LECTURE BLOCK 3C: FOURIER SERIES OF PERIODIC SIGNALS

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Fourier Analysis Techniques

We will learn:

- Express periodic signals in term of Fourier Series.
- Use Fourier Series to analyse system driven by continuous periodic signals.
- Apply Parseval's theorem to compute the power or energy contained in a signal.
- Study Fourier Series' examples, and properties,
- Look at Discrete Time Fourier Series.

Discrete Time Fourier Series (DTFS)

Fourier Series expansions of discrete-time periodic signals;
But, DTFS is <u>finite</u> in length (has a finite number of terms).

Computation of its expansion coefficients requires summations rather than integrals. The response of an LTI system to a discrete-time periodic signal can be determined by applying the superposition principle. The process entails the following steps:

Step 1: Compute the DTFS of the input signal.
Step 2: Compute the output response to each DTFS term using the system's frequency response $\mathbf{H}(e^{j\Omega})$, as defined by $\mathbf{H}(e^{j\Omega}) = \sum_{i=0}^{\infty} h[i]\, e^{-ji\Omega}$.

Step 3: Sum the results to obtain the total output signal.

Discrete vs. Cont. Time: Fourier Series

	Discrete Time:	Continuous Time:
Periodic:	$x[n] = x[n + N_0]$	$x(t) = x(t + T_0)$
Fourier Series:	$x[n] = \sum_{k=0}^{N_0-1} \mathbf{x}_k e^{jk\Omega_0 n}$	$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t}$
Computing Coefficient:	$\mathbf{x}_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n]\, e^{-jk\Omega_0 n}$	$\mathbf{x}_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, e^{-jn\omega_0 t}\, dt.$
# Terms:	N_0	Infinity, in general

A few notes:

▶ A periodic signal $x[n]$ of period N_0 is represented in its DTFS by the sum of N_0 complex exponentials at harmonic values of $\Omega_0 = 2\pi/N_0$. ◀

In general, the expansion coefficients are complex quantities,

$$\mathbf{x}_k = |\mathbf{x}_k| e^{j\phi_k}.$$

▶ Plots of $|\mathbf{x}_k|$ and ϕ_k as a function of k from $k = 0$ to $k = N_0 - 1$ constitute ***magnitude and phase line-spectra***. ◀

Orthogonality property

$$\sum_{n=0}^{N_0-1} e^{j2\pi[(k-m)/N_0]n} = N_0\delta[k-m]$$

$$= \begin{cases} N_0 & \text{if } k = m, \\ 0 & \text{if } k \neq m, \end{cases}$$

where k, m, and n are integers, and none are larger than N_0.

(a) $k = m$:

$$\sum_{n=0}^{N_0-1} e^0 = \sum_{n=0}^{N_0-1} 1 = N_0 \qquad (k = m).$$

(b) $k \neq m$:

If in the finite geometric series

$$\sum_{n=0}^{N_0-1} \mathbf{r}^n = \frac{\mathbf{r}^{N_0} - 1}{\mathbf{r} - 1} \qquad \text{for } \mathbf{r} \neq 1,$$

we replace $\mathbf{r}$ with $e^{j2\pi(k-m)/N_0}$, we have

$$\sum_{n=0}^{N_0-1} e^{j2\pi[(k-m)/N_0]n} = \frac{e^{j2\pi[(k-m)/N_0]N_0} - 1}{e^{j2\pi(k-m)/N_0} - 1}.$$

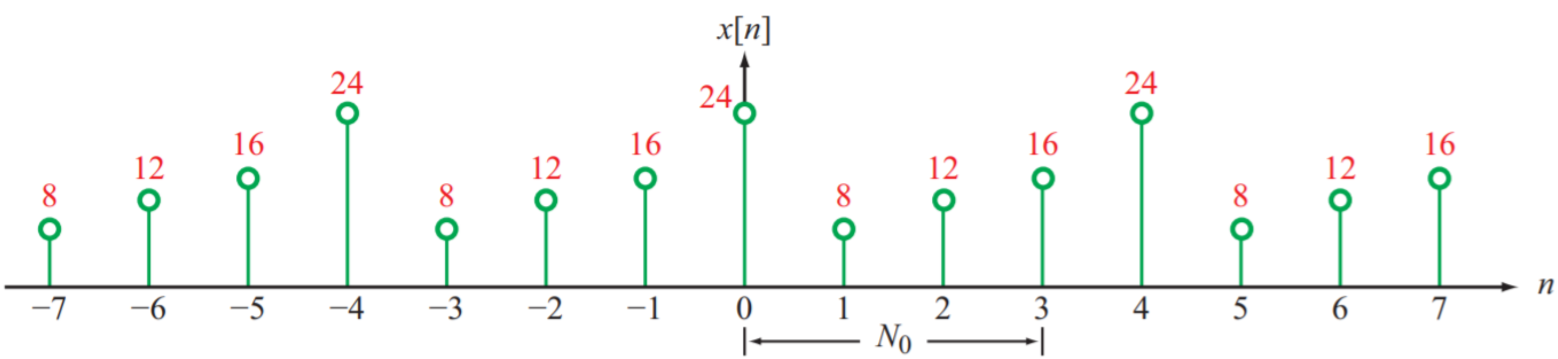
Since $k \neq m$, and k and m cannot be larger than N_0, the exponent $j2\pi(k-m)/N_0$ in the denominator cannot be a multiple of 2π. Hence, the denominator cannot be zero. The exponent in the numerator, on the other hand, is $j2\pi(k-m)$, and therefore, it is always a multiple of 2π. Consequently,

$$\sum_{n=0}^{N_0-1} e^{j2\pi[(k-m)/N_0]n} = 0 \qquad (k \neq m).$$

Example1: DTFS Computation

Compute the DTFS of the periodic signal

$$x[n] = \{\ldots, \underline{24}, 8, 12, 16, 24, 8, 12, 16, \ldots\}.$$



(a) $x[n]$

Example1: DTFS Computation

Solution: The period of x[n] is 4. So $N_0 = 4$

Plugging into $\mathbf{x}_k = \dfrac{1}{N_0} \sum_{n=0}^{N_0-1} x[n]\, e^{-jk\Omega_0 n}$ gives

$$x_0 = \tfrac{1}{4}\,(24 + 8 + 12 + 16) = 15.$$

$$\mathbf{x}_1 = \frac{1}{4}(24e^{0} + 8e^{-j\pi/2} + 12e^{-j\pi} + 16e^{-j3\pi/2}) = 3 + j2 = 3.6e^{j33.7^\circ}.$$

$$\mathbf{x}_2 = \frac{1}{4}(24 - 8 + 12 - 16) = 3.$$

$$\mathbf{x}_3 = \mathbf{x}_1^* = 3 - j2 = 3.6e^{-j33.7^\circ}.$$

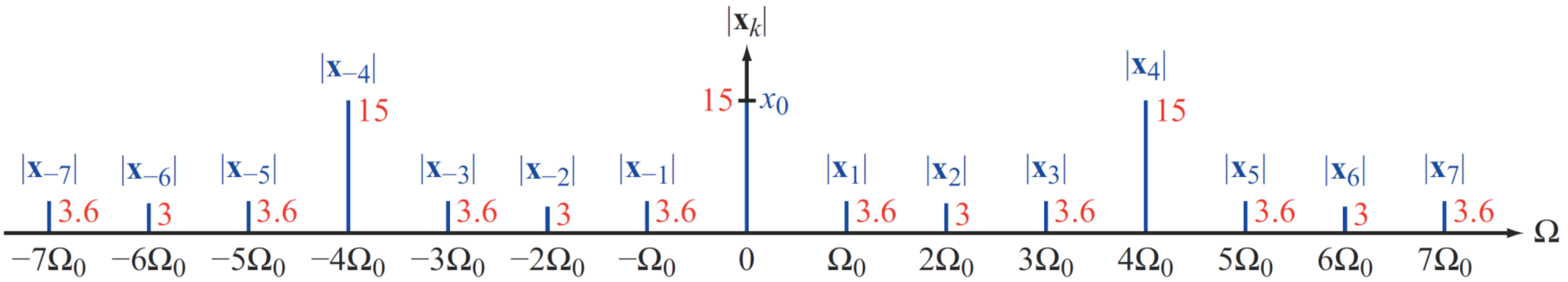
DTFS Example, Continued

The DTFS expansions of the given periodic signal x[n] is then

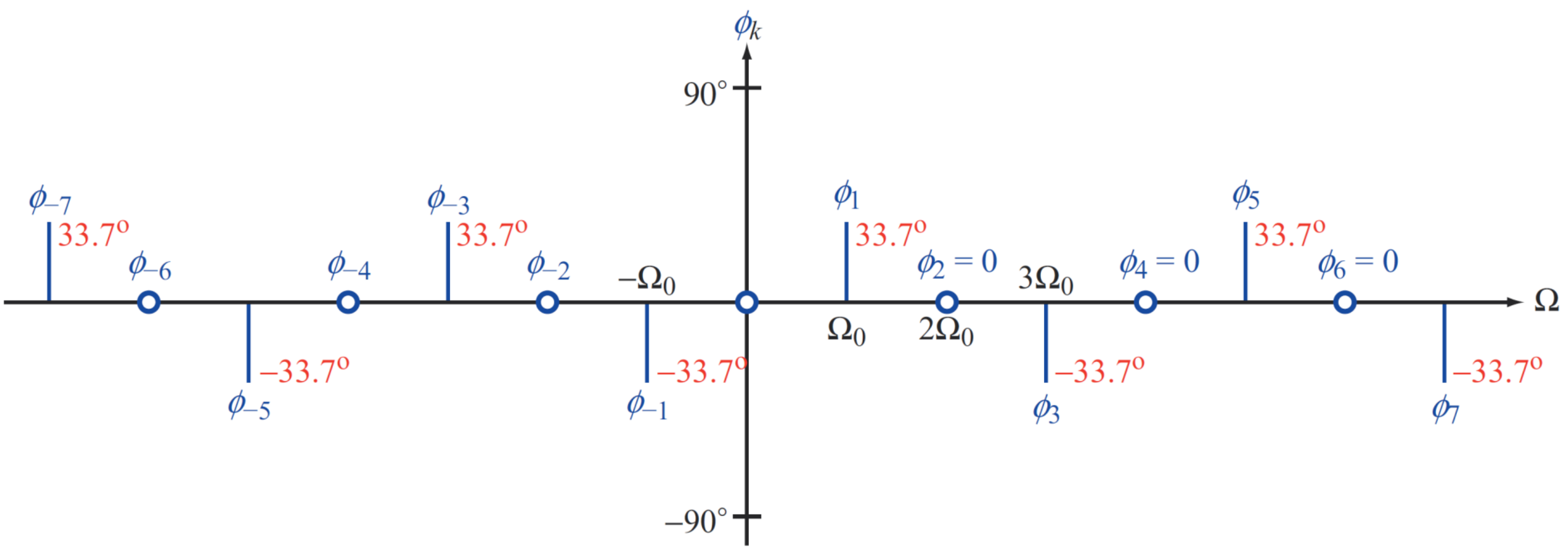
$$
\begin{aligned}
x[n] &= x_0 + \mathbf{x}_1 e^{j\pi n/2} + \mathbf{x}_2 e^{j\pi n} + \mathbf{x}_3 e^{j3\pi n/2} \\
&= 15 + 3.6e^{j33.7^\circ} e^{j\pi n/2} + 3e^{j\pi n} + 3.6e^{-j33.7^\circ} e^{j3\pi n/2}. \\
&= 15 + 7.2\cos\left(\frac{\pi n}{2} + 33.7^\circ\right) + 3\cos(\pi n).
\end{aligned}
$$

Magnitude and phase line spectra of x[n] on next slide.

DTFS Line Spectra for Example



(b) Magnitude line spectrum



(c) Phase line spectrum

Spectral Symmetry

- In Example 1, were we to use $\mathbf{x}_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]\, e^{-jk\Omega_0 n}$ to compute x_k for negative values of k, the outcome would have been $\mathbf{x}_{-k} = \mathbf{x}_k^*$,

which means that the magnitude line spectrum is an even function and the phase line spectrum is an odd function. This is consistent with CTFS.

- Were we to compute x_k for values of k greater than 3, we would discover that the line spectra repeats periodically at a period of four lines. This is very different from the spectra of periodic continuous-time signals.

So we notice that:

(1) Both spectra repeat every $N_0 = 4$ lines.

(2) Magnitude spectrum $|\mathbf{x}_k|$ is an even function.

(3) Phase spectrum ϕ_k is an odd function.

DTFS: Parseval's Theorem

Average power is the same
In time or frequency domain

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1}|x[n]|^2 = \sum_{k=0}^{N_0-1}|\mathbf{x}_k|^2.$$

For the previous example, we have:

Time Domain: $P_{\text{av}} = \frac{1}{4}\left[(24)^2 + (8)^2 + (12)^2 + (16)^2\right] = 260.$

Frequency Domain: $P_{\text{av}} = (15)^2 + (3.6)^2 + 3^2 + (3.6)^2 = 260.$

(Note that 3.6 is actually $\sqrt{13}$ rounded off.)

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t)$, $\alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$