LECTURE BLOCK 3B: FOURIER SERIES OF PERIODIC SIGNALS

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Fourier Analysis Techniques

We will learn:

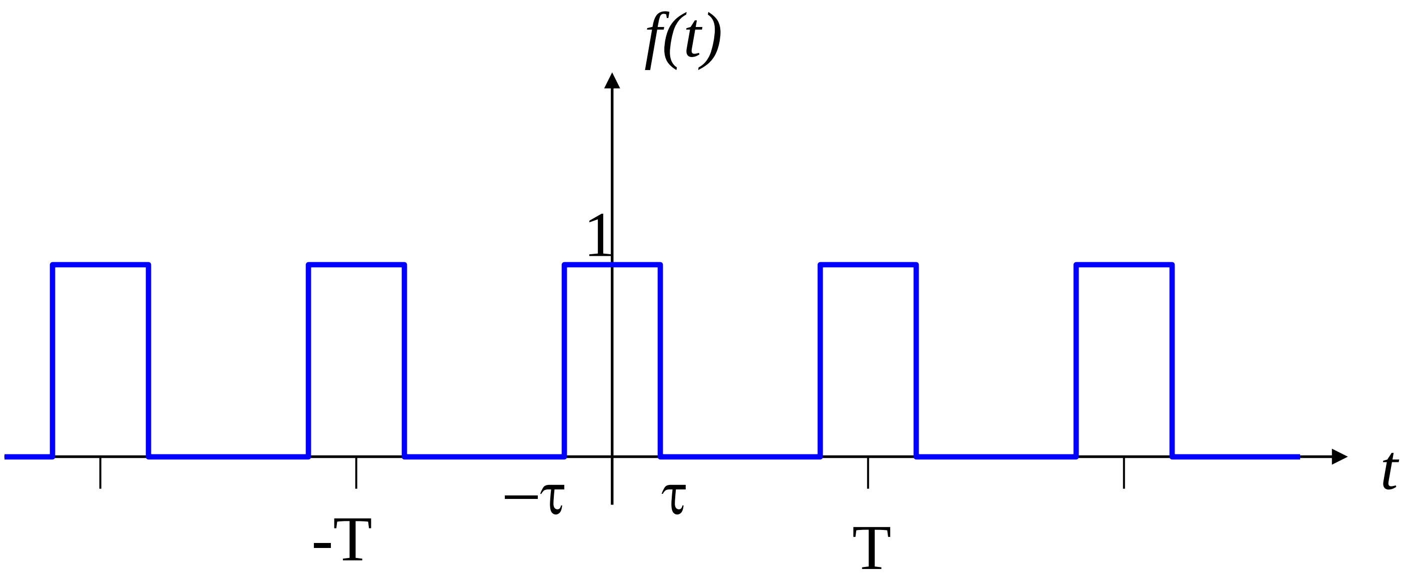
- Express periodic signals in term of Fourier Series.
- Use Fourier Series to analyse system driven by continuous periodic signals.
- Apply Parseval's theorem to compute the power or energy contained in a signal.
- Study Fourier Series' examples, and properties,
- Look at Discrete Time Fourier Series.

Fourier Series : Fourier Analysis - Example

Rectangular Pulse Train

As a useful example of Fourier analysis, consider the ideal periodic rectangular pulse train:

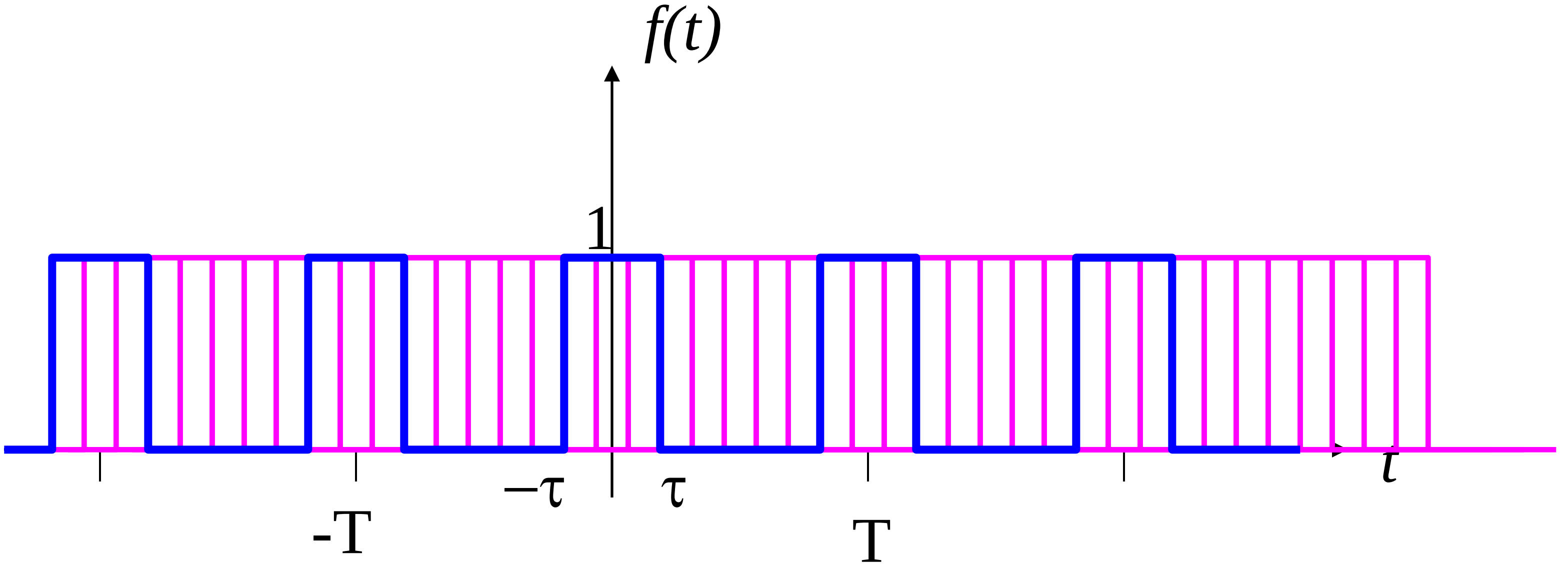
$$f(t) = \sum_{k=-\infty}^{\infty} P_{\tau}(t - kT) \quad ; \quad 0 < \tau < \frac{T}{2}$$



Fourier Series : Fourier Analysis - Example

Clearly this is periodic with period T:

$$f(t+T) = \sum_{k=-\infty}^{\infty} P_\tau(t+T-kT) = \sum_{k=-\infty}^{\infty} P_\tau(t-(k-1)T) = \sum_{k'=-\infty}^{\infty} P_\tau(t-k'T) = f(t)$$

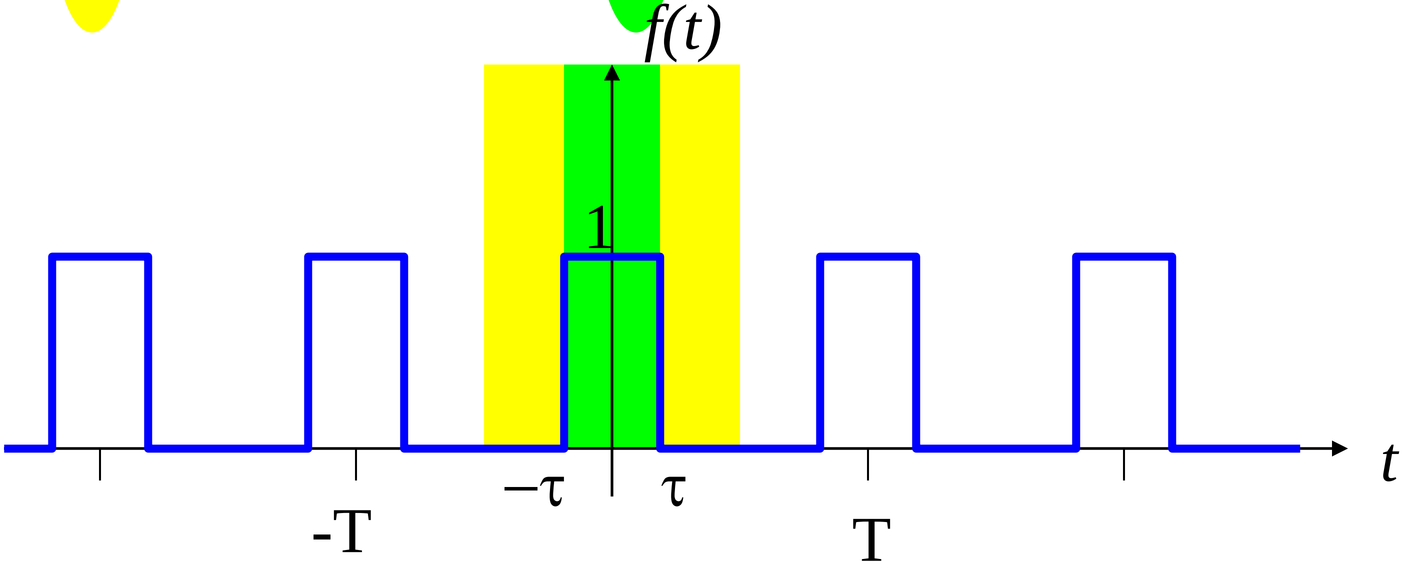


Fourier Series : Fourier Analysis - Example

So we can write

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn(2\pi/T)t}$$

$$F_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} P_\tau(t) e^{-jn(2\pi/T)t}\,dt = \frac{1}{T}\int_{-\tau}^{\tau} e^{-jn(2\pi/T)t}\,dt = \frac{1}{-jn2\pi}\left[e^{jn(2\pi/T)}\right]_{-\tau}^{\tau}$$



Fourier Series : Fourier Analysis - Example

Hence

$$F_n = \frac{1}{-jn2\pi}\left[e^{-jn(2\pi/T)\tau} - e^{jn(2\pi/T)\tau}\right] = \frac{\sin(n2\pi\tau / T)}{n\pi}$$

It is convenient to tidy this up a little by defining the *sinc* function

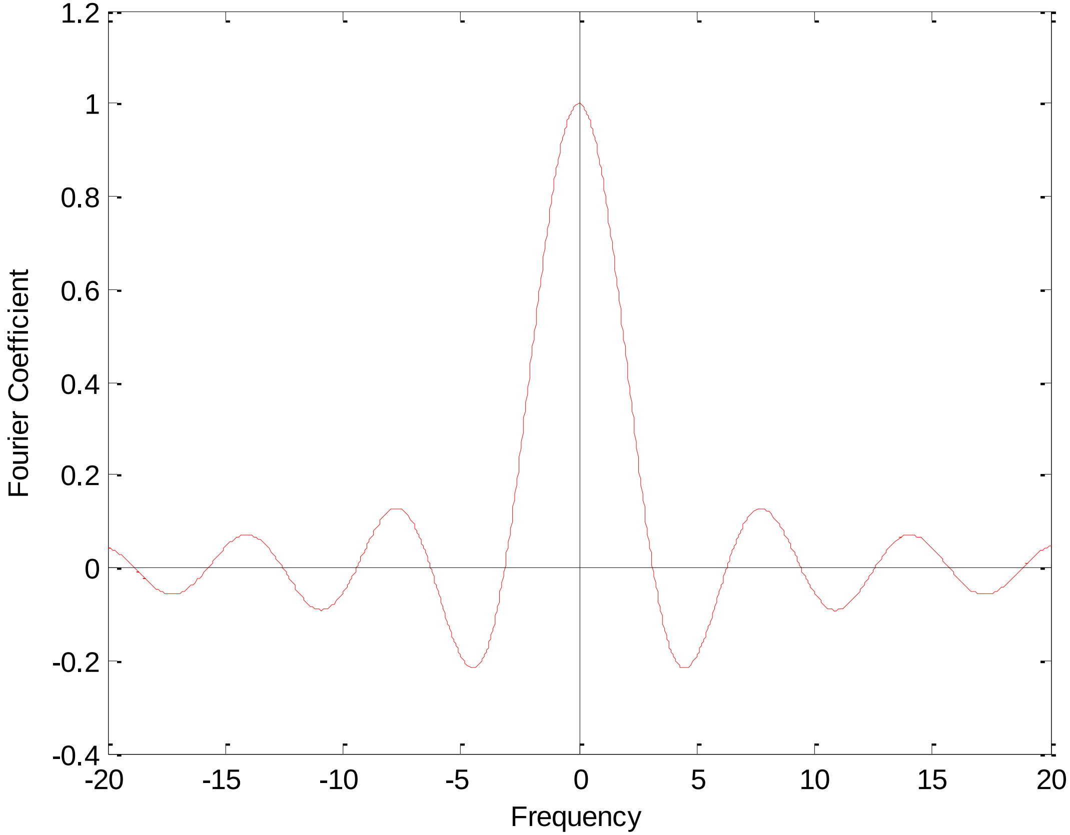
$$\operatorname{sinc}(x) = \frac{\sin x}{x}$$

So that

$$F_n = \frac{2\tau}{T}\operatorname{sinc}(n2\pi\tau / T)$$

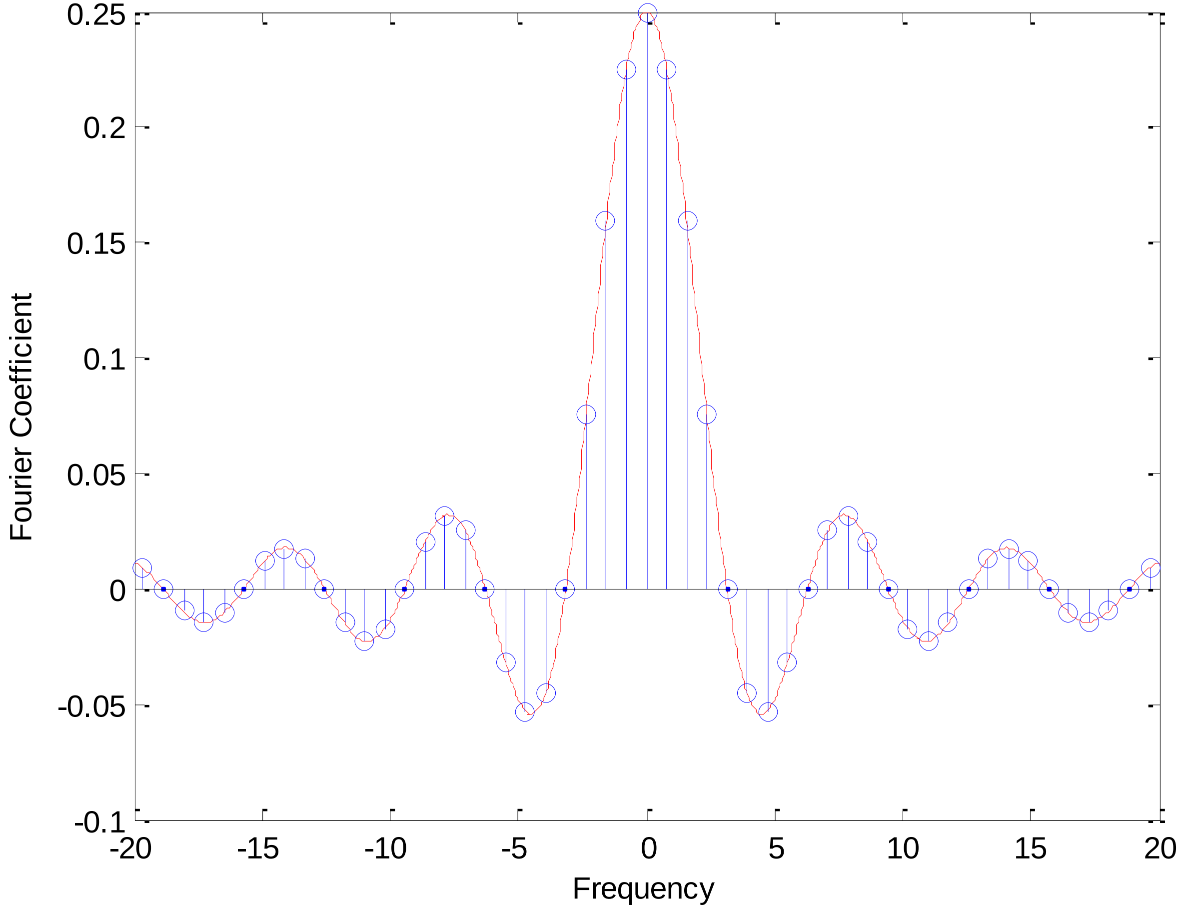
Fourier Series : Fourier Analysis - Example

The *sinc* function gives the relative amplitude of the Fourier coefficients for a rectangular pulse train. The period of the wave determines where the harmonics lie on the frequency axis.



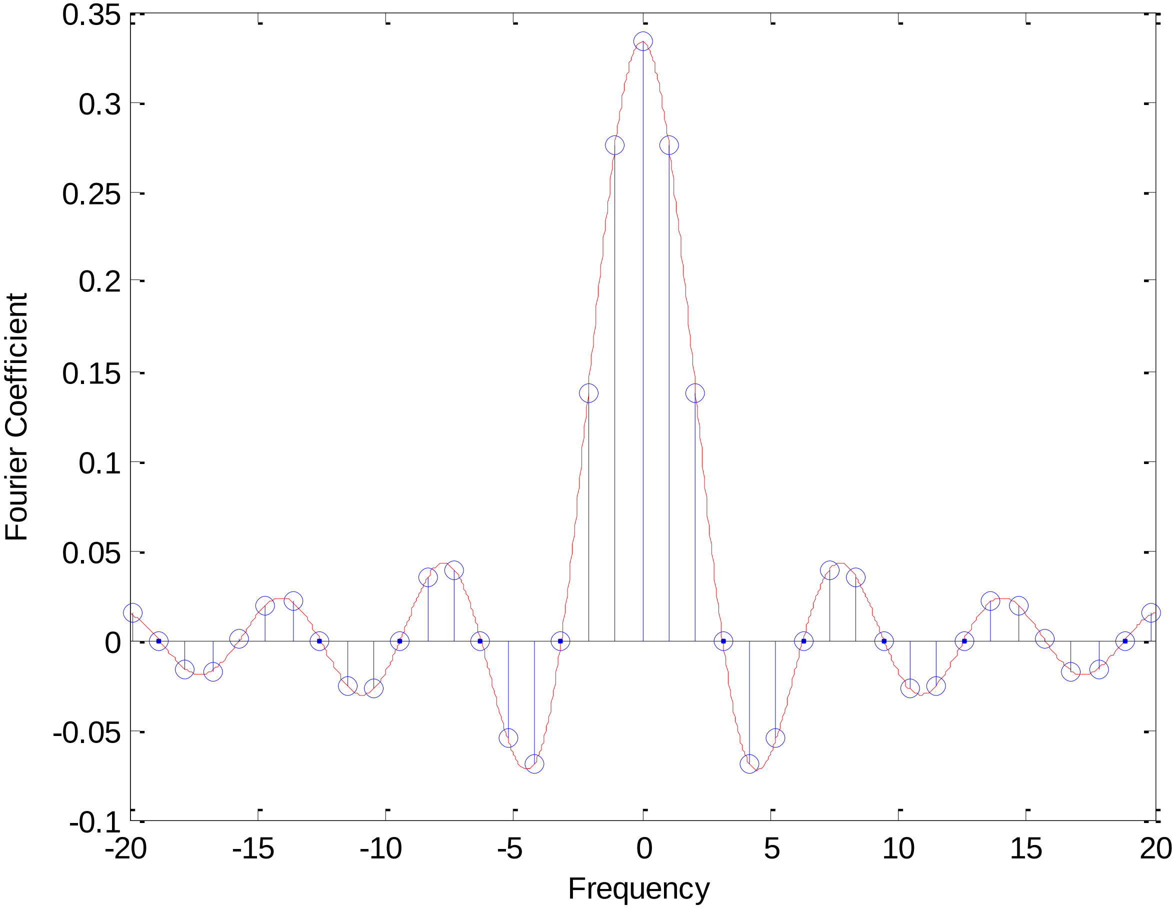
Fourier Series : Fourier Analysis - Example

If τ = T/8 there are four harmonics per lobe of the *sinc* function. (One of which has zero amplitude.)



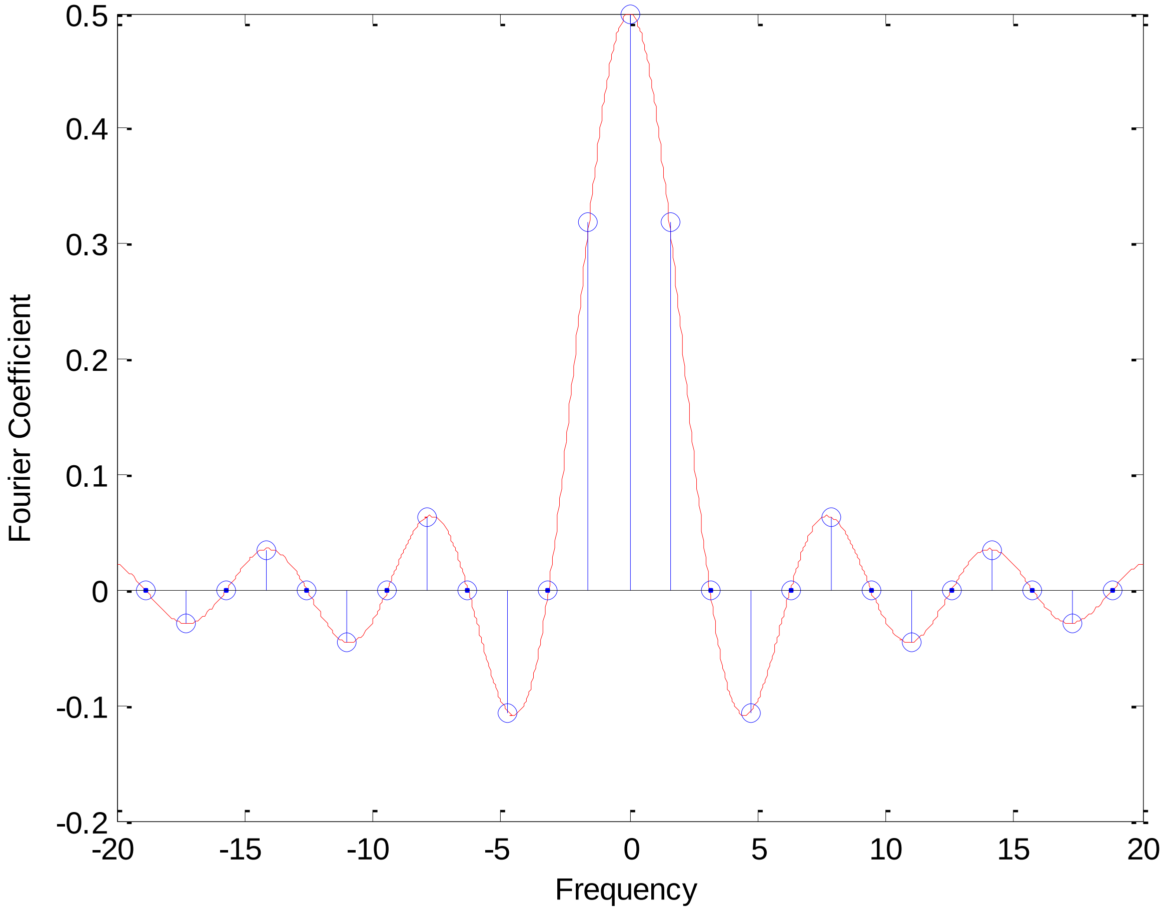
Fourier Series : Fourier Analysis - Example

If τ = T/6 there are three harmonics per lobe of the *sinc* function. (One of which also has zero amplitude.)



Fourier Series : Fourier Analysis - Example

For the square wave τ = T/4 and there are two harmonics per lobe of the *sinc* function. (One of which also has zero amplitude.)



Fourier Series : Fourier Analysis - Example

The square wave (τ = T/4) is of special interest since it occurs so frequently in practice.

$$F_n = \frac{1}{2}\operatorname{sinc}(n\pi/2) = \begin{cases} \frac{1}{2} & : \quad n = 0 \\ 0 & : \quad n = 2m \\ \dfrac{(-1)^m}{(2m+1)\pi} & : \quad n = 2m+1 \end{cases}$$

Thus all the even harmonics are zero (except the DC term n = 0). The average signal power in the square wave is

$$\overline{P} = \frac{1}{T}\int_{-T/2}^{T/2} f^2(t)dt = \frac{1}{T}\int_{-T/2}^{T/2} P_\tau^2(t)dt = \frac{1}{T}\int_{-\tau}^{\tau} dt = \frac{2\tau}{T} = \frac{1}{2}$$

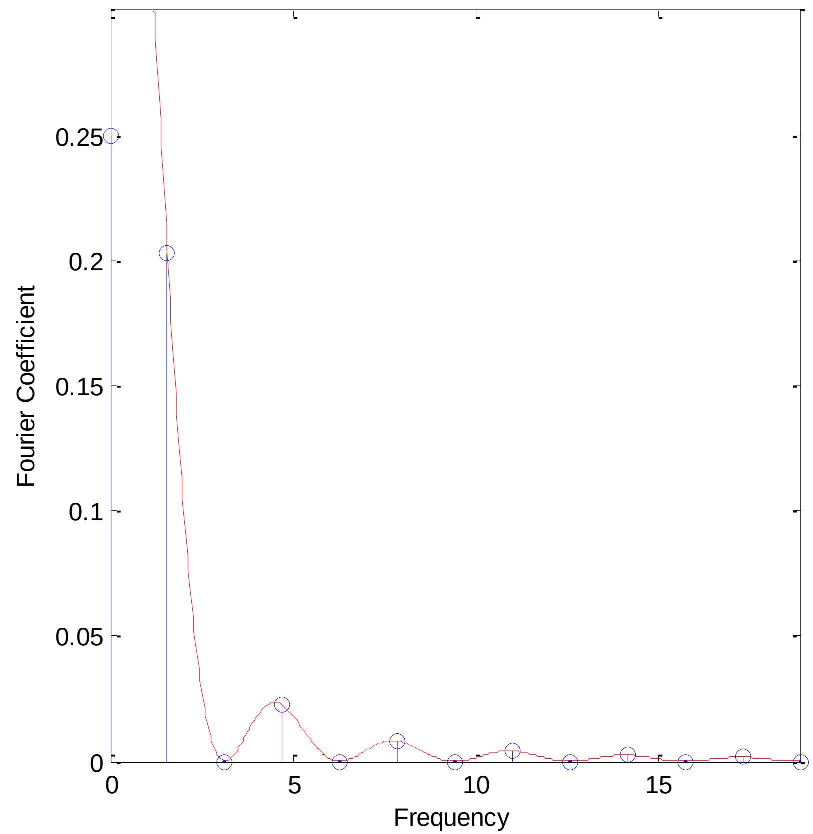
Fourier Series : Fourier Analysis - Example

The power in the DC term of the expansion is

$$S_0 = \left|F_0\right|^2 = \left(\frac{2\tau}{T}\right)^2 = \frac{1}{4}$$

The power in the n-th harmonic (counting both positive and negative n as contributing) is

$$S_n = 2\left|F_n\right|^2 = \begin{cases} 0 & : \quad \text{n even} \\ \dfrac{2}{n^2\pi^2} & : \quad \text{n odd} \end{cases}$$

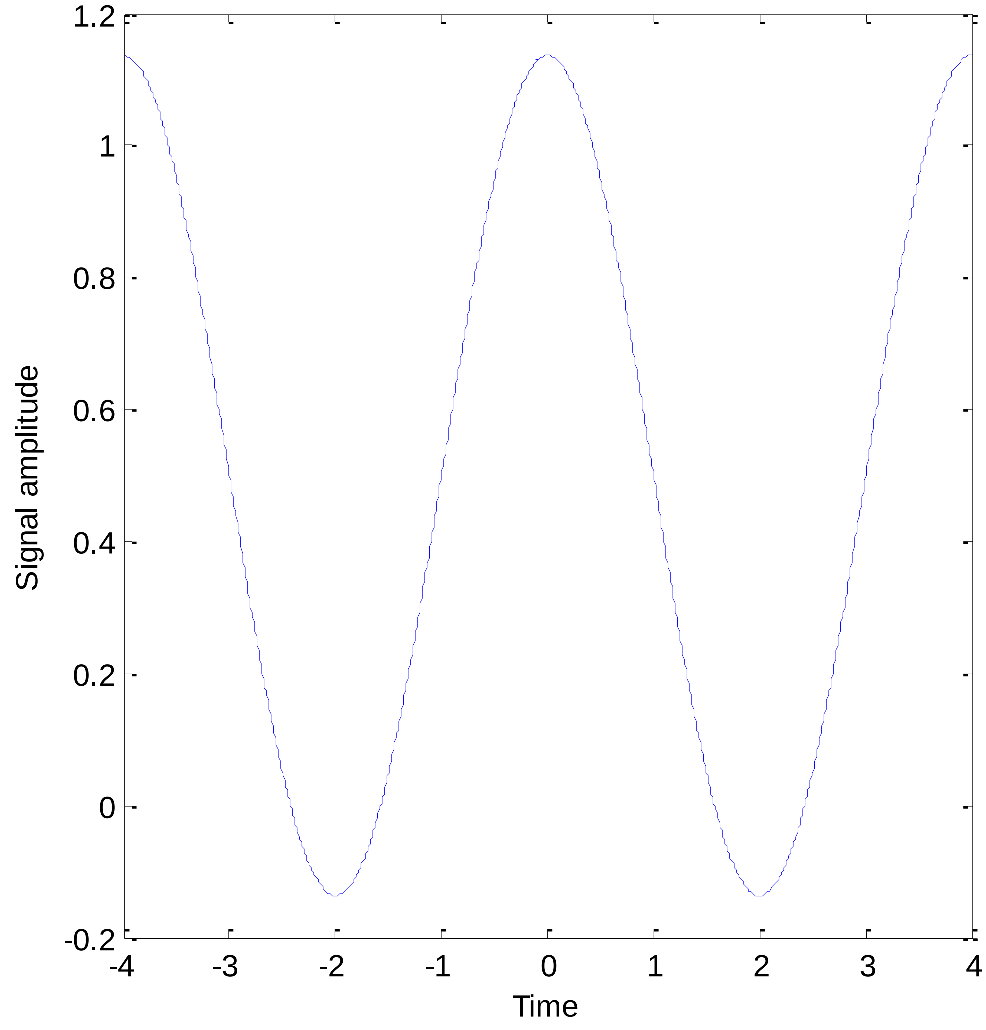


The fact that the square wave contains considerable high frequency power is shown by the fact that 2% of total power (and 4% of non-DC power) is contained in harmonics higher than the tenth.

Fourier Series : Fourier Analysis - Example

The relatively slow convergence for the square wave is shown in the progressive synthesis of the signal from its Fourier components.

The peculiar "ringing" around the edges of the partially reconstructed pulses is also a feature ("Gibbs phenomenon") of the behaviour of Fourier series near discontinuities

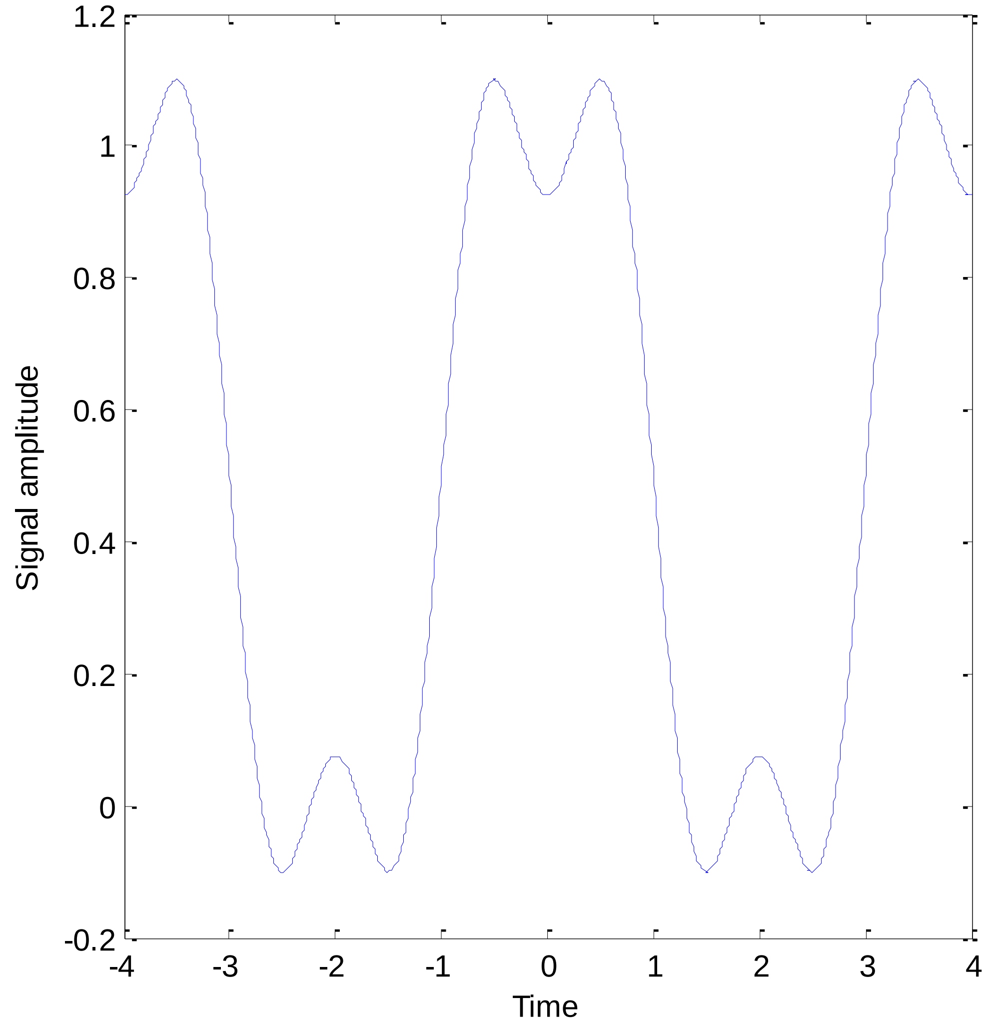


Up to 1st Harmonic

Fourier Series : Fourier Analysis - Example

The relatively slow convergence for the square wave is shown in the progressive synthesis of the signal from its Fourier components.

The peculiar "ringing" around the edges of the partially reconstructed pulses is also a feature ("Gibbs phenomenon") of the behaviour of Fourier series near discontinuities

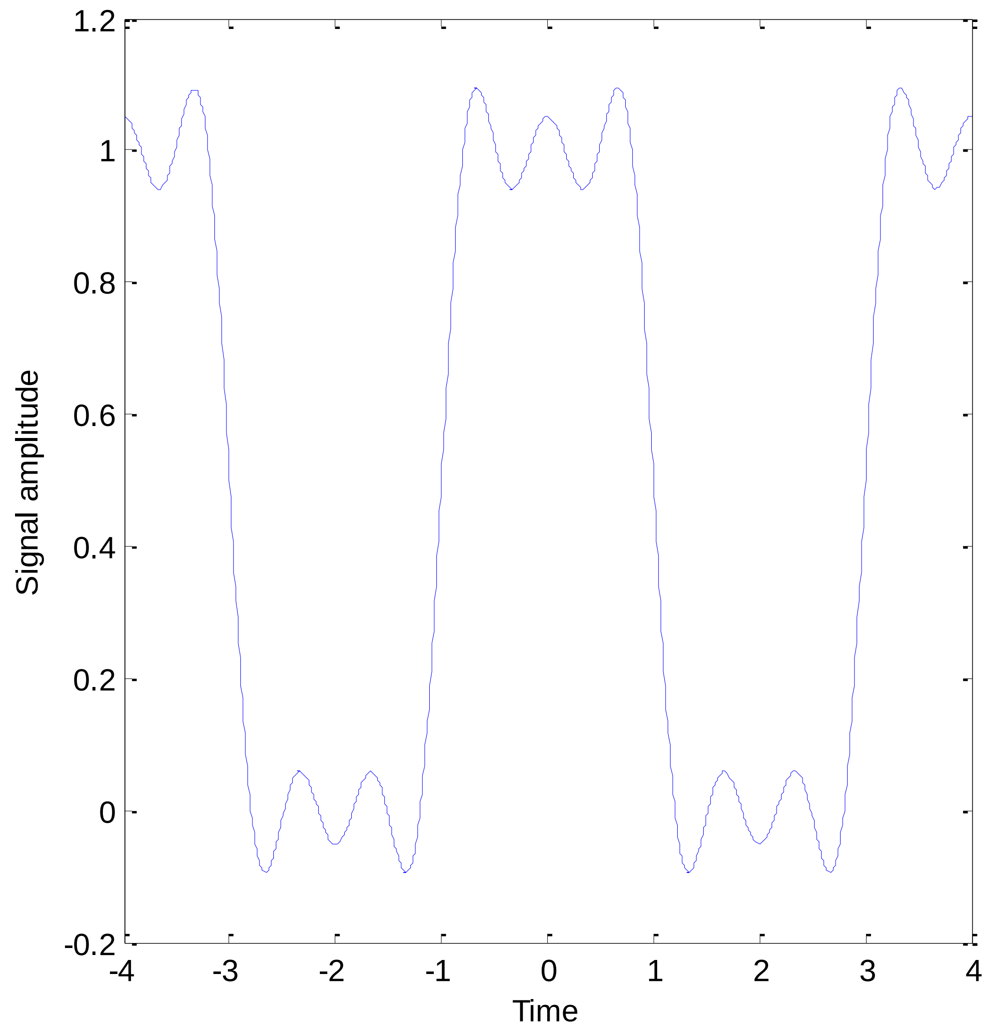


Up to 3rd Harmonic

Fourier Series : Fourier Analysis - Example

The relatively slow convergence for the square wave is shown in the progressive synthesis of the signal from its Fourier components.

The peculiar "ringing" around the edges of the partially reconstructed pulses is also a feature ("Gibbs phenomenon") of the behaviour of Fourier series near discontinuities

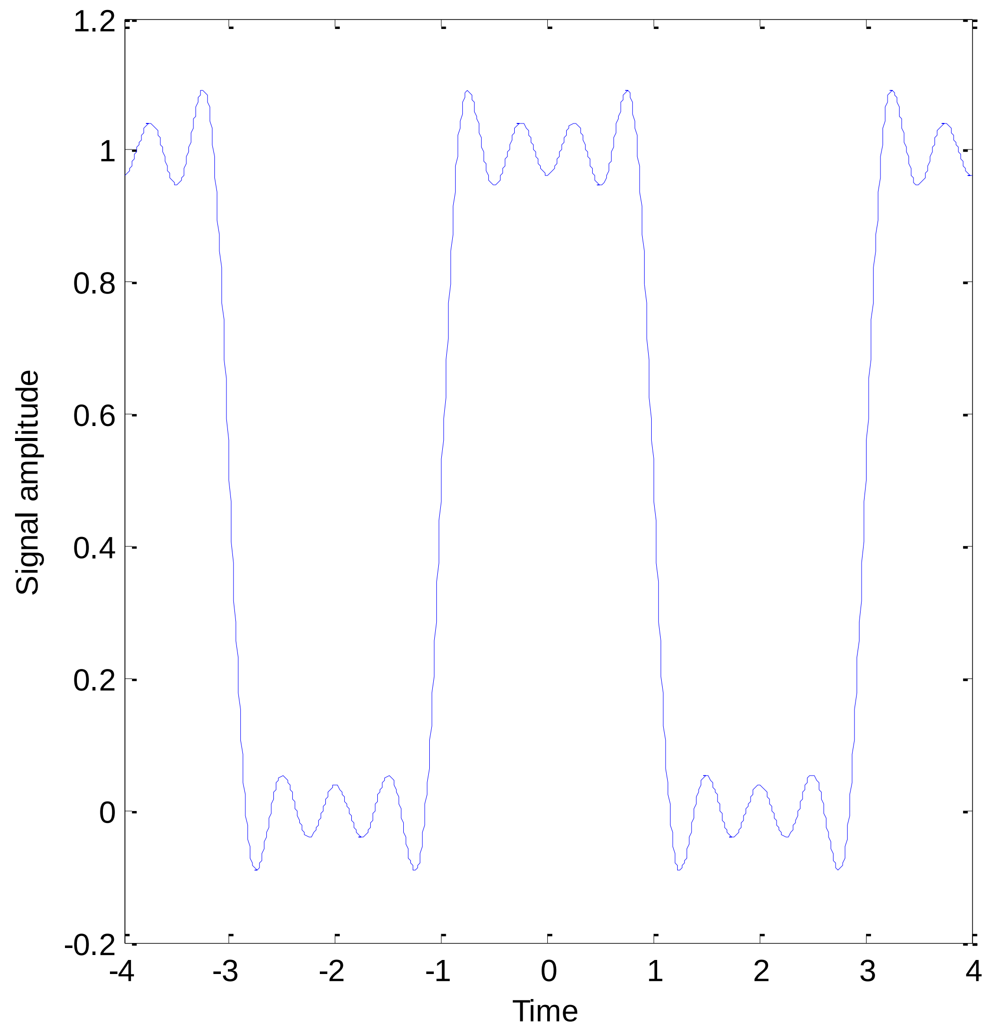


Up to 5th Harmonic

Fourier Series : Fourier Analysis - Example

The relatively slow convergence for the square wave is shown in the progressive synthesis of the signal from its Fourier components.

The peculiar "ringing" around the edges of the partially reconstructed pulses is also a feature ("Gibbs phenomenon") of the behaviour of Fourier series near discontinuities

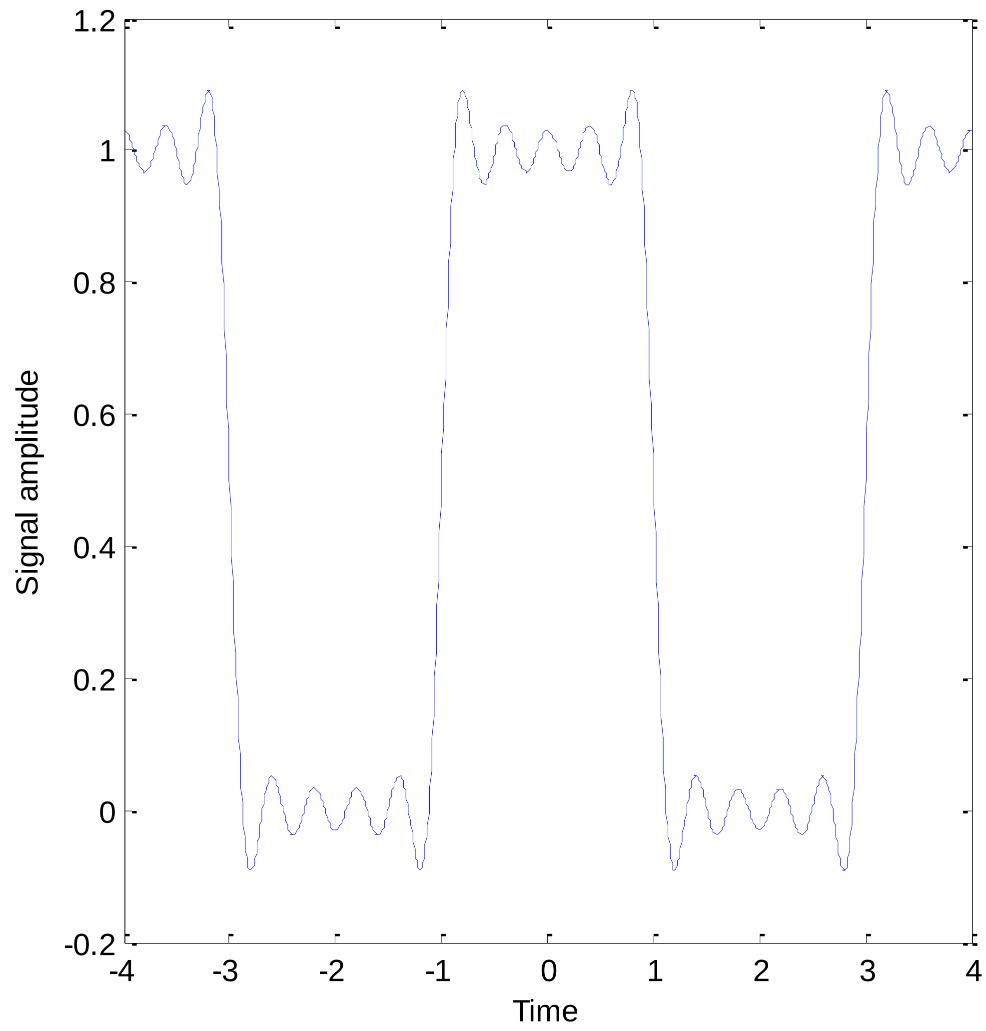


Up to 7th Harmonic

Fourier Series : Fourier Analysis - Example

The relatively slow convergence for the square wave is shown in the progressive synthesis of the signal from its Fourier components.

The peculiar "ringing" around the edges of the partially reconstructed pulses is also a feature ("Gibbs phenomenon") of the behaviour of Fourier series near discontinuities

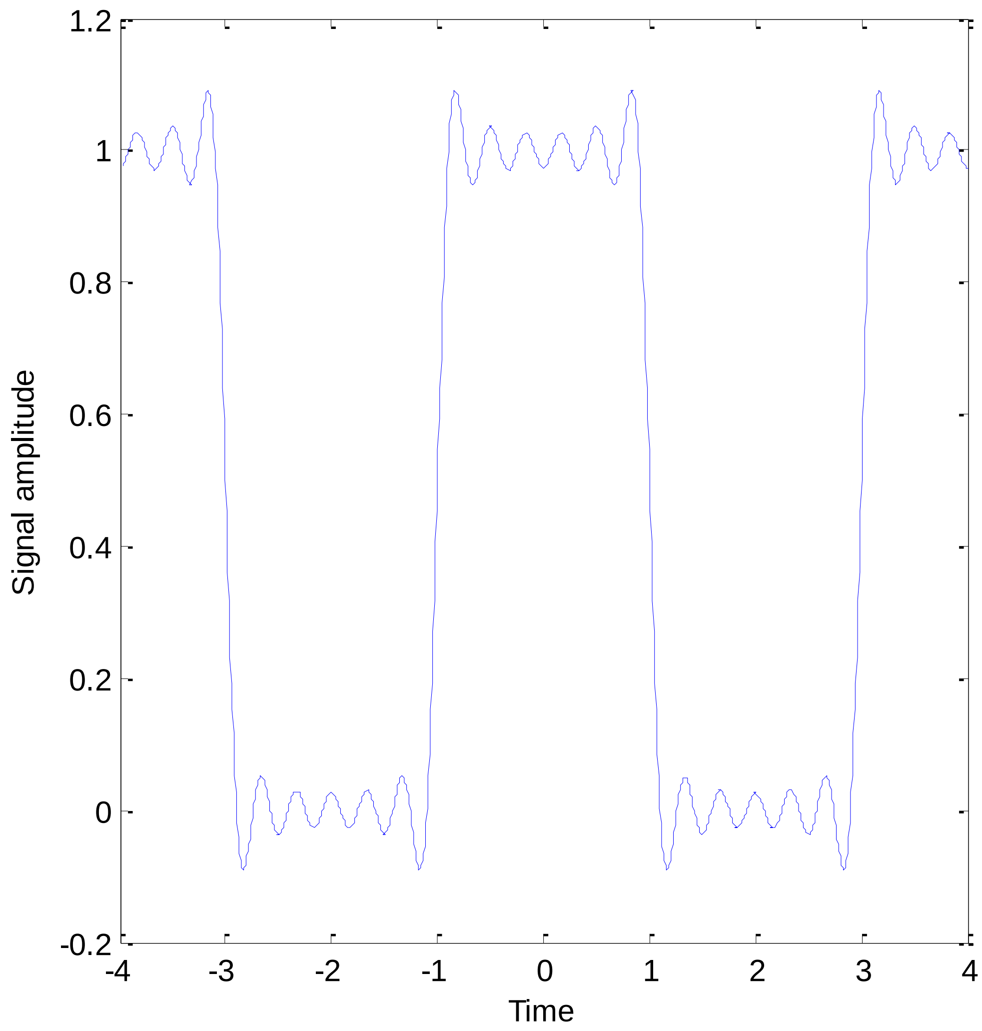


Up to 9th Harmonic

Fourier Series : Fourier Analysis - Example

The relatively slow convergence for the square wave is shown in the progressive synthesis of the signal from its Fourier components.

The peculiar "ringing" around the edges of the partially reconstructed pulses is also a feature ("Gibbs phenomenon") of the behaviour of Fourier series near discontinuities

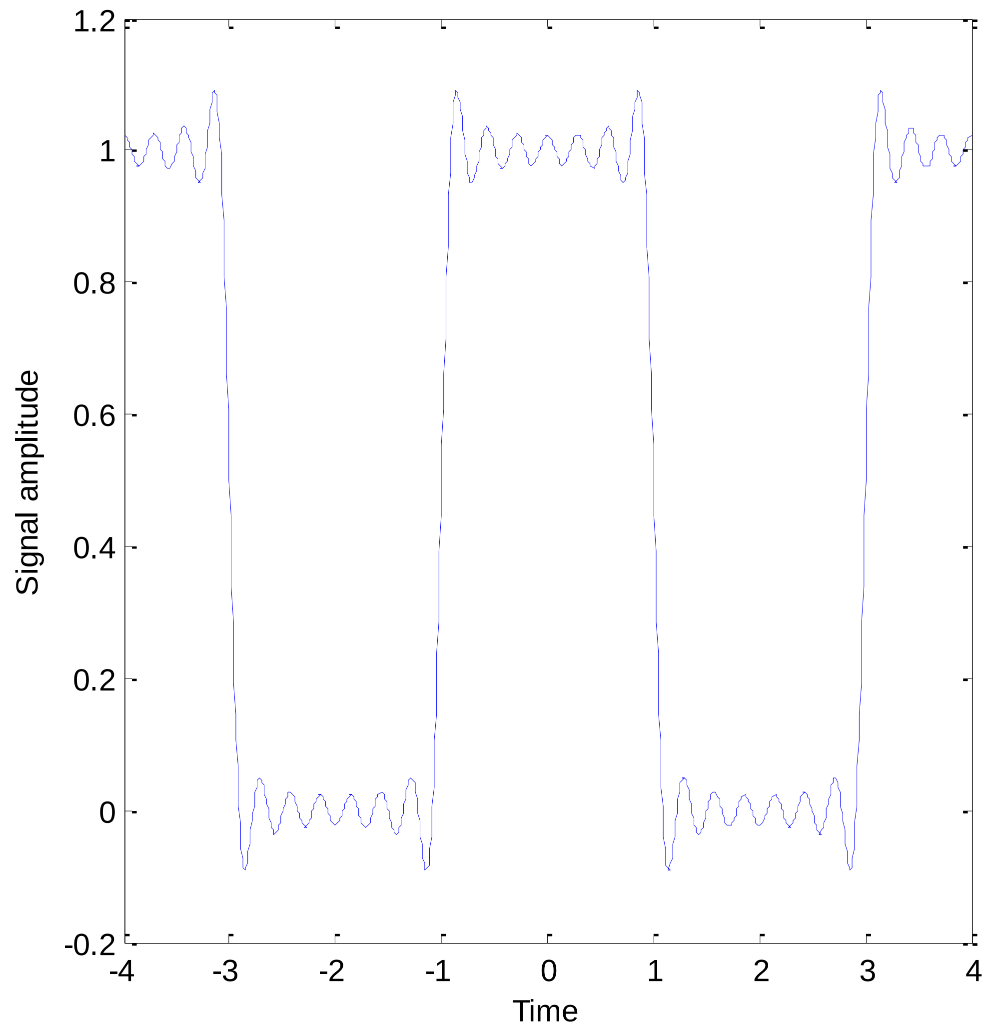


Up to 11th Harmonic

Fourier Series : Fourier Analysis - Example

The relatively slow convergence for the square wave is shown in the progressive synthesis of the signal from its Fourier components.

The peculiar "ringing" around the edges of the partially reconstructed pulses is also a feature ("Gibbs phenomenon") of the behaviour of Fourier series near discontinuities

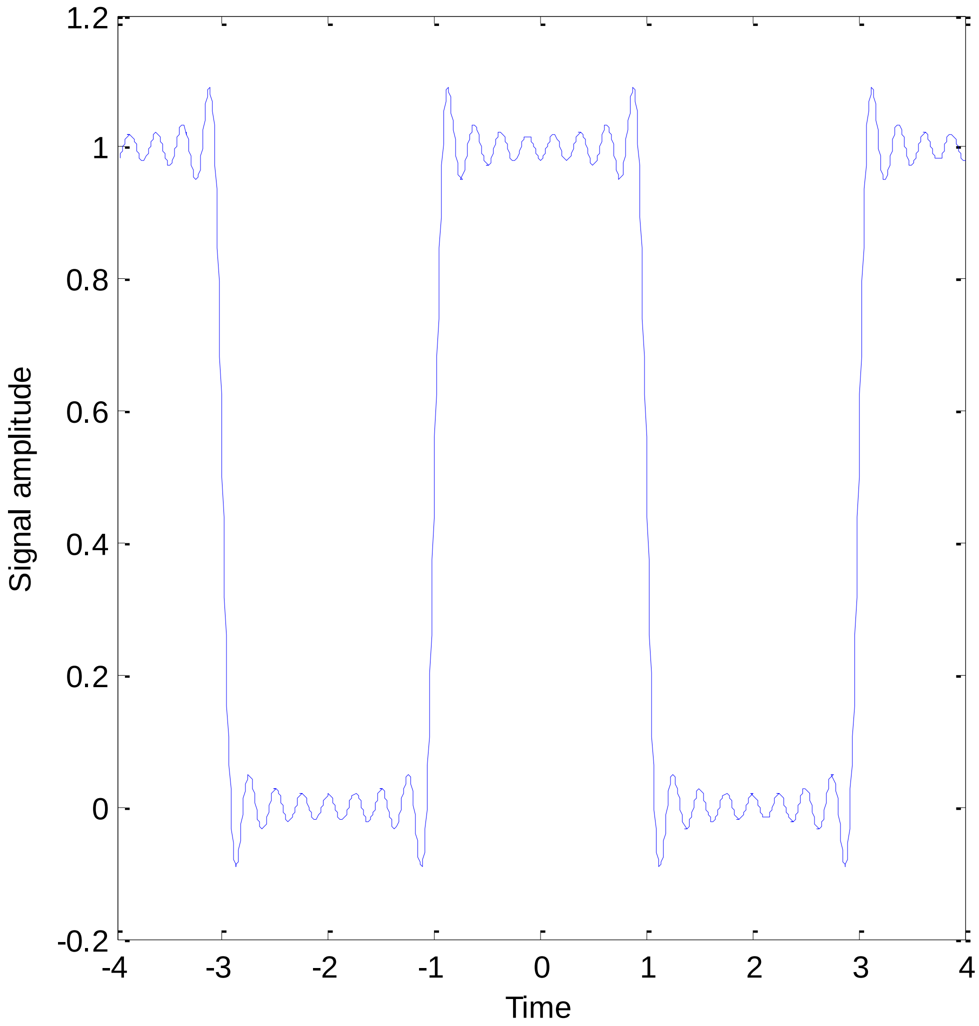


Up to 13th Harmonic

Fourier Series : Fourier Analysis - Example

The relatively slow convergence for the square wave is shown in the progressive synthesis of the signal from its Fourier components.

The peculiar "ringing" around the edges of the partially reconstructed pulses is also a feature ("Gibbs phenomenon") of the behaviour of Fourier series near discontinuities

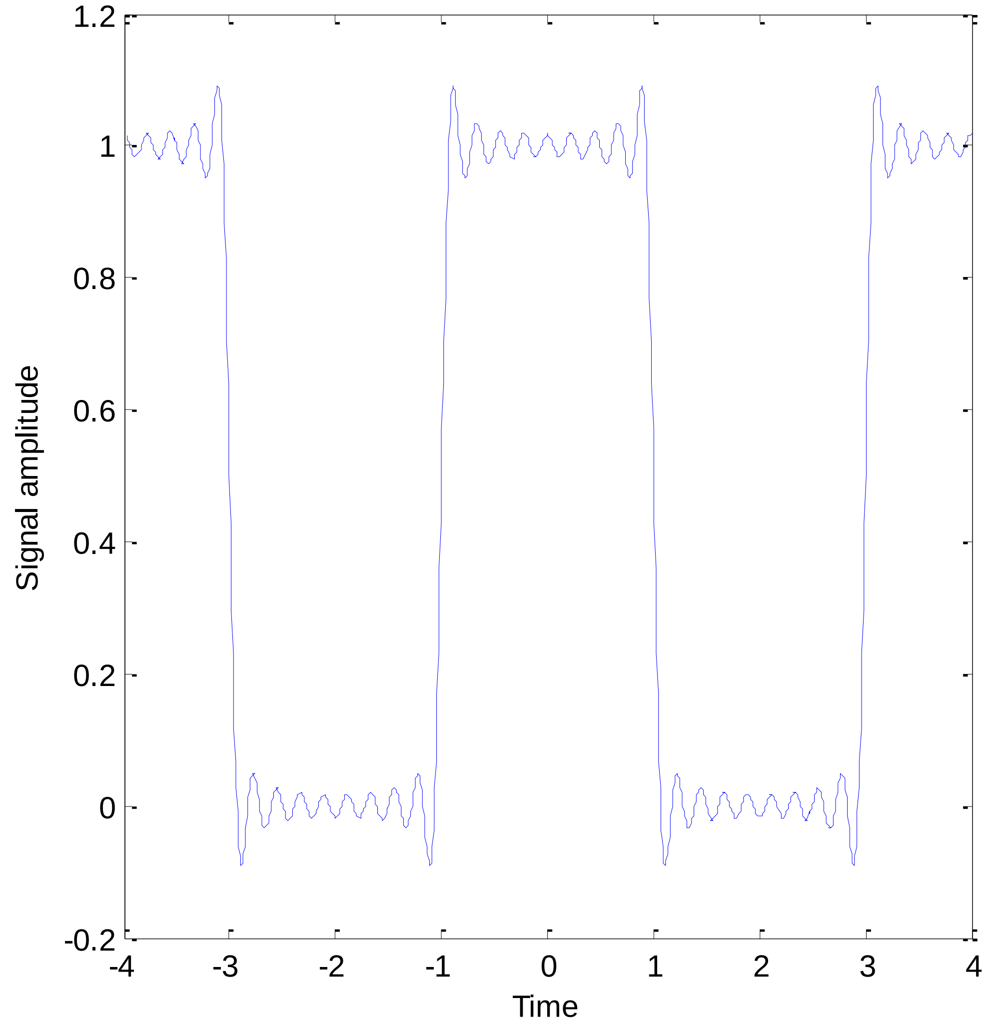


Up to 15th Harmonic

Fourier Series : Fourier Analysis - Example

The relatively slow convergence for the square wave is shown in the progressive synthesis of the signal from its Fourier components.

The peculiar "ringing" around the edges of the partially reconstructed pulses is also a feature ("Gibbs phenomenon") of the behaviour of Fourier series near discontinuities

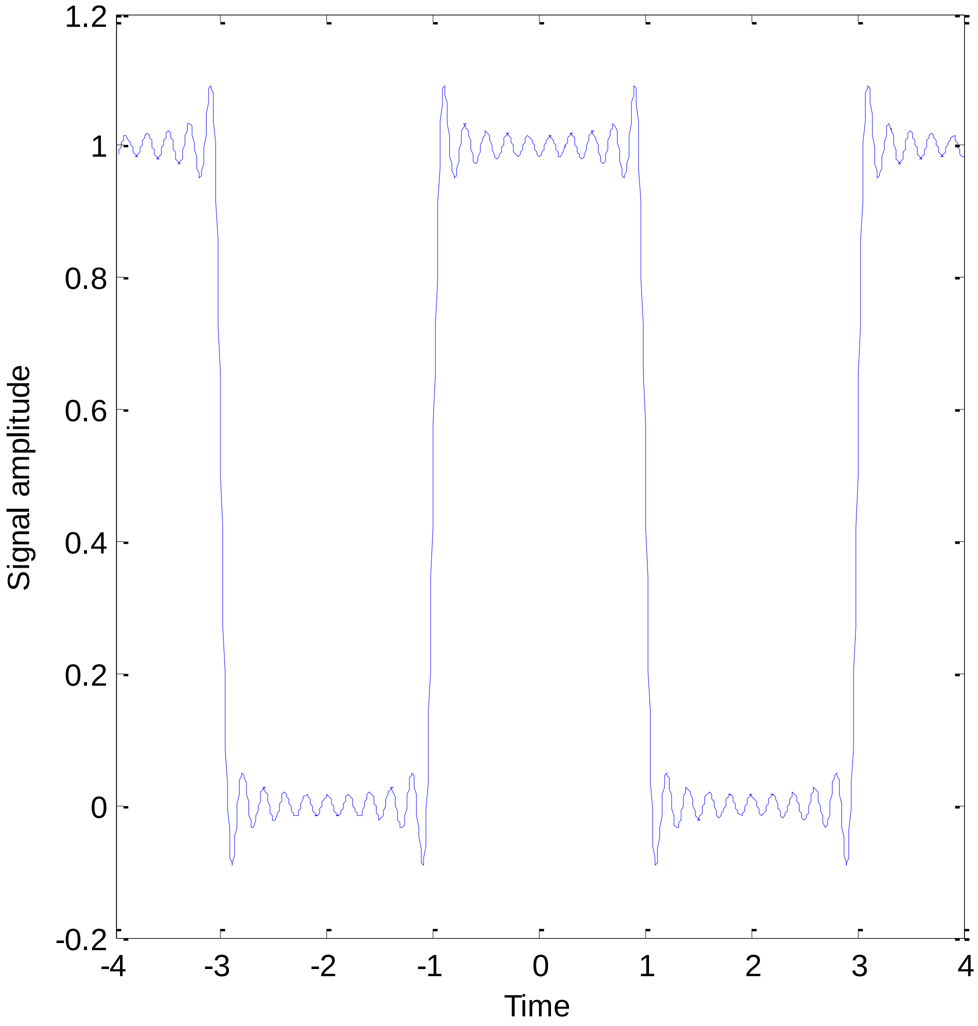


Up to 17th Harmonic

Fourier Series : Fourier Analysis - Example

The relatively slow convergence for the square wave is shown in the progressive synthesis of the signal from its Fourier components.

The peculiar "ringing" around the edges of the partially reconstructed pulses is also a feature ("Gibbs phenomenon") of the behaviour of Fourier series near discontinuities



Up to 19th Harmonic

Fourier Series : Properties - Convergence

The Fourier coefficients have a number of useful general properties. These will be discussed in more detail in the lectures on the Fourier Transform, but a selection is presented here to complete our introduction to Fourier Series.
The convergence property is of mainly mathematical interest:

If a periodic function satisfies some conditions, then the Fourier series converges to the function at any time t at which the function is continuous.
A sufficient set of conditions is the Dirichlet conditions :

i. The function can have at most a finite number of discontinuities in any cycle.
ii. The function can have at most a finite number of maxima and minima in any cycle.
iii. The function must be absolutely integrable over any cycle. i.e.

$$\int_{\tau}^{\tau+T} \left| f(t) \right| dt < \infty$$

Fourier Series : Properties - linearity

Linearity is a more immediately useful, if fairly obvious property of Fourier Series:

If *f(t)* and *g(t)* are two functions with the same period, then the Fourier coefficients of the sum of the two functions is the sum of the Fourier coefficients of the individual signals.

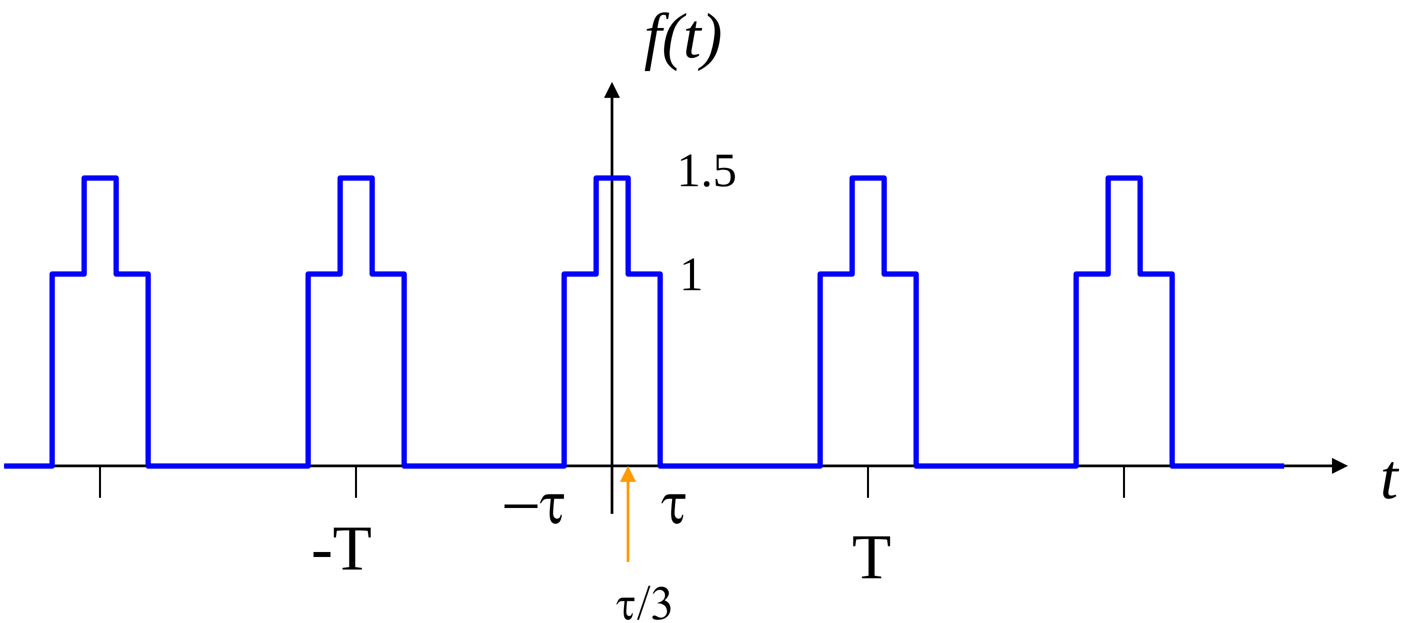
Slightly more generally where α and β are arbitrary constants

$$h(t) = \alpha\, f(t) + \beta\, g(t) \Rightarrow H_n = \alpha\, F_n + \beta\, G_n$$

In words, if you add two signals, then the Fourier coefficients of the sum are just the sums of the Fourier coefficients of the individual signals.

Fourier Series : Properties - linearity

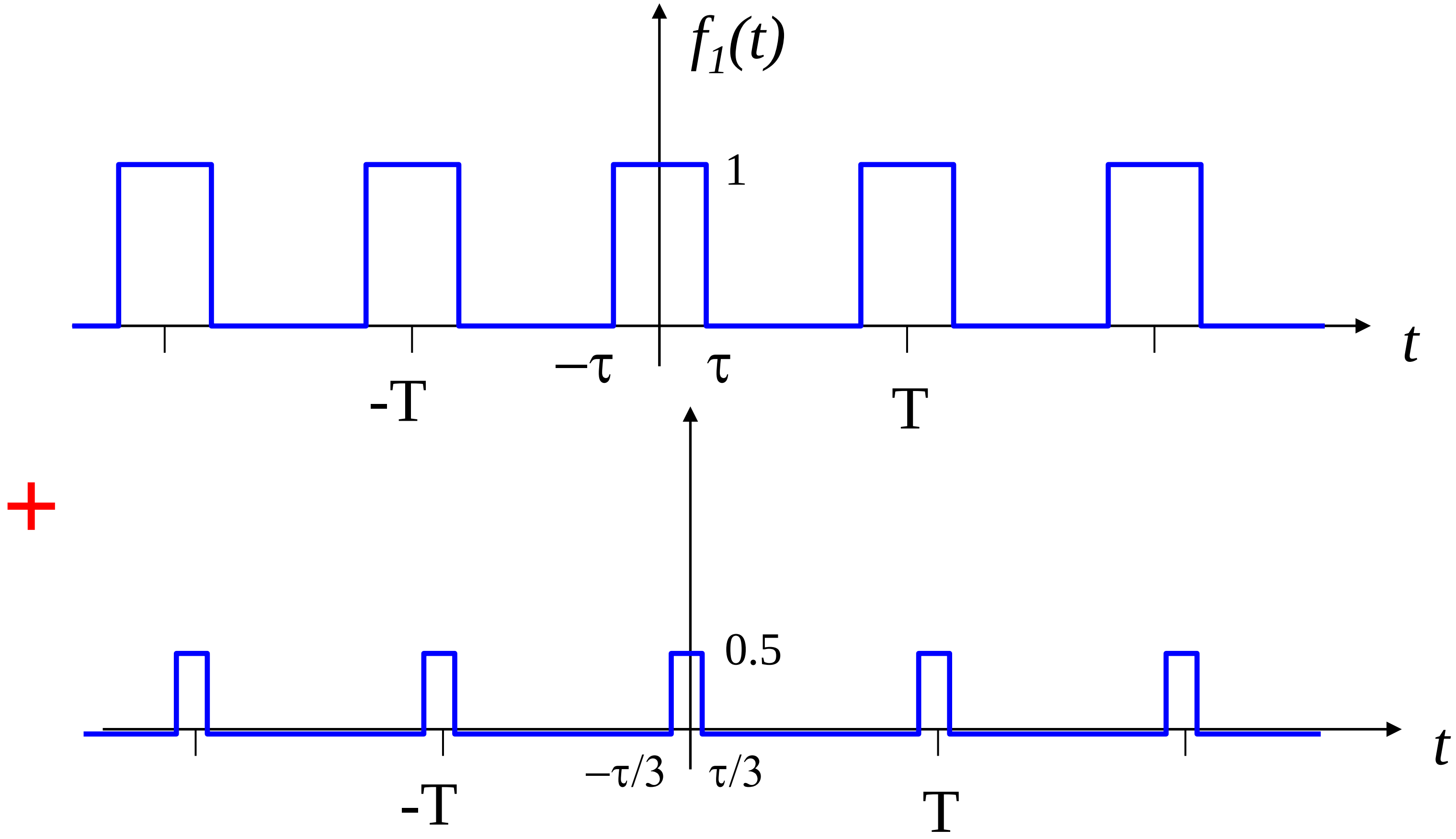
For example if we have a pulse train of the type shown below



$$f(t) = \sum_{k=-\infty}^{\infty} P_{\tau}(t-kT) + 0.5 \sum_{k=-\infty}^{\infty} P_{\frac{1}{3}\tau}(t-kT)$$

Fourier Series : Properties - linearity

This is most easily handled by recognising that the signal is the sum of two independent rectangular pulse trains.



Fourier Series : Properties - linearity

So the n-th Fourier coefficient for the sum is the sum of the n-th coefficient for the 2t width pulse train and 0.5 times the n-th coefficient for the 2t/3 width pulse train.

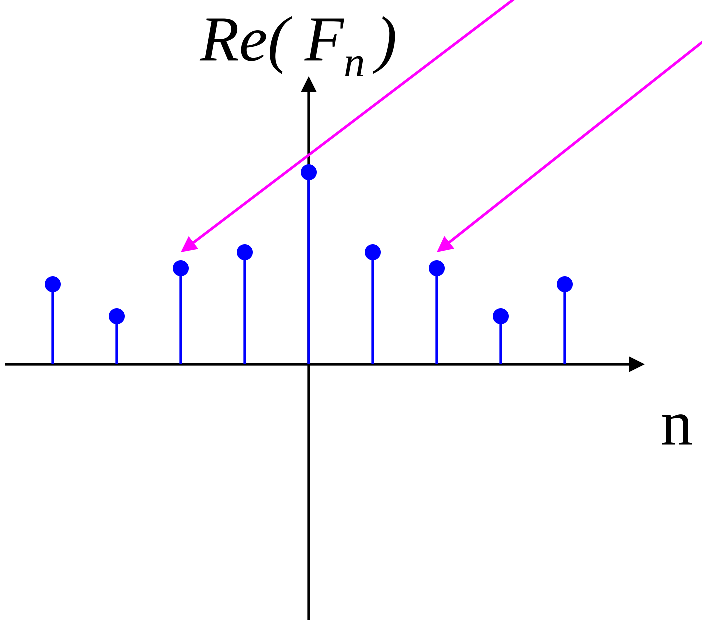
$$F_n = \frac{2\tau}{T}\operatorname{sinc}\left(\frac{2n\pi\tau}{T}\right) + \frac{\tau}{3T}\operatorname{sinc}\left(\frac{2n\pi\tau}{3T}\right)$$

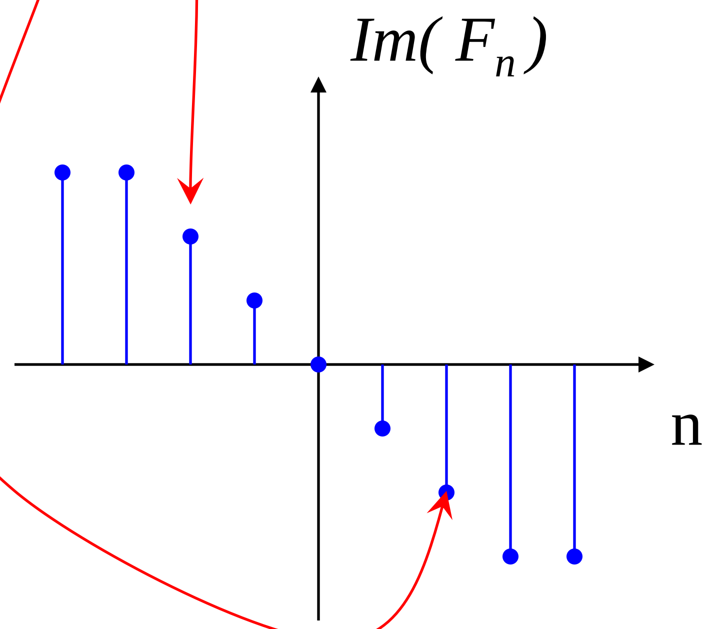
Fourier Series : Properties - symmetry

If the signal f(t) has some intrinsic symmetry this is reflected in the Fourier coefficients.

Thus if f(t) is a real function (no imaginary part) then

$$F_{-n} = F_n^{*}$$





Fourier Series : Properties - symmetry

Thus if f(t) is a real (no imaginary part) and even ($f(-t) = f(t)$) function then

$$\operatorname{Im} F_n = 0$$

If f(t) is a real (no imaginary part) and odd ($f(-t) = -f(t)$) function then

$$\operatorname{Re} F_n = 0$$

If g$(t) = f(-t)$ then (Time reversal)

$$G_n = F_{-n}$$

Fourier Series : Properties - shift

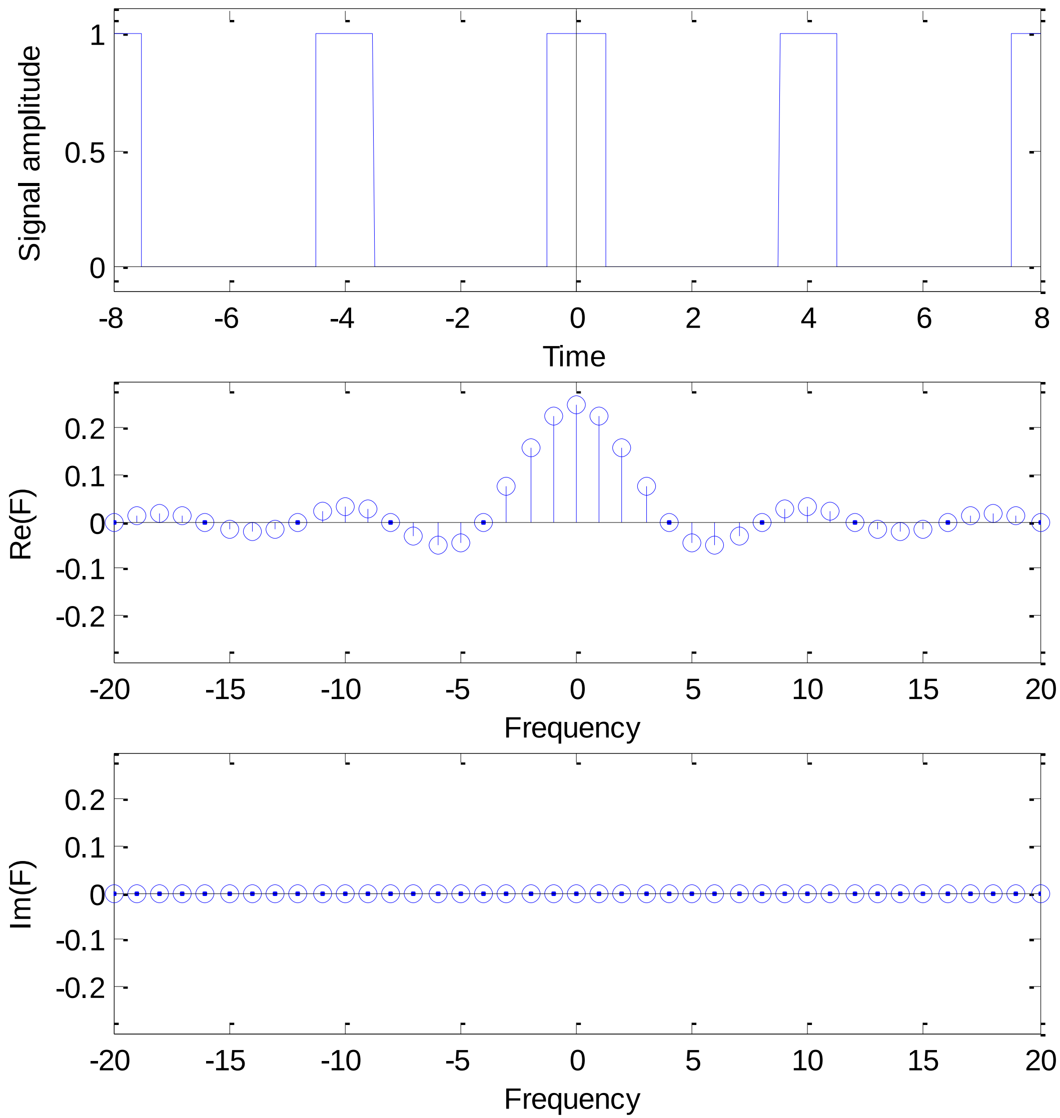
If a signal f(t) is shifted then the Fourier coefficients of the shifted function can be related to the coefficients of the un-shifted signal

If $\quad g(t) = f(t-\tau)$

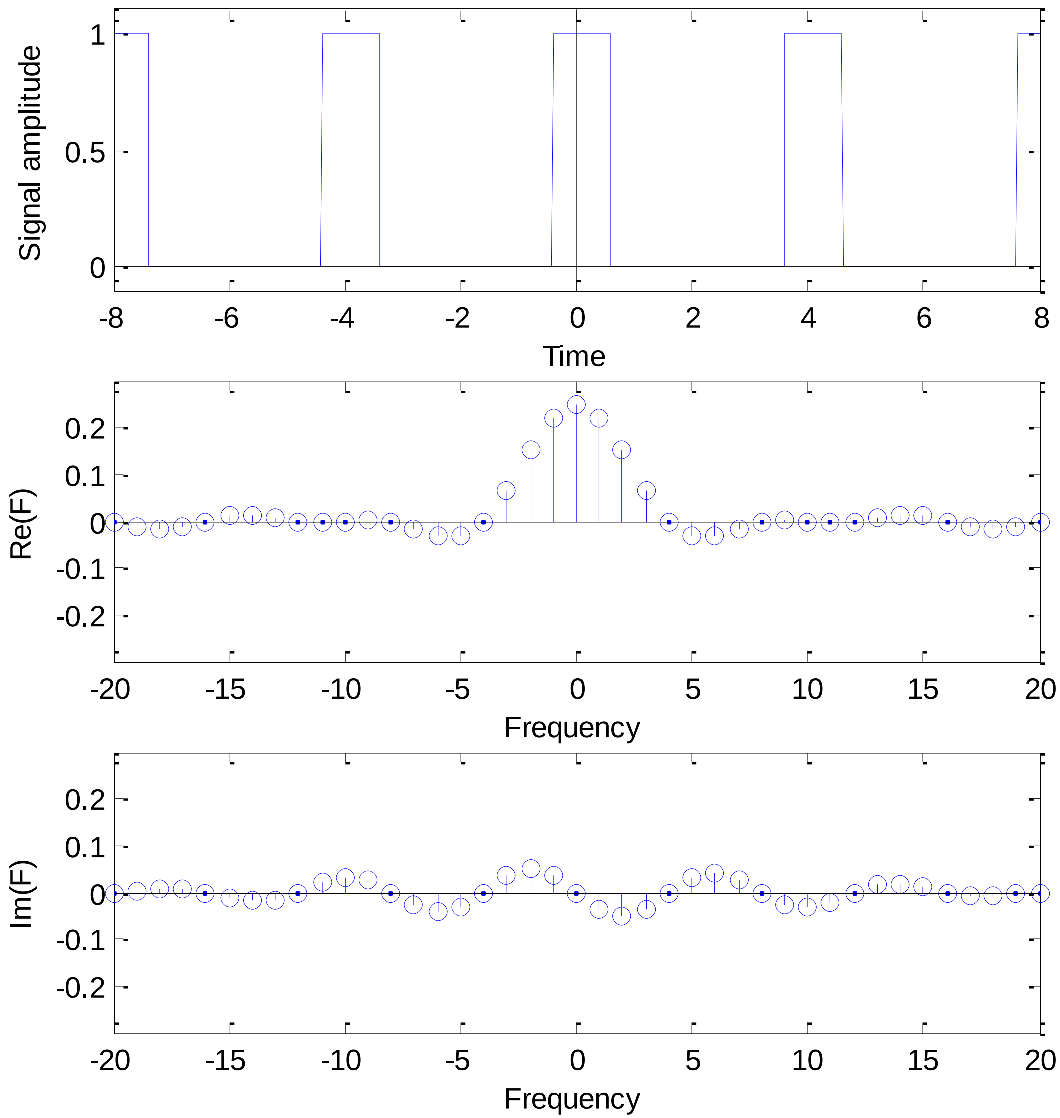
then $\quad G_n = e^{-jn\omega_1\tau} F_n$

$$= \left[\cos\left(\frac{2n\pi}{T}\tau\right) - j\sin\left(\frac{2n\pi}{T}\tau\right)\right] F_n$$

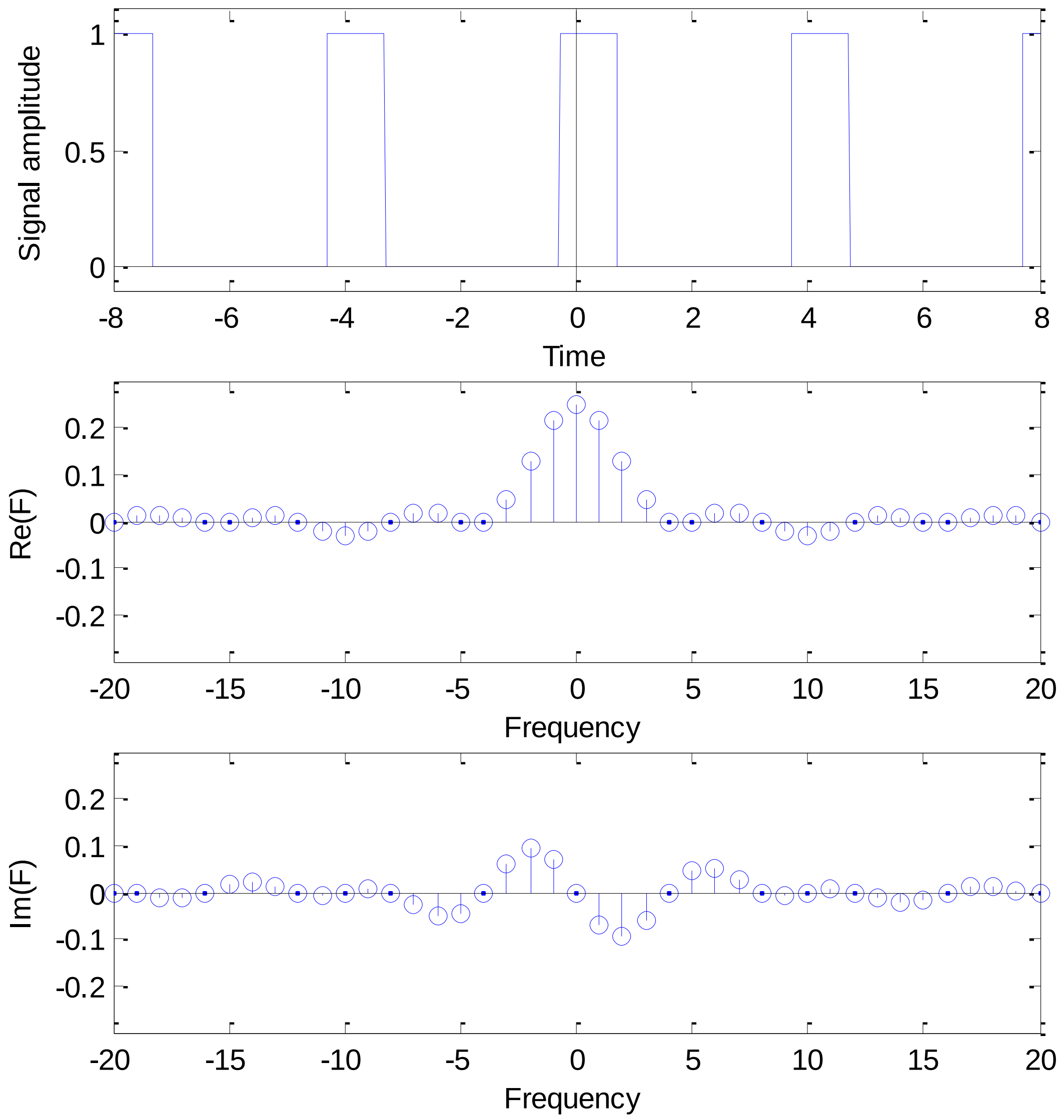
Fourier Series : Properties - shift



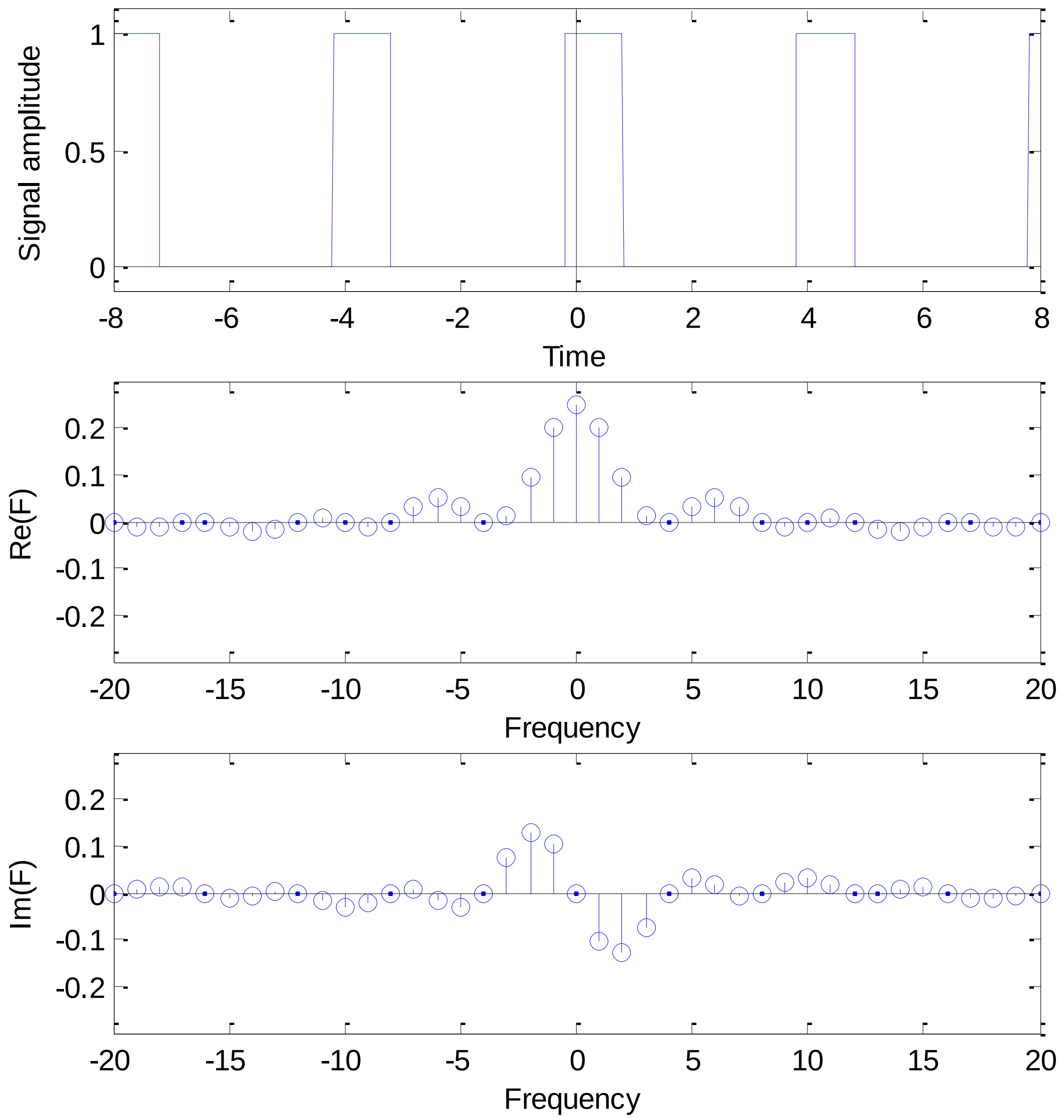
Fourier Series : Properties - shift



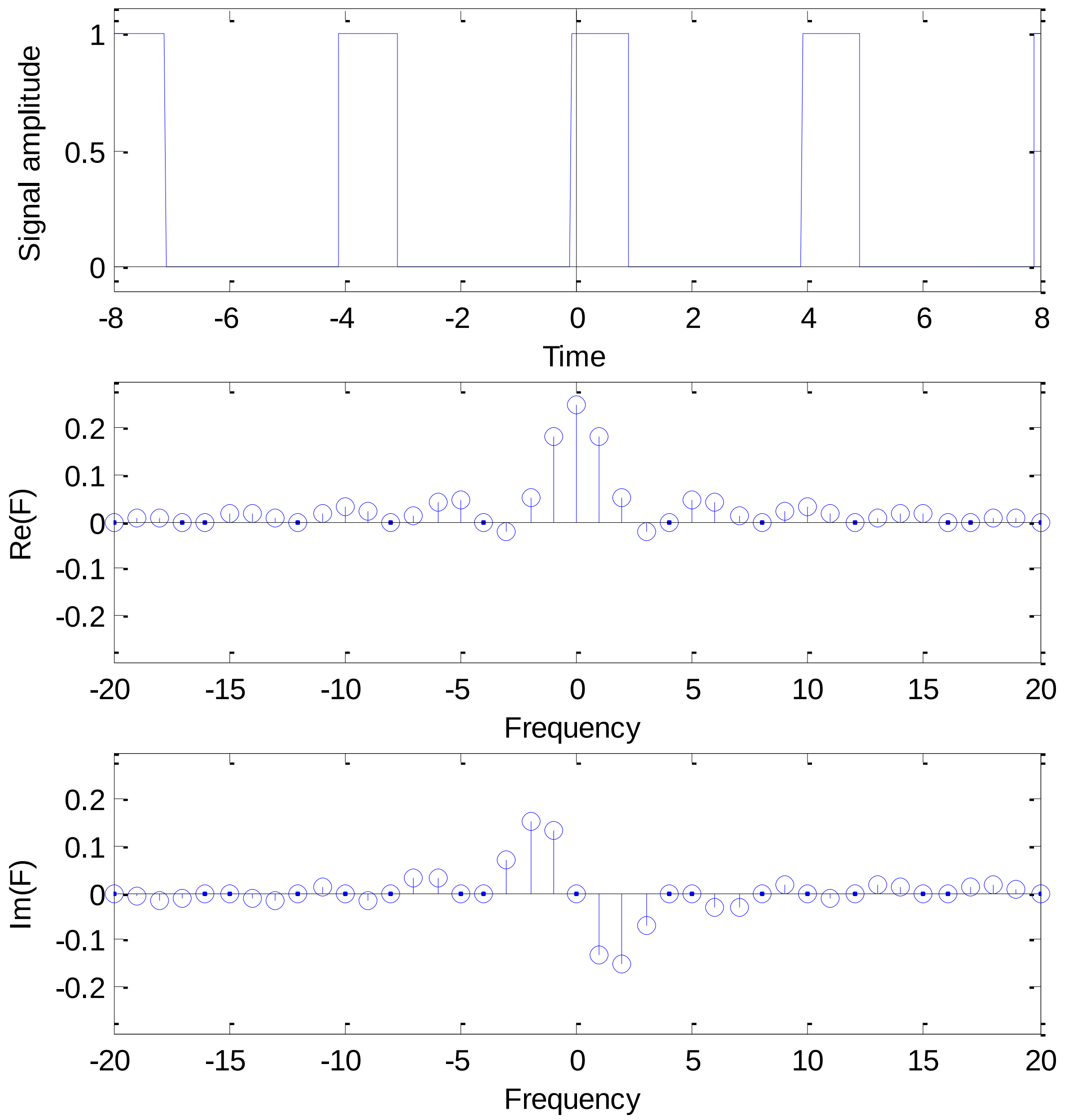
Fourier Series : Properties - shift



Fourier Series : Properties - shift



Fourier Series : Properties - shift



Discrete vs. Cont. Time: Fourier Series

	Discrete Time:	Continuous Time:
Periodic:	$x[n] = x[n + N_0]$	$x(t) = x(t + T_0)$
Fourier Series:	$x[n] = \sum_{k=0}^{N_0-1} \mathbf{x}_k e^{jk\Omega_0 n}$	$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t}$
Computing Coefficient:	$\mathbf{x}_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n]\, e^{-jk\Omega_0 n}$	$\mathbf{x}_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, e^{-jn\omega_0 t}\, dt.$
# Terms:	N_0	Infinity, in general

Discrete-Time Fourier Series (DTFS)

Compute the DTFS of the periodic signal

$$x[n] = \{\ldots, \underline{24}, 8, 12, 16, 24, 8, 12, 16, \ldots\}.$$

Solution: The period of x[n] is 4. So $N_0 = 4$

Plugging into $\mathbf{x}_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n]\, e^{-jk\Omega_0 n}$ gives

$$x_0 = \tfrac{1}{4}(24 + 8 + 12 + 16) = 15.$$

$$\mathbf{x}_1 = \frac{1}{4}(24e^0 + 8e^{-j\pi/2} + 12e^{-j\pi} + 16e^{-j3\pi/2}) = 3 + j2 = 3.6e^{j33.7^\circ}.$$

$$\mathbf{x}_2 = \frac{1}{4}(24 - 8 + 12 - 16) = 3.$$

$$\mathbf{x}_3 = \mathbf{x}_1^* = 3 - j2 = 3.6e^{-j33.7^\circ}.$$

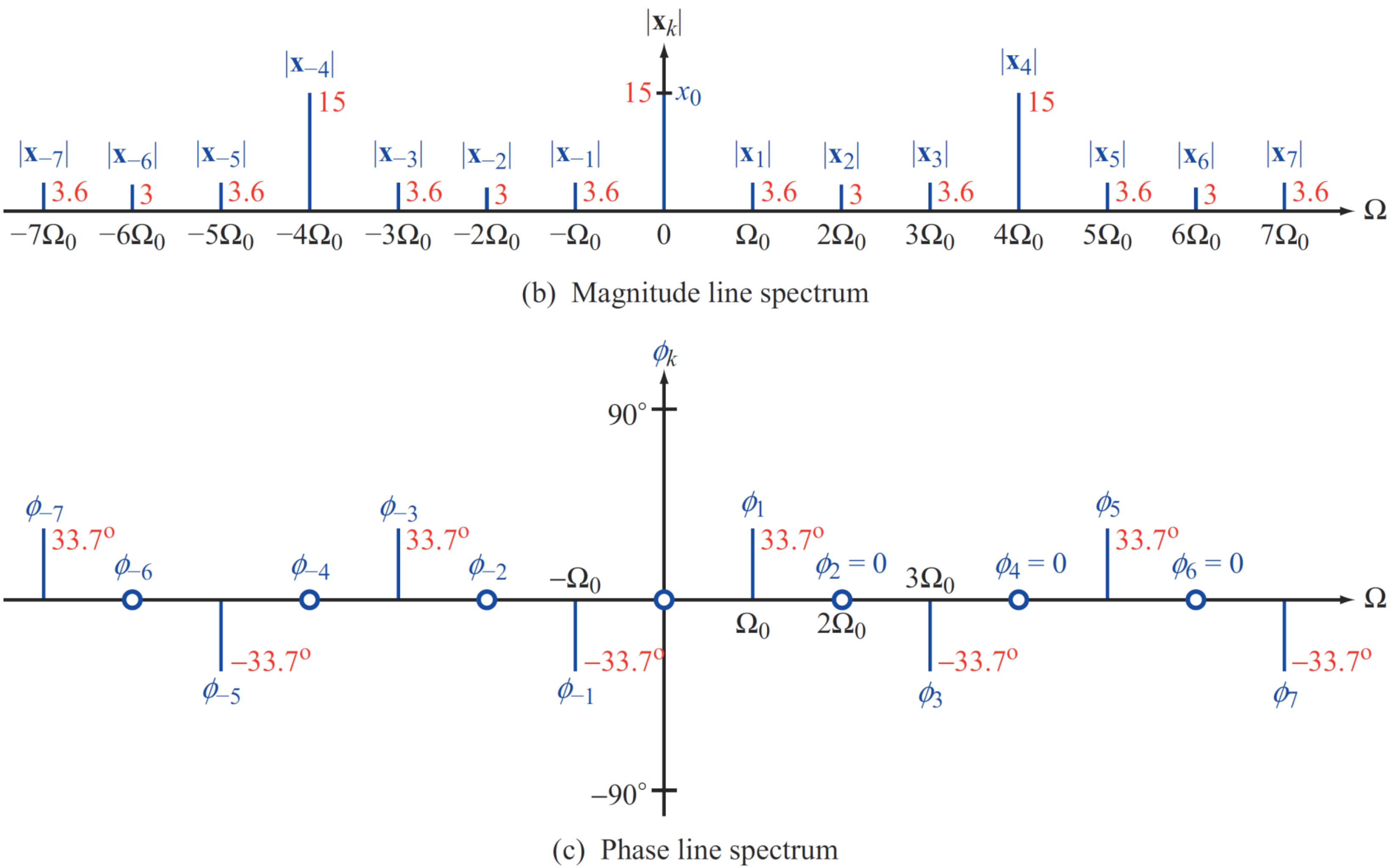
DTFS Example, Continued

The DTFS expansions of the given periodic signal x[n] is then

$$
\begin{aligned}
x[n] &= x_0 + \mathbf{x}_1 e^{j\pi n/2} + \mathbf{x}_2 e^{j\pi n} + \mathbf{x}_3 e^{j3\pi n/2} \\
&= 15 + 3.6 e^{j33.7^\circ} e^{j\pi n/2} + 3 e^{j\pi n} + 3.6 e^{-j33.7^\circ} e^{j3\pi n/2}. \\
&= 15 + 7.2 \cos\left(\frac{\pi n}{2} + 33.7^\circ\right) + 3\cos(\pi n).
\end{aligned}
$$

Magnitude and phase line spectra of x[n] on next slide.

DTFS Line Spectra for Example



DTFS: Parseval's Theorem

Average power is the same
In time or frequency domain

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1}|x[n]|^2=\sum_{k=0}^{N_0-1}|\mathbf{x}_k|^2.$$

For the previous example, we have:

Time Domain: $P_{\text{av}}=\frac{1}{4}\left[(24)^2+(8)^2+(12)^2+(16)^2\right]=260.$

Frequency Domain: $P_{\text{av}}=(15)^2+(3.6)^2+3^2+(3.6)^2=260.$

(Note that 3.6 is actually $\sqrt{13}$ rounded off.)

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t)$, $\alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau) y(t - \tau) d\tau$	$T a_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\, dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right) a_k = \left(\frac{1}{jk(2\pi/T)}\right) a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \|a_k\| = \|a_{-k}\| \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} \quad [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} \quad [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n - r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n - 1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$