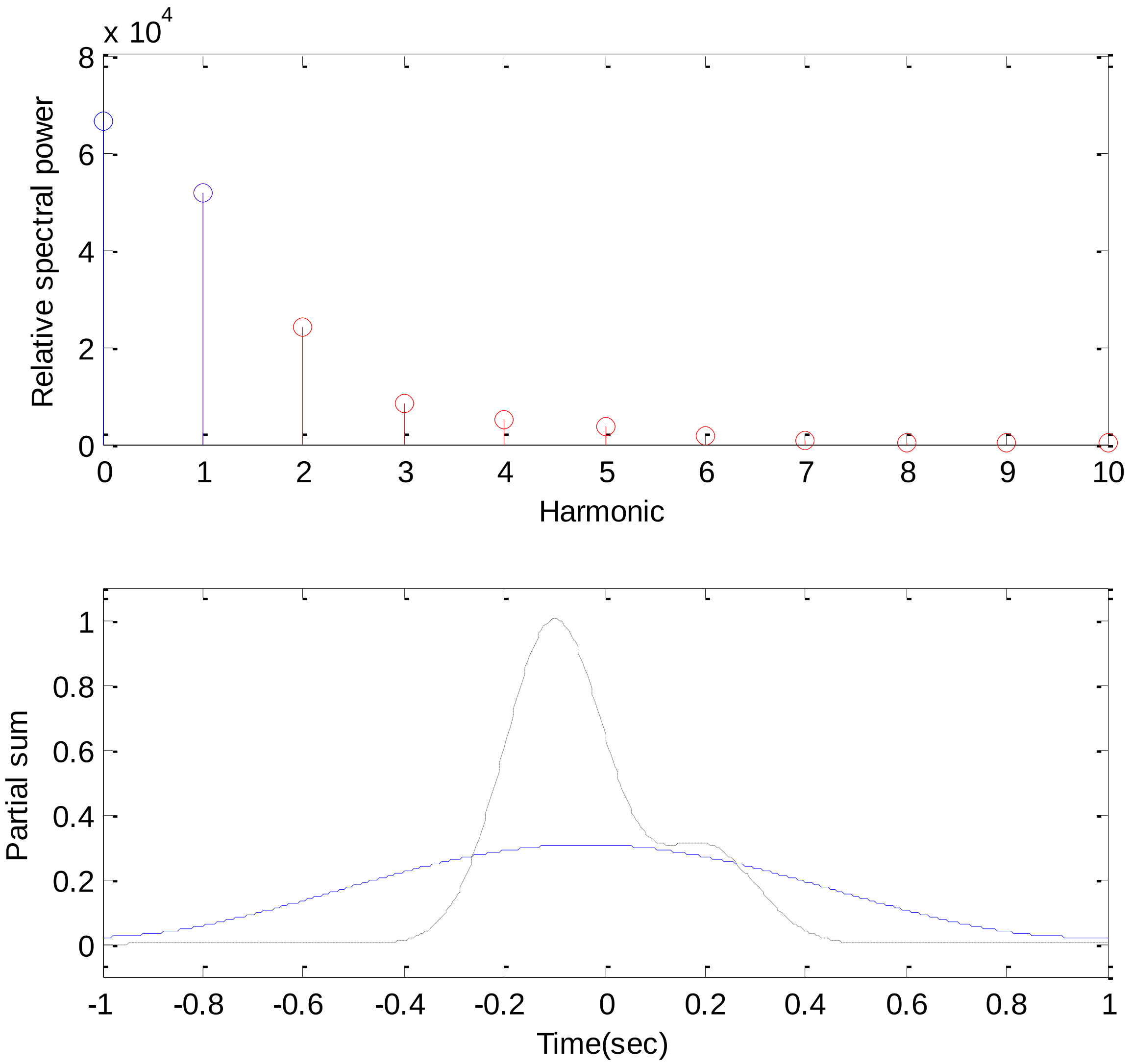

DC + 1 harmonic

Fourier Series : Fourier Synthesis

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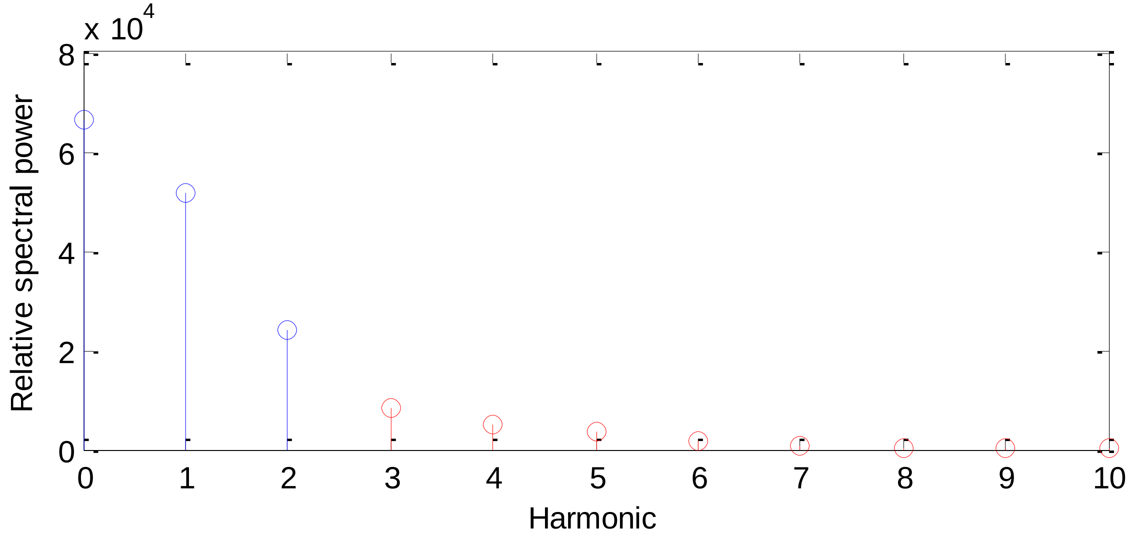
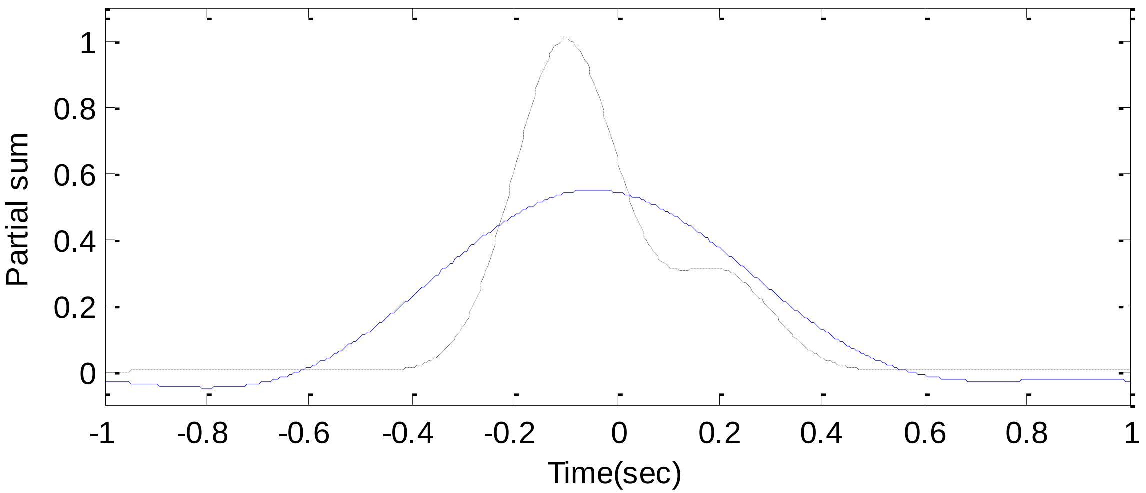


DC + 2 harmonics



Fourier Series : Fourier Synthesis

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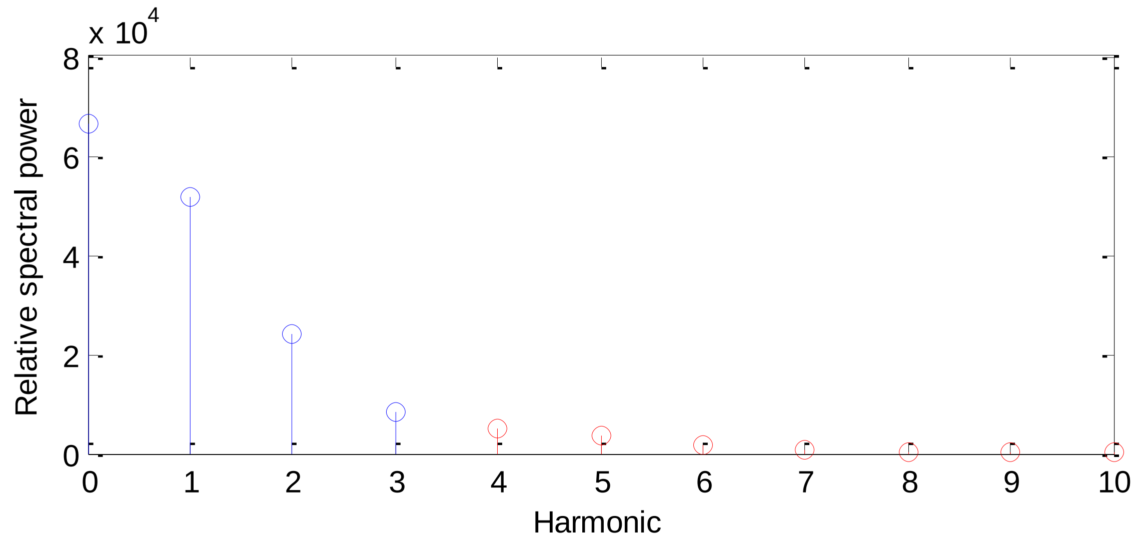
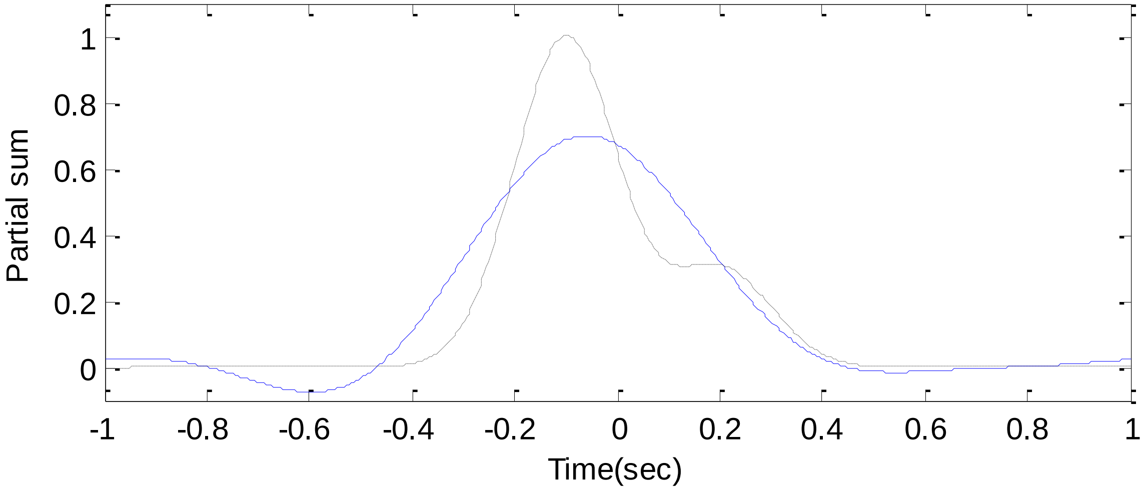


DC + 3 harmonics



Fourier Series : Fourier Synthesis

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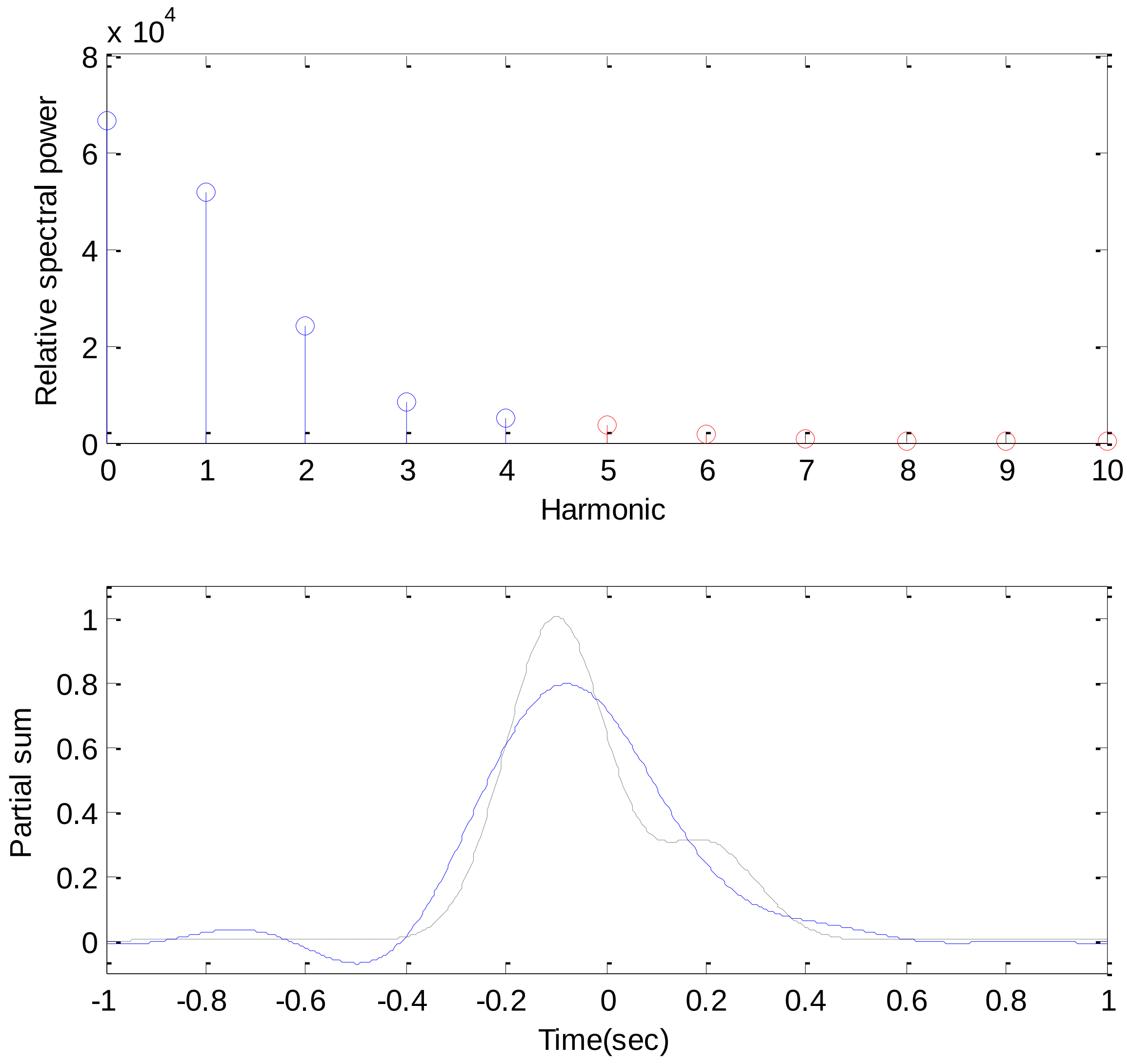

DC + 4 harmonics

Fourier Series : Fourier Synthesis

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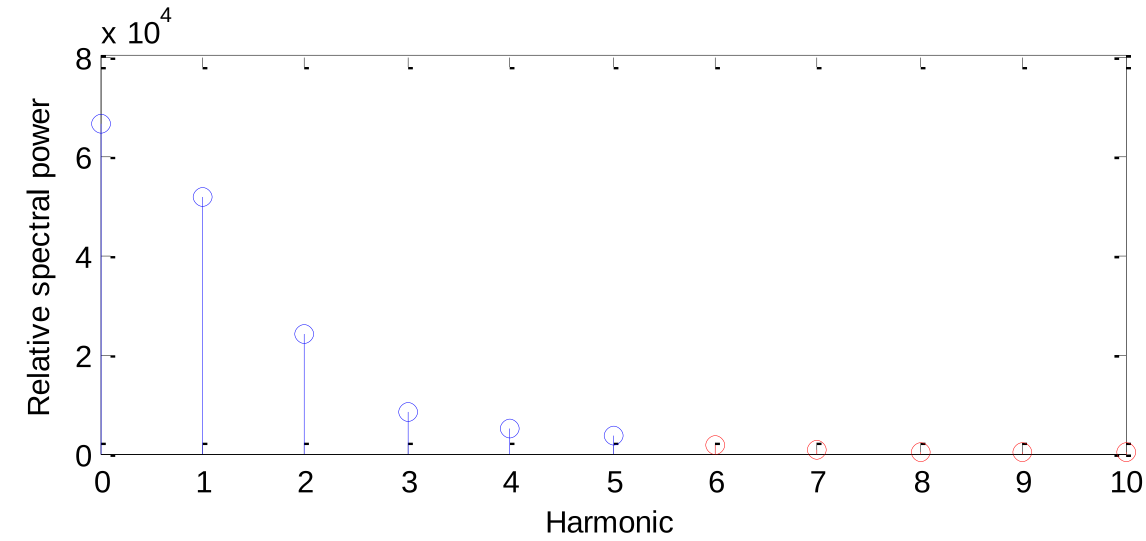
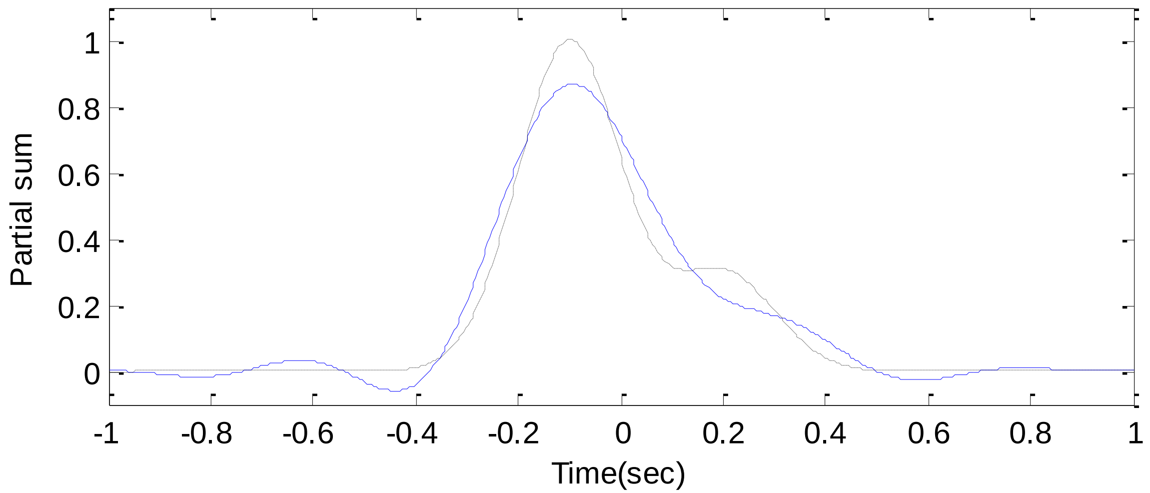


DC + 5 harmonics



Fourier Series : Fourier Synthesis

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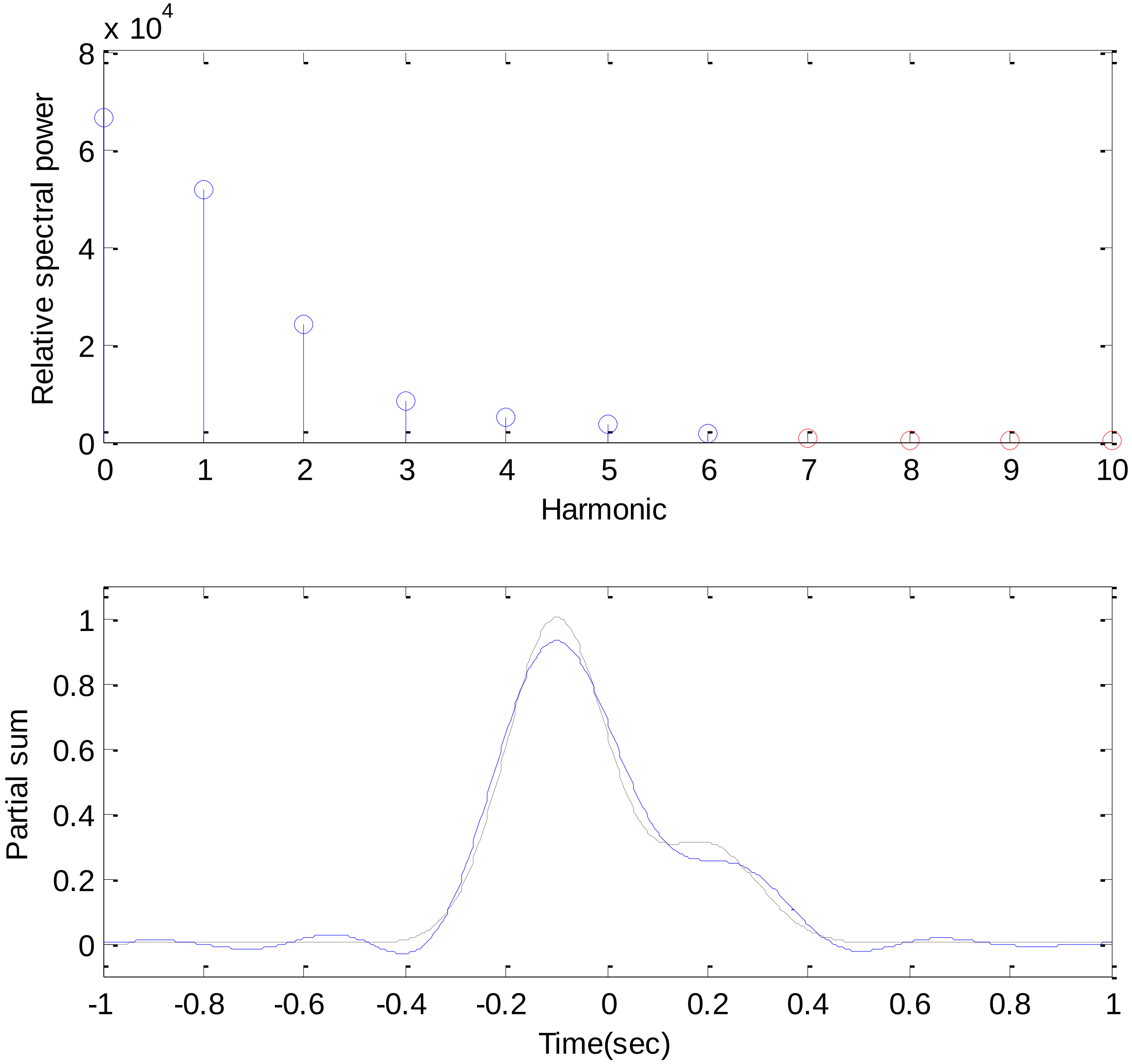
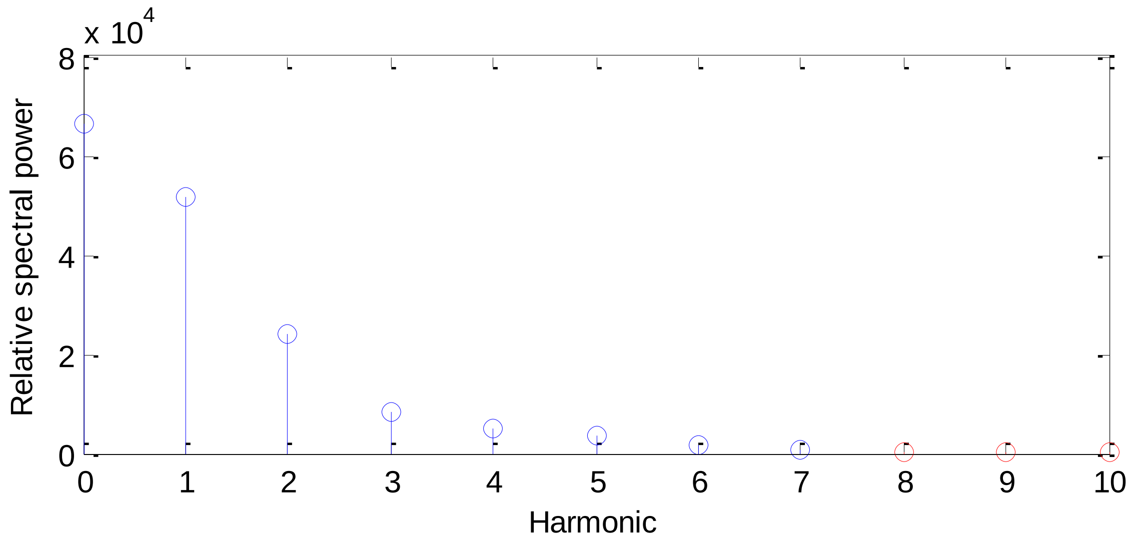


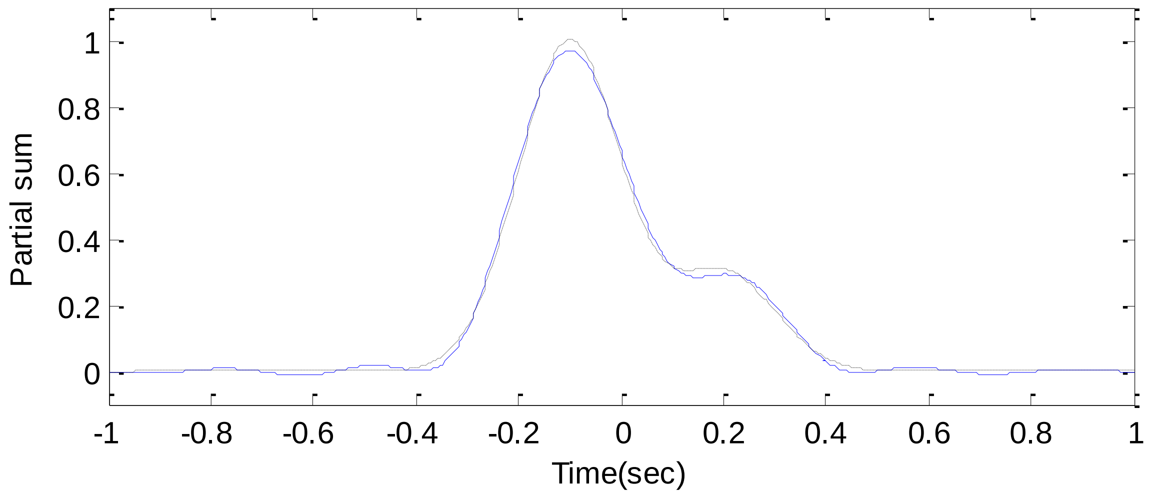
DC + 6 harmonics

Fourier Series : Fourier Synthesis

Periodic signals (which in addition satisfy some not-very-restrictive extra conditions) can be approximated arbitrarily closely by a sum of the harmonics of a fundamental that has the same frequency as the signal.



DC + 7 harmonics



Fourier Series : Fourier Synthesis

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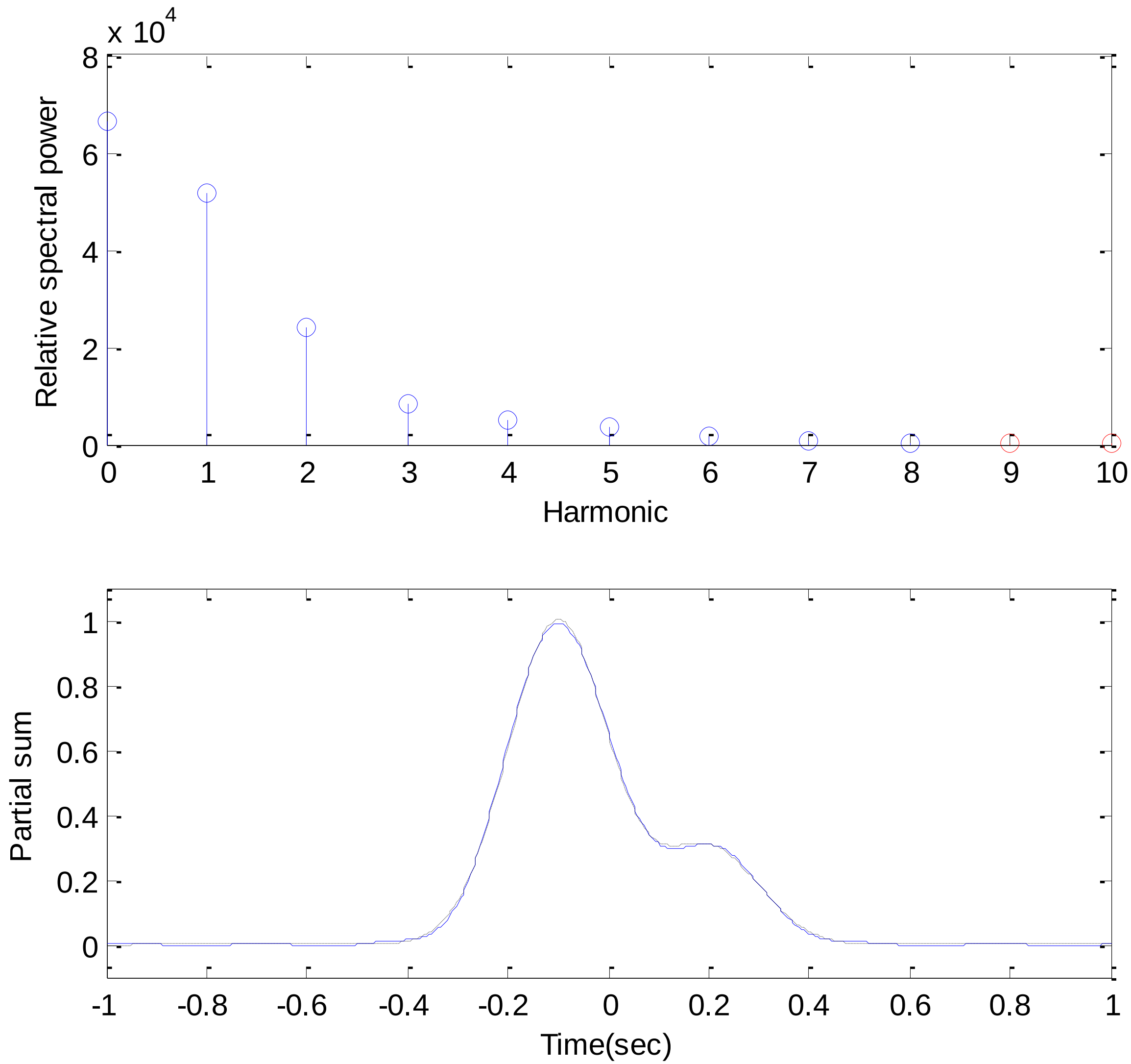
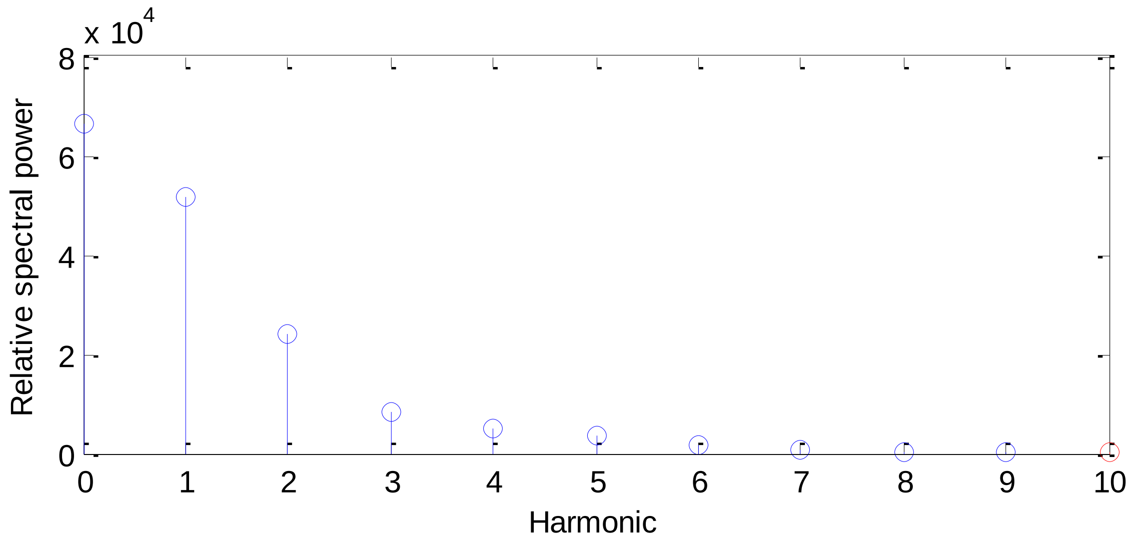
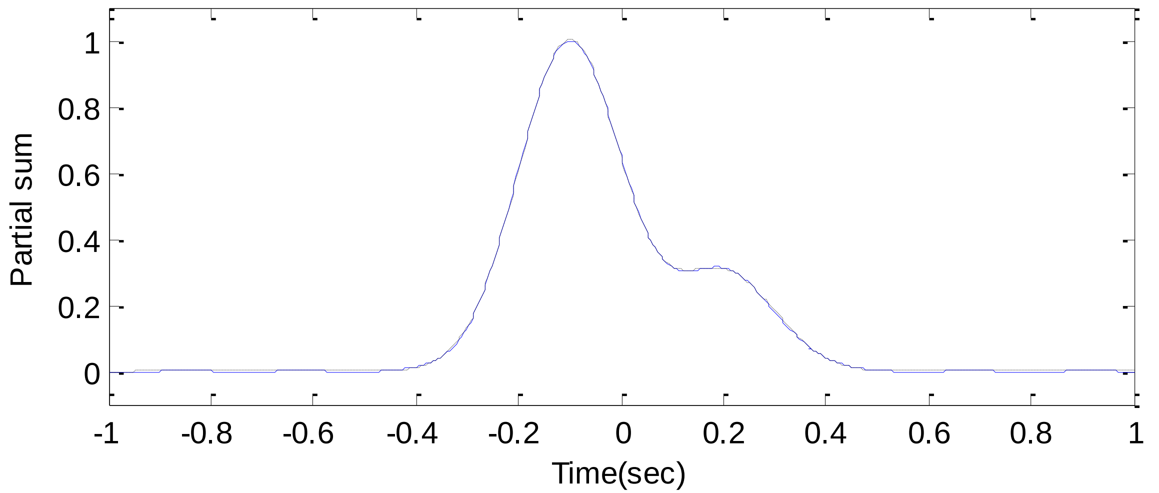


DC + 8 harmonics

Fourier Series : Fourier Synthesis

Periodic signals (which in addition satisfy some not-very-restrictive extra conditions) can be approximated arbitrarily closely by a sum of the harmonics of a fundamental that has the same frequency as the signal.

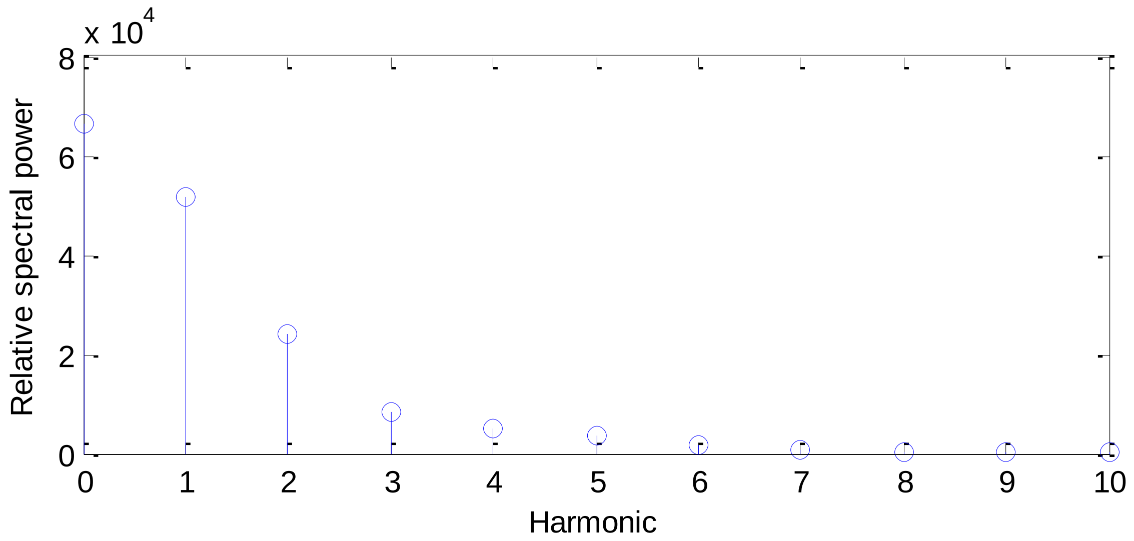




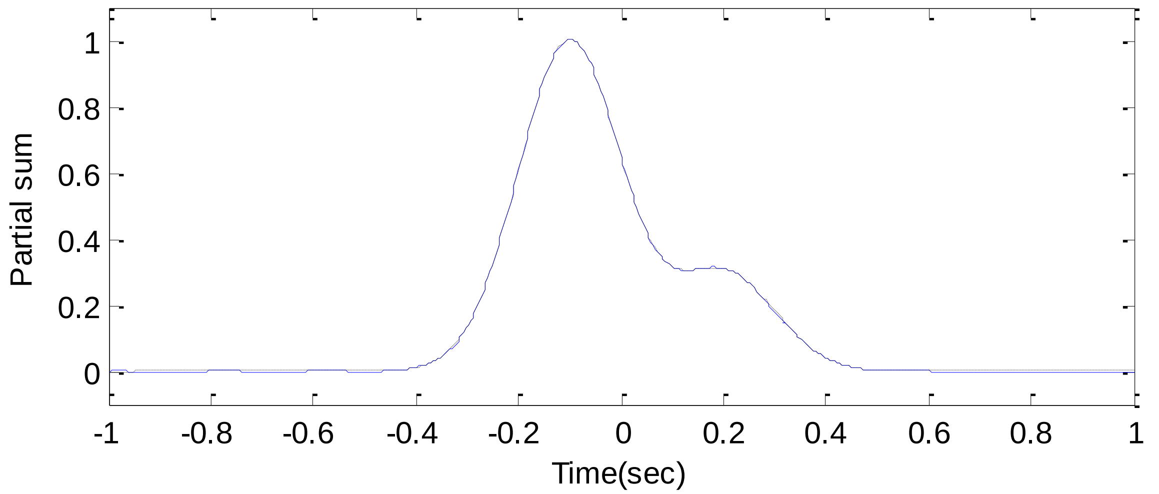
DC + 9 harmonics

Fourier Series : Fourier Synthesis

Periodic signals (which in addition satisfy some not-very-restrictive extra conditions) can be approximated arbitrarily closely by a sum of the harmonics of a fundamental that has the same frequency as the signal.



DC + 10 harmonics



Three Forms of Fourier Series Expansion:

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)].$$

(sine/cosine representation) (5.27)

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n),$$

(amplitude/phase representation)

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t},$$

(exponential representation)

Computing Coefficients of Fourier Series: Sine/Cosine Form of Fourier Series Expansion

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]. \tag{5.27}$$

(sine/cosine representation) (5.27)

Compute coefficients from a single period of x(t) using the formulae

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt,$$

$$a_n = \frac{2}{T_0} \int_0^{T_0} x(t)\, \cos(n\omega_0 t)\, dt,$$

and $$b_n = \frac{2}{T_0} \int_0^{T_0} x(t)\, \sin(n\omega_0 t)\, dt.$$

Computing Coefficients of Fourier Series: Amplitude/Phase Form of Fourier Series Expansion

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n),$$

(amplitude/phase representation)

Compute coefficients from a single period of x(t) using the formula

$$c_n = \sqrt{a_n^2 + b_n^2}$$

and

$$\phi_n = \begin{cases} -\tan^{-1}\left(\frac{b_n}{a_n}\right), & a_n > 0 \\ \pi - \tan^{-1}\left(\frac{b_n}{a_n}\right), & a_n < 0 \end{cases}$$

Computing Coefficients of Fourier Series:
Exponential Form of Fourier Series Expansion

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t},$$

(exponential representation).

Compute coefficients from a single period of x(t) using the formula

$$\mathbf{x}_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, e^{-jn\omega_0 t}\, dt.$$

Note this is much simpler than using the other two representations. <u>We will usually use this form</u>.

Summary of the Three Fourier Series Representations: Relations Between the Coefficients of the Representations

Table 5-3: Fourier series representations for a real-valued periodic function $x(t)$.

Cosine/Sine	Amplitude/Phase	Complex Exponential
$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$	$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n)$	$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t}$
$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt$	$c_n e^{j\phi_n} = a_n - jb_n$	$\mathbf{x}_n = \lvert\mathbf{x}_n\rvert e^{j\phi_n};\ \mathbf{x}_{-n} = \mathbf{x}_n^*;\ \phi_{-n} = -\phi_n$
$a_n = \frac{2}{T_0} \int_0^{T_0} x(t)\, \cos n\omega_0 t\, dt$	$c_n = \sqrt{a_n^2 + b_n^2}$	$\lvert\mathbf{x}_n\rvert = c_n/2;\ x_0 = c_0$
$b_n = \frac{2}{T_0} \int_0^{T_0} x(t)\, \sin n\omega_0 t\, dt$	$\phi_n = \begin{cases} -\tan^{-1}(b_n/a_n), & a_n > 0 \\ \pi - \tan^{-1}(b_n/a_n), & a_n < 0 \end{cases}$	$\mathbf{x}_n = \frac{1}{T_0} \int_0^{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$
$a_0 = c_0 = x_0;\ a_n = c_n \cos\phi_n;\ b_n = -c_n \sin\phi_n;\ \mathbf{x}_n = \frac{1}{2}(a_n - jb_n)$		

Fourier Series : Fourier Analysis - Orthogonality

Summary: 2 complex exponential harmonics with different frequencies are orthogonal (having a zero inner product) to each other.

Let's look at the reasoning:

If $f(t)$ and $g(t)$ are two (generally complex) periodic functions with period T, then a scalar product may be defined by the integral:

$$(f,g) = \int_0^T f^*(t)g(t)dt$$

For two complex exponential harmonics the scalar product becomes:

$$(e^{j\frac{2n\pi}{T}t}, e^{j\frac{2m\pi}{T}t}) = \int_0^T e^{-j\frac{2n\pi}{T}t} e^{j\frac{2m\pi}{T}t} dt$$

Fourier Series : Fourier Analysis - Orthogonality

With this definition it is not too hard to show that harmonics with different frequencies are orthogonal.

$$(e^{j\frac{2n\pi}{T}t}, e^{j\frac{2m\pi}{T}t}) = \int_0^T e^{-j\frac{2n\pi}{T}t} e^{j\frac{2m\pi}{T}t} dt$$

Product of exponentials

$$= \int_0^T e^{j\frac{2(m-n)\pi}{T}t} dt$$

Integral of exponential

$$= \frac{T}{j2(m-n)\pi} \left[e^{j\frac{2(m-n)\pi}{T}t} \right]_0^T$$

Expand expression

$$= \frac{T}{j2(m-n)\pi} \left[e^{j2(m-n)\pi} - e^0 \right]$$

Fourier Series : Fourier Analysis - Orthogonality

If n and m are integers then

$$e^{j2(m-n)\pi} = \cos(2(m-n)\pi) + \sin(2(m-n)\pi) = 1$$

So the terms inside the square brackets cancel

$$\left[e^{j2(m-n)\pi} - e^{0}\right] = [1-1] = 0$$

And provided n and m are different (so the denominator is non-zero) we get

$$\int_0^T e^{-j\frac{2n\pi}{T}t} e^{j\frac{2m\pi}{T}t} dt = 0 \text{ for } n \neq m$$

In words: harmonics with different frequencies are orthogonal.

Fourier Series : Fourier Analysis - Orthogonality

If n and m are equal then

$$\int_0^T e^{-j\frac{2n\pi}{T}t} e^{j\frac{2n\pi}{T}t} dt = \int_0^T dt = T$$

Thus the full orthogonality relation for the complex exponential harmonics is

$$\int_0^T e^{-j\frac{2n\pi}{T}t} e^{j\frac{2m\pi}{T}t} dt = \begin{cases} 0 & ; n \neq m \\ T & ; n = m \end{cases}$$

Fourier Series : Fourier Analysis - Fourier Coefficients

The formula relating the function *f(t)* to the Fourier coefficients F_n is the same as that giving vector components in terms of basis vectors.

The simplest way to derive the expression is to use the orthogonality relations for complex exponentials.

By multiplying both sides of the complex exponential Fourier expansion by each harmonic in turn and integrating over one cycle of the signal we get

$$\int_0^T e^{-j\frac{2m\pi}{T}t} f(t)dt = \int_\tau^{\tau+T} e^{-j\frac{2m\pi}{T}t} \sum_{n=-\infty}^{\infty} F_n e^{j\frac{2n\pi}{T}t} dt$$

Interchange order of integration and summation

$$= \sum_{n=-\infty}^{\infty} F_n \int_\tau^{\tau+T} e^{-j\frac{2m\pi}{T}t} e^{j\frac{2n\pi}{T}t} dt$$

Orthogonality

$$= TF_m$$

Fourier Series : Fourier Analysis - Fourier Coefficients

The expression for the complex exponential Fourier coefficients is thus

$$F_m = \frac{1}{T}\int_0^T f(t)e^{-j\frac{2m\pi}{T}t}dt$$

The limits on the integral can be any pair of times separated by one period. Another common choice is -T/2 and T/2.

Sine/cosine series coefficients can be derived from this result:

$$a_0 = \frac{1}{T}\int_0^T f(t)dt \qquad b_m = \frac{1}{2T}\int_0^T f(t)\sin\left(\frac{2m\pi}{T}t\right)dt$$

$$a_m = \frac{1}{2T}\int_0^T f(t)\cos\left(\frac{2m\pi}{T}t\right)dt \qquad m = 1,2,3,...$$

Fourier Series : Fourier Analysis - Power

Other properties of ordinary vectors also extend to Fourier series.

An important example follows from the definition of the length or *norm* (or magnitude) of a vector.

A commonly used norm for ordinary vectors is defined by

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$$

If the vector **a** is written as a sum of orthogonal basis vectors, then then this formula can be written as:

$$\|\mathbf{a}\|^2 = \mathbf{a}\cdot\mathbf{a} = \sum_k a_k^2 \mathbf{e}_k \cdot \mathbf{e}_k = \sum_k a_k^2 \|\mathbf{e}_k\|^2$$

Fourier Series : Fourier Analysis - Power

Suppose we replace the vectors with functions and the dot product with the scalar product defined earlier, then

$$\|\mathbf{a}\|^2 \to \|f\|^2 = (f, f) = \int_0^T |f(t)|^2 dt$$

ie. the squared norm of the signal is just the energy in a single cycle.

Also

$$\|\mathbf{e}_n\|^2 \to \left\| e^{j\frac{2n\pi}{N}t} \right\|^2 = \left(e^{j\frac{2n\pi}{N}t}, e^{j\frac{2n\pi}{N}t} \right) = T$$

and

$$a_n^2 \to |F_n|^2$$

Fourier Series : Fourier Analysis - Parseval's relation

Then, if we use the vector result we get

$$\int_0^T |f(t)|^2 dt = T \sum_n |F_n|^2$$

Equivalently, on dividing by T

$$\frac{1}{T} \int_0^T |f(t)|^2 dt = \sum_n |F_n|^2$$

This is Parseval's relation. Its physical interpretation is that the power in a signal is equal to the sum of the powers in each harmonic (it is not too difficult to show that $|F_n|^2$ is the power in the n-th complex harmonic).

This may seem obvious, but it is usually not true that the power in the sum of two or more signals is equal to the sum of the powers of the component signals.

Fourier Series : Fourier Analysis - Spectra

The quantities

$$S_0 = \left|F_0\right|^2 \quad , \quad S_n = \left|F_n\right|^2 + \left|F_{-n}\right|^2 \quad ; \quad n = 1,2,3,\cdots$$

are proportional to the power contained in the component with frequency ,

$$\omega_n = n\frac{2\pi}{T} \; ; n = 0,1,2,3,\cdots$$

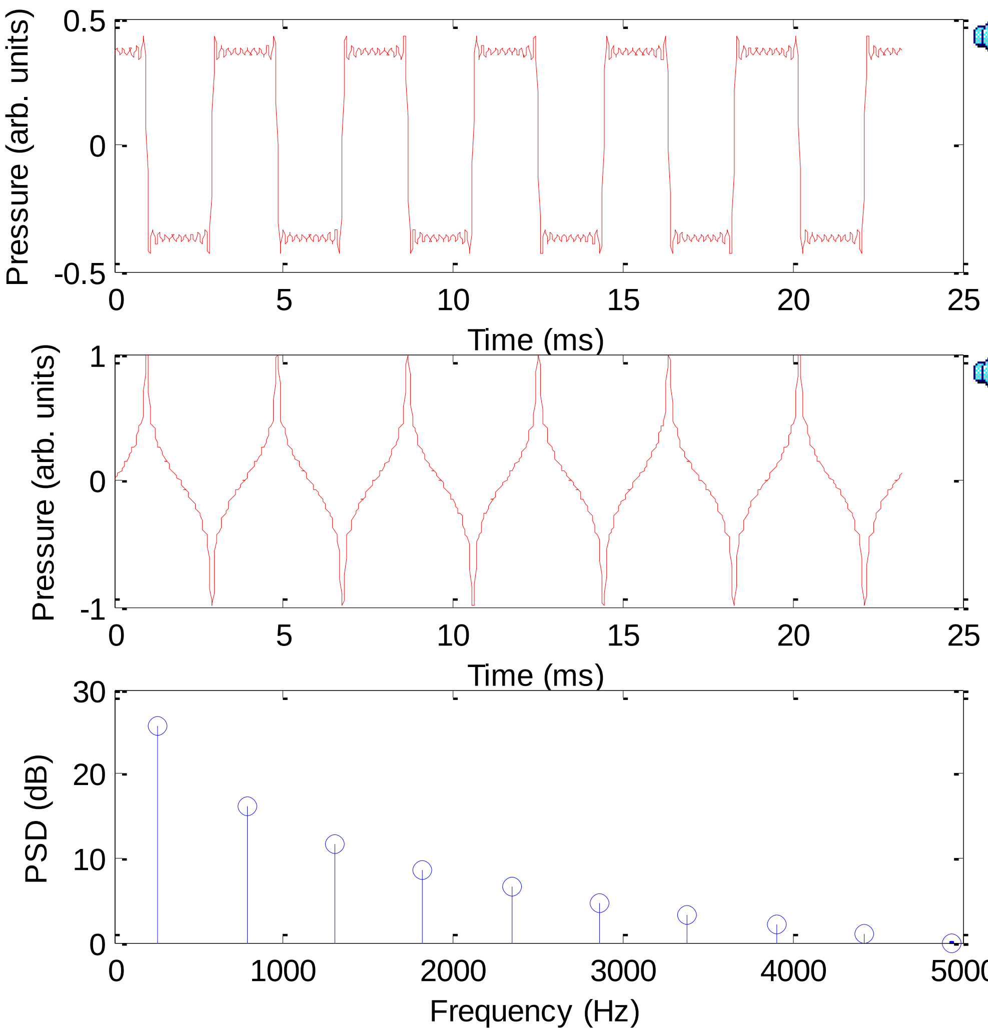
and thus give the (one-sided) distribution of power with frequency.

The set of such quantities is the (one-sided) power spectrum of the signal.

Fourier Series : Fourier Analysis - Spectra

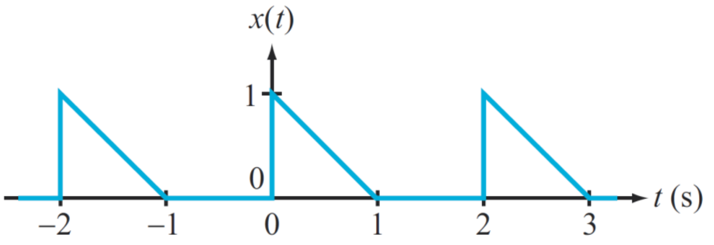
The information in many signals is more easily extracted from its power spectrum than from the original signal (sound signals are an example - the human ear appears to be insensitive to the phases of the harmonics).

Note, however, that the power spectrum is not a complete description of the signal, information about the relative phases of the harmonics is lost in forming the power spectrum



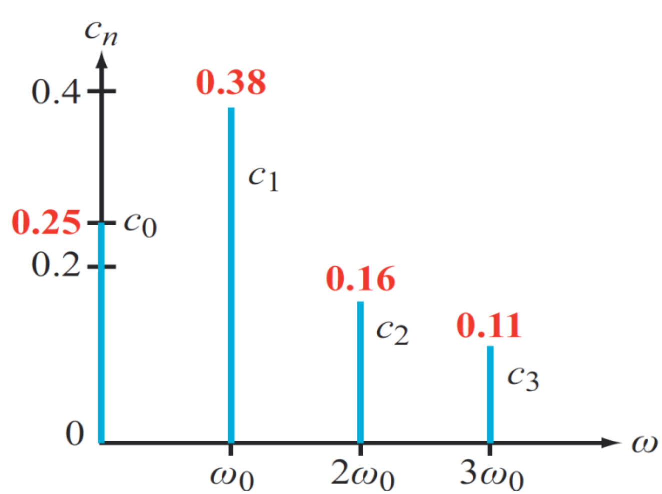
One-Sided Line Spectrum of x(t)

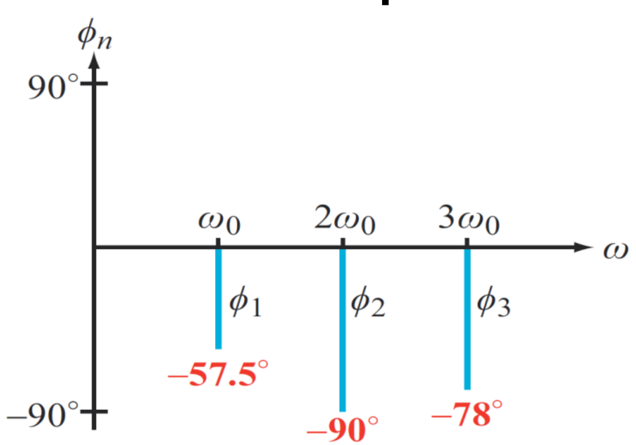
- Plot of amplitudes c_n and phases ϕ_n vs. frequency ω.
- Example: x(t) has period=2.



$$c_n = \begin{cases} \left(\dfrac{4}{n^4\pi^4} + \dfrac{1}{n^2\pi^2}\right)^{1/2} & \text{for } n = \text{odd}, \\ \dfrac{1}{n\pi} & \text{for } n = \text{even} \end{cases}$$

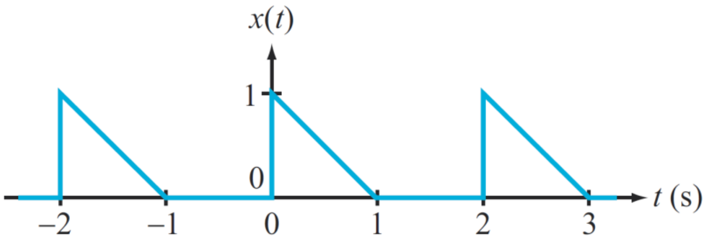
- Amplitude spectrum: Phase spectrum:





Two-Sided Line Spectrum of x(t)

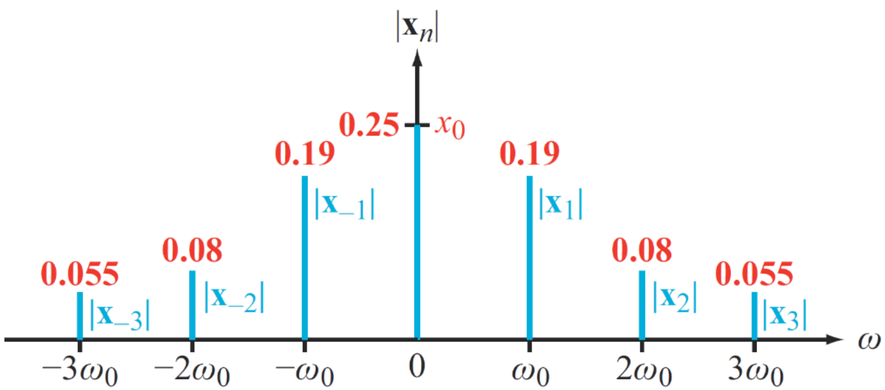
- Plot of magnitudes $|\mathbf{x}_n|$ and phases ϕ_n vs. frequency ω .
- Example: x(t) has period=2.

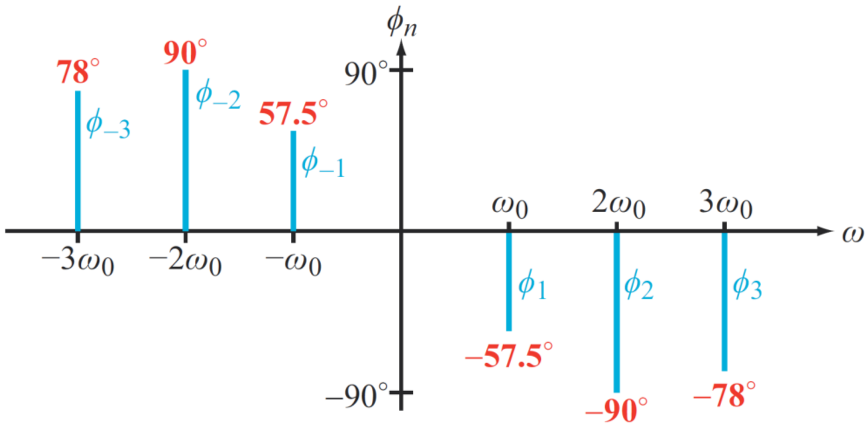


$$c_n = \begin{cases} \left(\dfrac{4}{n^4\pi^4} + \dfrac{1}{n^2\pi^2} \right)^{1/2} & \text{for } n = \text{odd}, \\ \dfrac{1}{n\pi} & \text{for } n = \text{even} \end{cases}$$

- Magnitude spectrum:

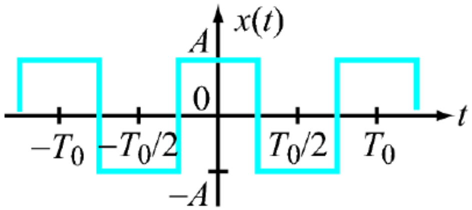
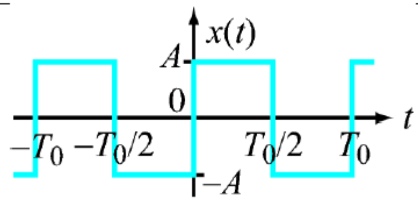
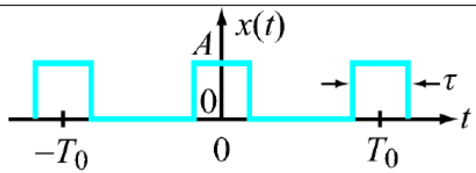
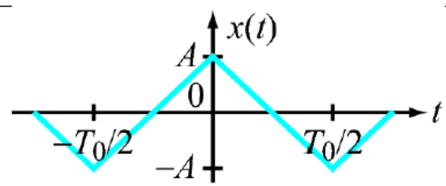
Phase spectrum:





Fourier Series Expansions for Some General Forms of Periodic Signals

Table 5-4: Fourier series expressions for a select set of periodic waveforms.

	Waveform	Fourier Series
1. Square Wave		$x(t) = \sum_{n=1}^{\infty} \frac{4A}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{2n\pi t}{T_0}\right)$
2. Time-Shifted Square Wave		$x(t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{4A}{n\pi} \sin\left(\frac{2n\pi t}{T_0}\right)$
3. Pulse Train		$x(t) = \frac{A\tau}{T_0} + \sum_{n=1}^{\infty} \frac{2A}{n\pi} \sin\left(\frac{n\pi\tau}{T_0}\right) \cos\left(\frac{2n\pi t}{T_0}\right)$
4. Triangular Wave		$x(t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{8A}{n^2\pi^2} \cos\left(\frac{2n\pi t}{T_0}\right)$

Symmetry Simplifies Computation

Even Symmetry: $x(t) = x(-t)$

$$a_0 = \frac{2}{T_0} \int_0^{T_0/2} x(t)\, dt$$

$$a_n = \frac{4}{T_0} \int_0^{T_0/2} x(t)\ \cos(n\omega_0 t)\, dt$$

$$b_n = 0$$

$$c_n = |a_n|, \qquad \phi_n = \begin{cases} 0 & \text{if } a_n > 0 \\ 180^\circ & \text{if } a_n < 0 \end{cases}$$

Odd Symmetry: $x(t) = -x(-t)$

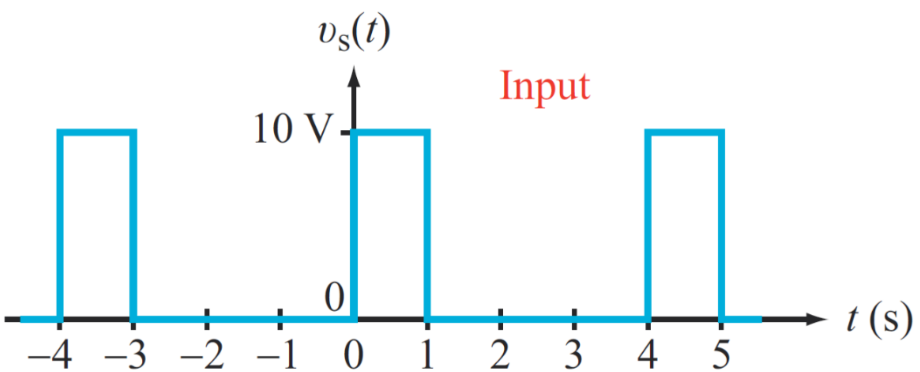
$$a_0 = 0 \qquad a_n = 0$$

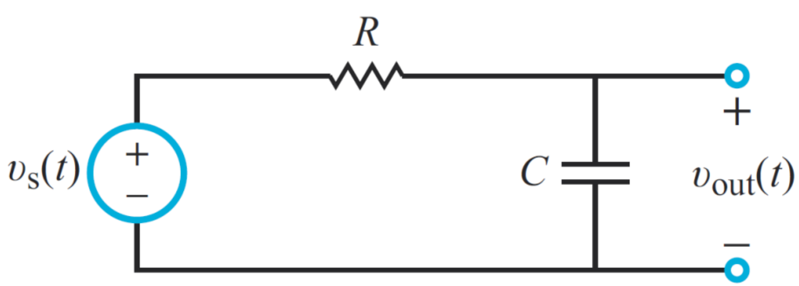
$$b_n = \frac{4}{T_0} \int_0^{T_0/2} x(t)\ \sin(n\omega_0 t)\, dt$$

$$c_n = |b_n| \qquad \phi_n = \begin{cases} -90^\circ & \text{if } b_n > 0 \\ 90^\circ & \text{if } b_n < 0 \end{cases}$$

Circuit Analysis using Fourier Series

- Compute the voltage response of the RC circuit shown to the input square wave shown, using Fourier series. The time constant RC of the circuit is RC=2 s.





Circuit Analysis using Fourier Series

- The input waveform has the Fourier series expansion

$$v_s(t) = a_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n)$$

$$c_n \angle \phi_n = a_n - jb_n = \frac{10}{n\pi}\left[\sin\frac{n\pi}{2} - j\left(1 - \cos\frac{n\pi}{2}\right)\right]$$

- The RC circuit has the transfer function (RC=2 s)

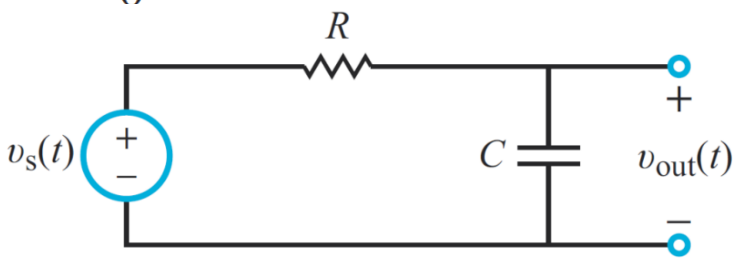
$$\mathbf{H}(\omega) = \frac{1}{\sqrt{1+4\omega^2}}\, e^{-j\tan^{-1}(2\omega)}$$

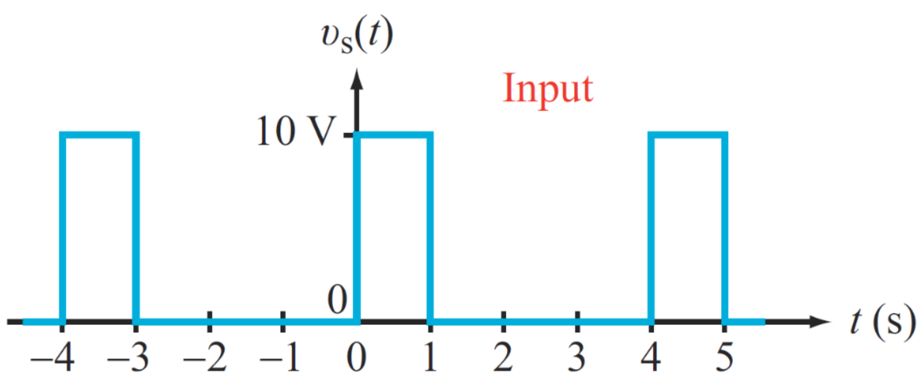
Circuit Analysis using Fourier Series

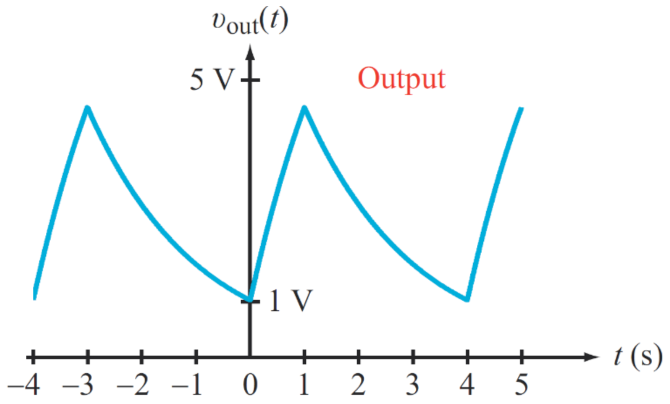
- The output waveform has the Fourier series expansion

$$v_{\text{out}}(t) = 2.5 + \sum_{n=1}^{\infty} \Re\mathfrak{e}\left\{ c_n \frac{1}{\sqrt{1+4n^2\omega_0^2}} e^{j[n\omega_0 t + \phi_n - \tan^{-1}(2n\omega_0)]} \right\}$$

- Plot of the output waveform







Parseval's Theorem for Fourier Series:

Average Power is the same whether computed in the time domain

$$P_x = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\, dt = a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)/2, \tag{5.58a}$$

$$P_x = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\, dt = c_0^2 + \sum_{n=1}^{\infty} c_n^2/2, \tag{5.58b}$$

$$P_x = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |\mathbf{x}_n|^2. \tag{5.58c}$$

Recall that average power of $c_n \cos(n\omega_0 t + \phi_n)$ is $c_n^2/2$

Phasor Analysis: Basics

- If an LTI system is described by a differential equation with an eternal sinusoidal input, then phasor analysis is a simple procedure for computing the system response.
- The system response is also an eternal sinusoidal signal.
- The phasor associated with signal $\upsilon(t) = V_0 \cos(\omega t + \phi)$ is the complex number $\mathbf{V} = V_0 e^{j\phi}$

$$\upsilon(t) = V_0 \cos(\omega t + \phi) \quad \longleftrightarrow \quad \mathbf{V} = V_0 e^{j\phi}$$

- If $\mathbf{X} = |\mathbf{X}| e^{j\phi}$ then $\mathfrak{Re}[|\mathbf{X}| e^{j\phi} e^{j\omega t}] = |\mathbf{X}| \cos(\omega t + \phi)$

Phasor Analysis: Basics

- The effects of differentiation and integration are:

$x(t)$		$\mathbf{X}$
$A\cos\omega t$	$\leftrightarrow$	A
$A\cos(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j\phi}$
$-A\cos(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j(\phi\pm\pi)}$
$A\sin\omega t$	$\leftrightarrow$	$Ae^{-j\pi/2}=-jA$
$A\sin(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j(\phi-\pi/2)}$
$-A\sin(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j(\phi+\pi/2)}$
$\frac{d}{dt}[A\cos(\omega t+\phi)]$	$\leftrightarrow$	$j\omega Ae^{j\phi}$
$\int A\cos(\omega t'+\phi)\,dt'$	$\leftrightarrow$	$\frac{1}{j\omega}Ae^{j\phi}$

$$i(t)=\Re\mathfrak{e}[\mathbf{I}e^{j\omega t}]$$

$$\frac{di}{dt} \leftrightarrow j\omega\mathbf{I}.$$

$$\int i\,dt' \leftrightarrow \frac{\mathbf{I}}{j\omega}$$

- Use these properties to convert manipulations of sines and cosines into manipulations of complex numbers.

Phasor Analysis: Example

- Use phasor analysis to solve the differential equation (a=300 and b=50000)

$$\frac{d^2y}{dt^2} + a\,\frac{dy}{dt} + by = 10\sin(100t + 60°)$$

- The phasor associated with the input is $\mathbf{X} = 10e^{-j30°}$
- The differential equation becomes algebraic

$$(j\omega)^2\mathbf{Y} + j\omega a\mathbf{Y} + b\mathbf{Y} = 10e^{-j30°}$$

- The phasor associated with the output is then found as

$$\mathbf{Y} = \frac{10e^{-j30°}}{b - \omega^2 + j\omega a} = \frac{10e^{-j30°}}{10^4(4 + j3)} = \frac{10^{-3}e^{-j30°}}{5e^{j36.87°}} = 0.2 \times 10^{-3}e^{-j66.87°}$$

- The response is then $y(t) = 0.2 \times 10^{-3}\cos(100t - 66.87°)$