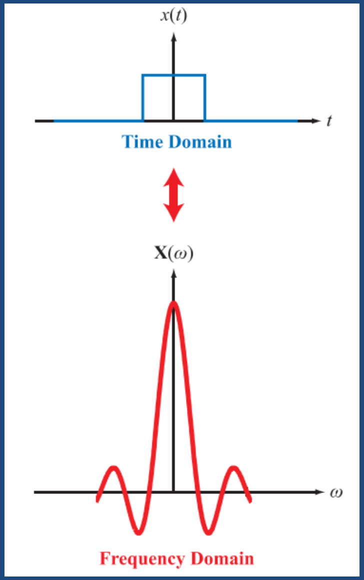
LECTURE BLOCK 4A: FOURIER TRANSFORM (OF NON-PERIODIC SIGNALS)



Dr. Yau Hee Kho

yauhee.kho@vuw.ac.nz

VICTORIA UNIVERSITY OF WELLINGTON TE HERENGA WAKA

Fourier Transform

We will learn:

- Compute Fourier Transform of nonperiodic signals,
- Compute Fourier Transform of periodic signals.
- Discrete time Fourier Transform.

Fourier Transform

Fourier analysis via the Fourier series is limited to periodic signals.

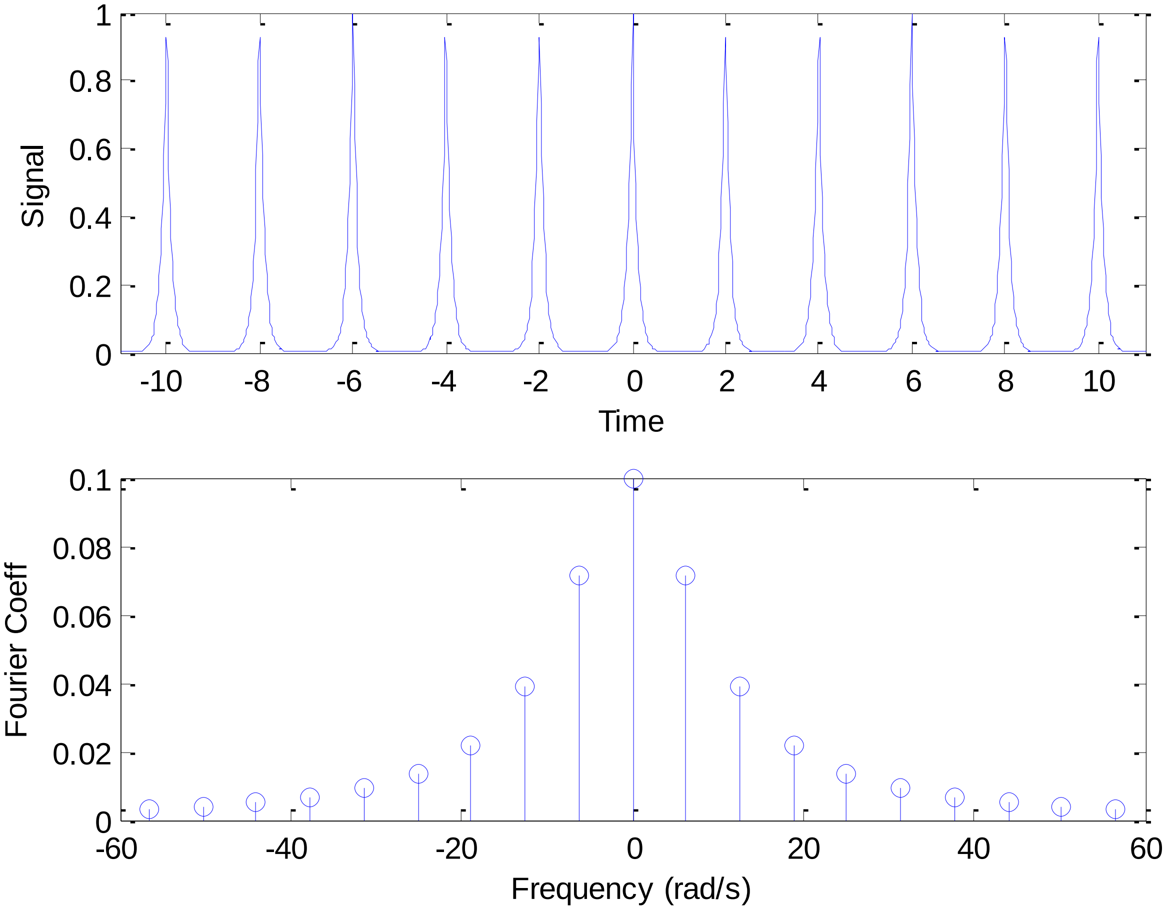
Most signals are not periodic and it is important to extend the idea of Fourier analysis to such signals.

The aim is to find a way of defining a spectrum for non-periodic signals.

We first treat the case of <u>pulse-like finite energy signals</u>.

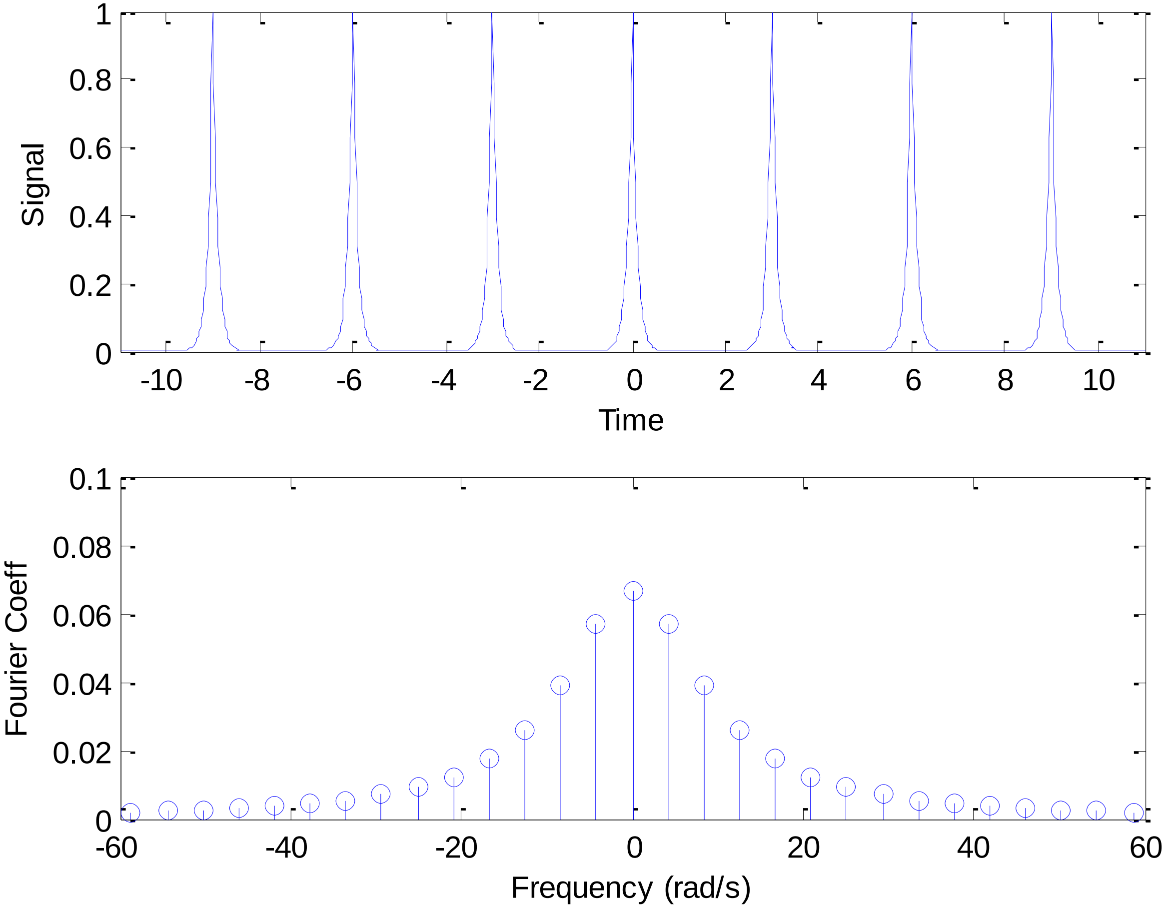
Fourier Transform

Let's start with a periodic signal.



Fourier Transform

Increase the period while keeping the pulse shape fixed.

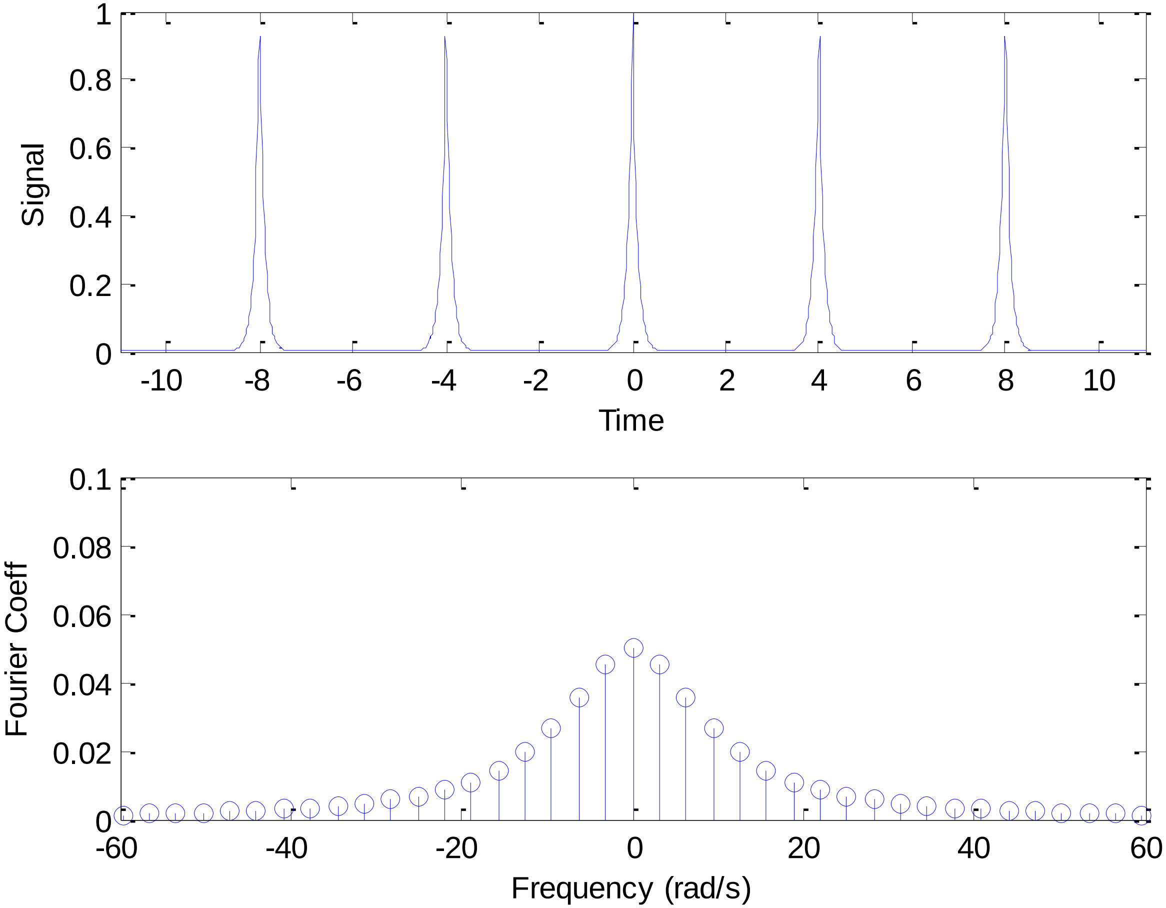


Note that the number of harmonics on a given frequency interval increases as the period increases.

Also note that amplitude of the harmonics decreases.

Fourier Transform

Keep increasing the period (T) while keeping the pulse shape fixed.

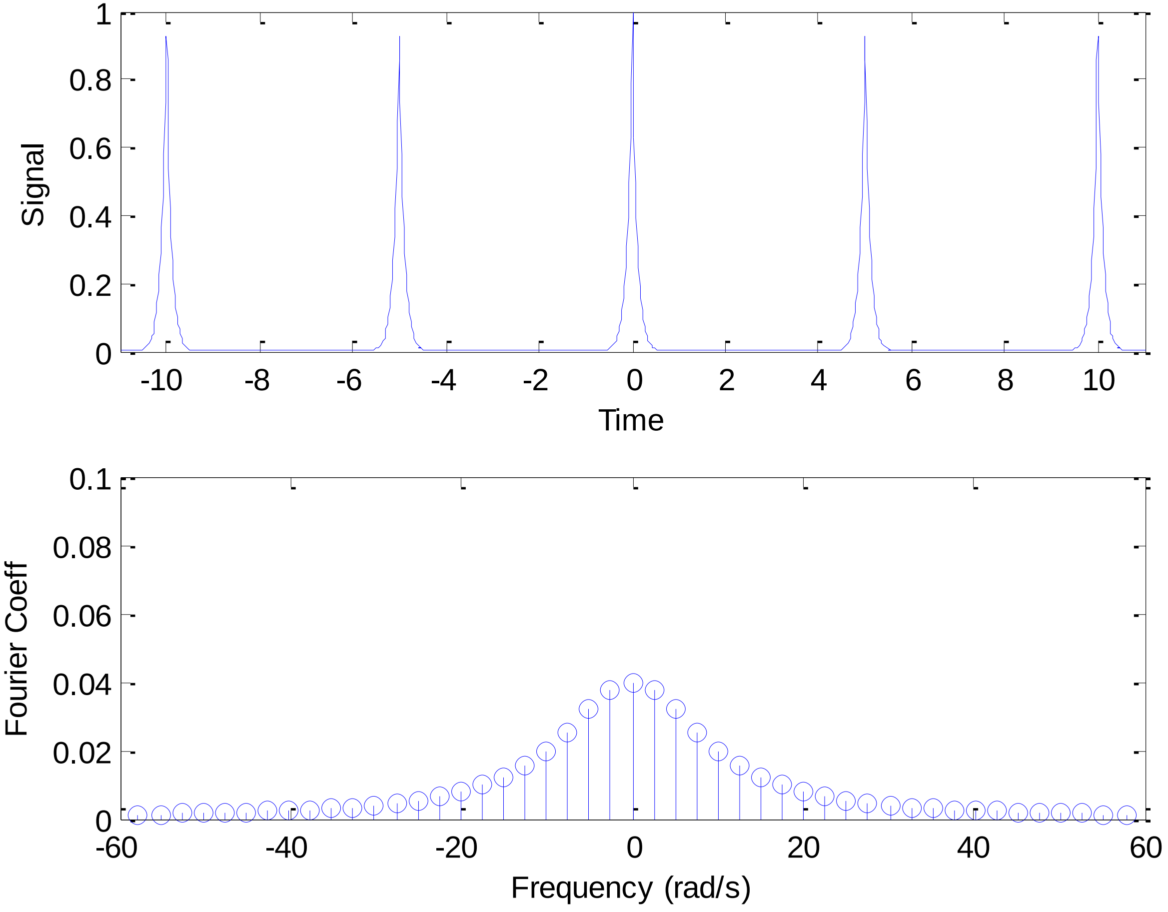


The interval between harmonics decreases in proportion to 1/T.

As does the amplitude.

Fourier Transform

Keep increasing the period (T) while keeping the pulse shape fixed.



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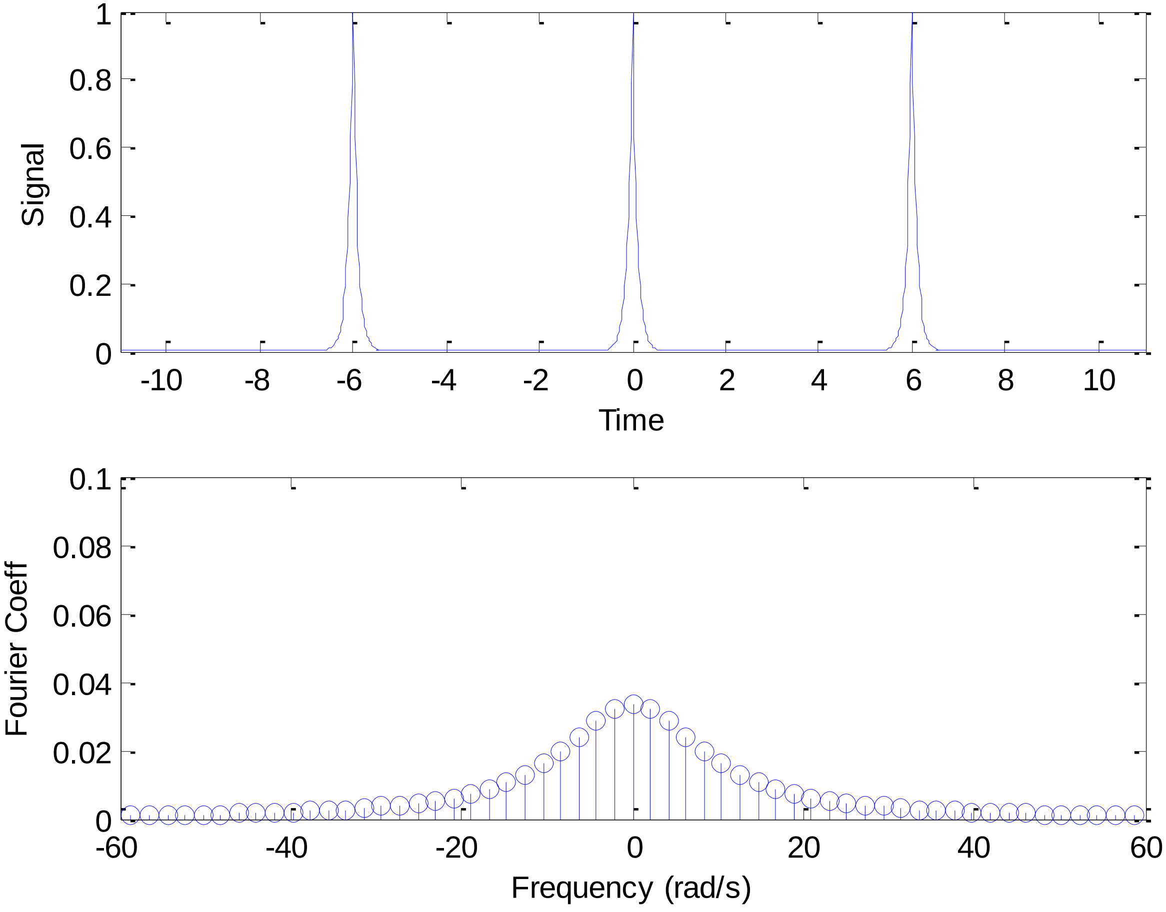
Fourier Transform

Keep increasing the period (T) while keeping the pulse shape fixed.

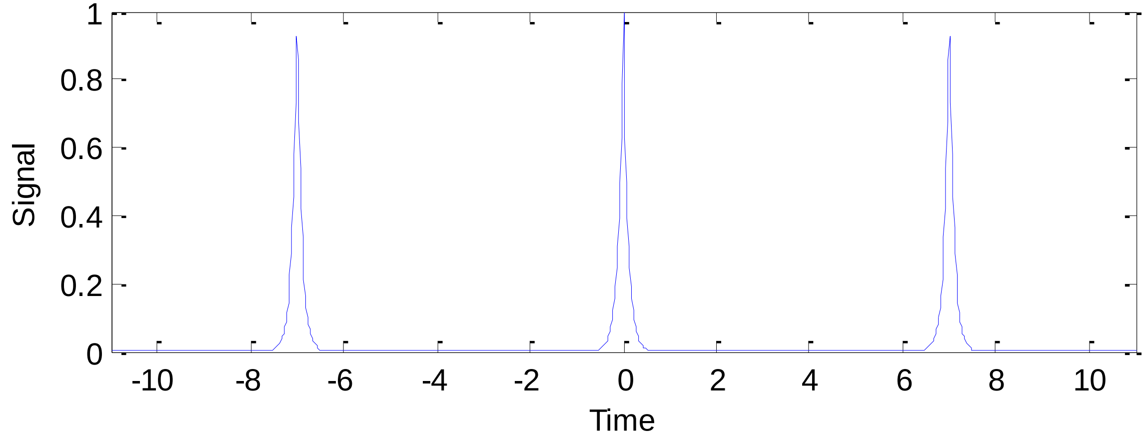


The interval between harmonics decreases to zero as the period goes to infinity.

And the amplitude approaches zero.

Fourier Transform

Keep increasing the period (T) while keeping the pulse shape fixed.



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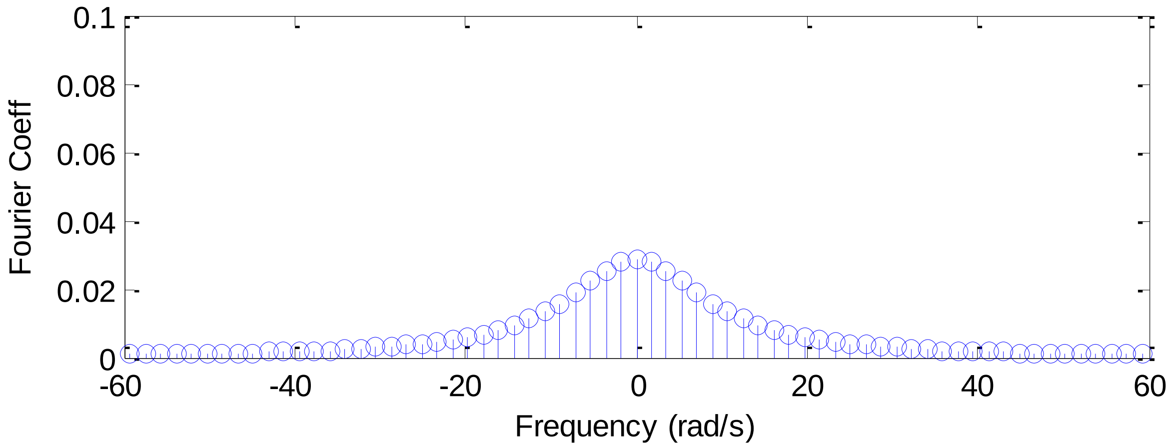
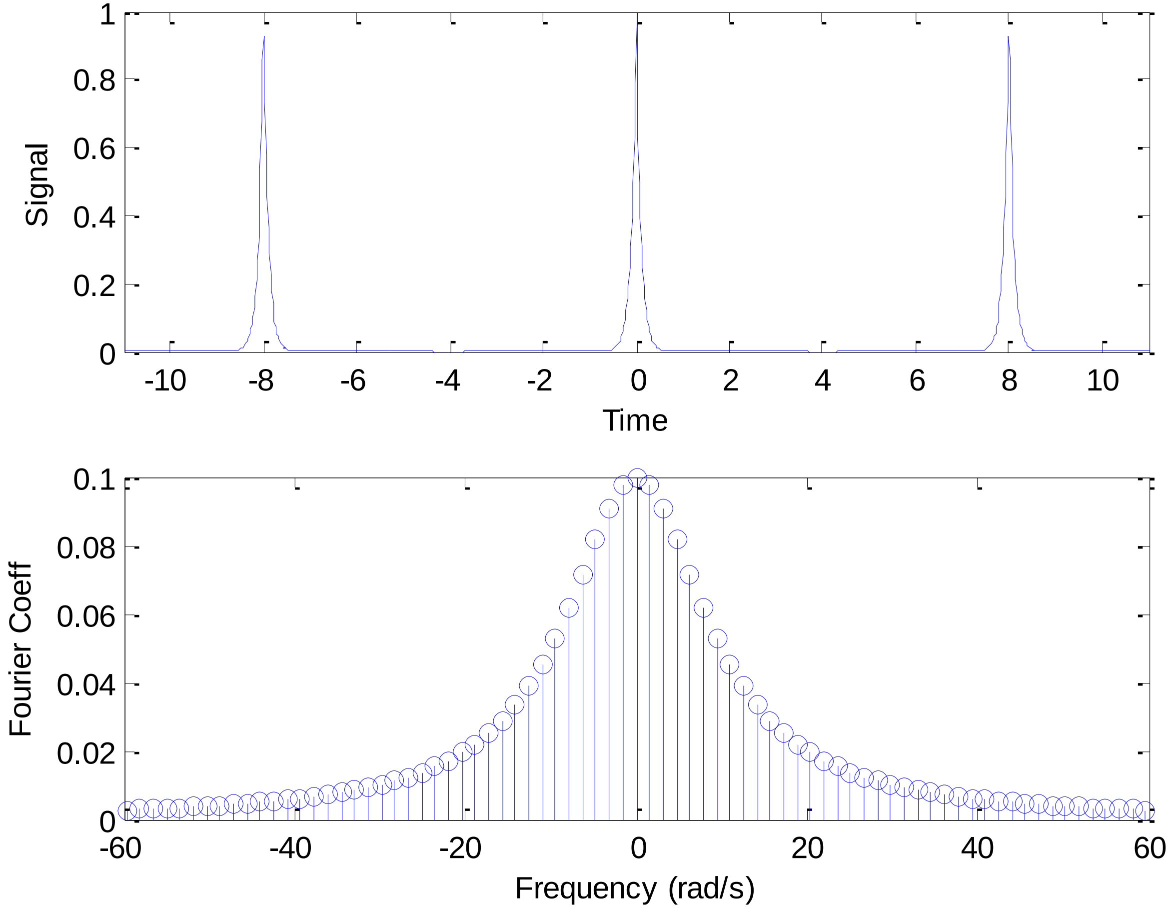


Fourier Transform

To counteract the decrease in amplitude, multiply the coefficients by T.



The interval between harmonics decreases to zero as the period goes to infinity.

But the scaled amplitude envelope now remains constant.

Fourier Transform

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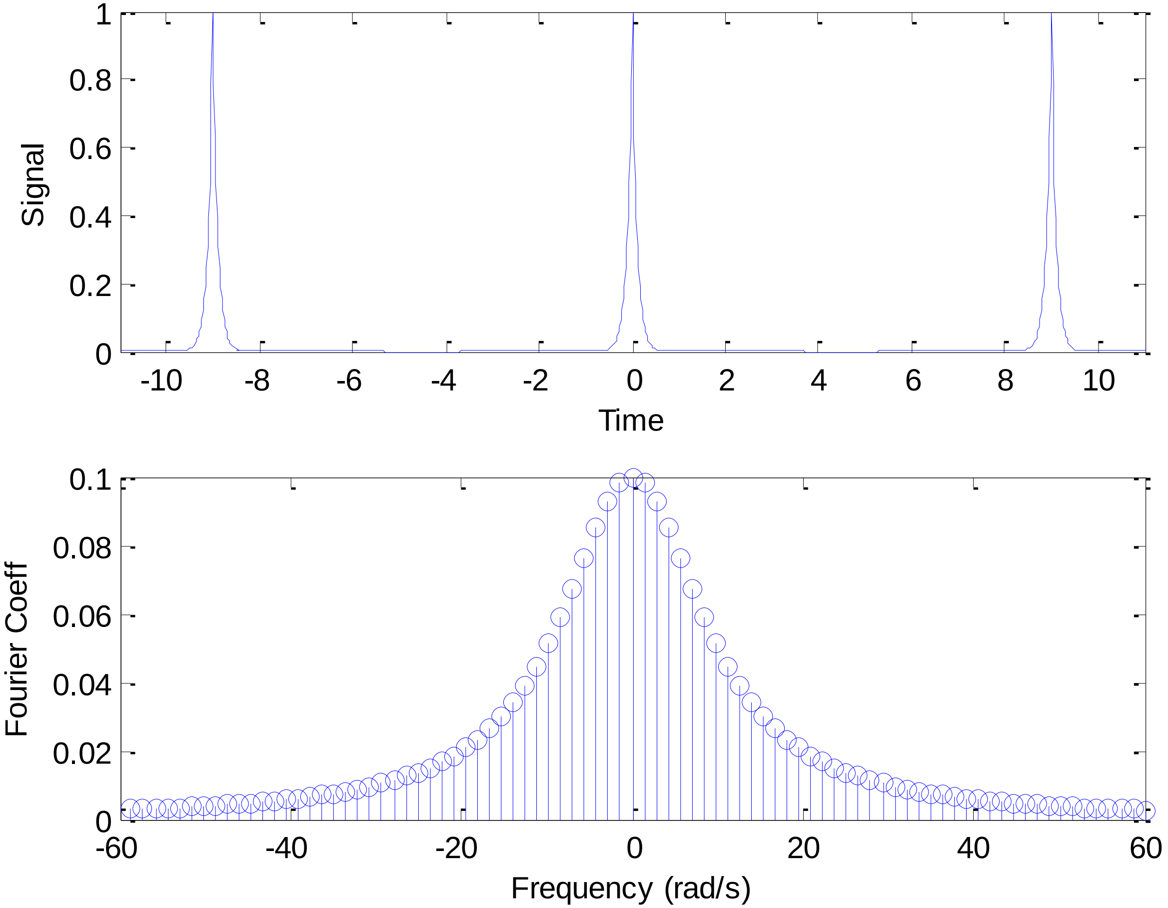
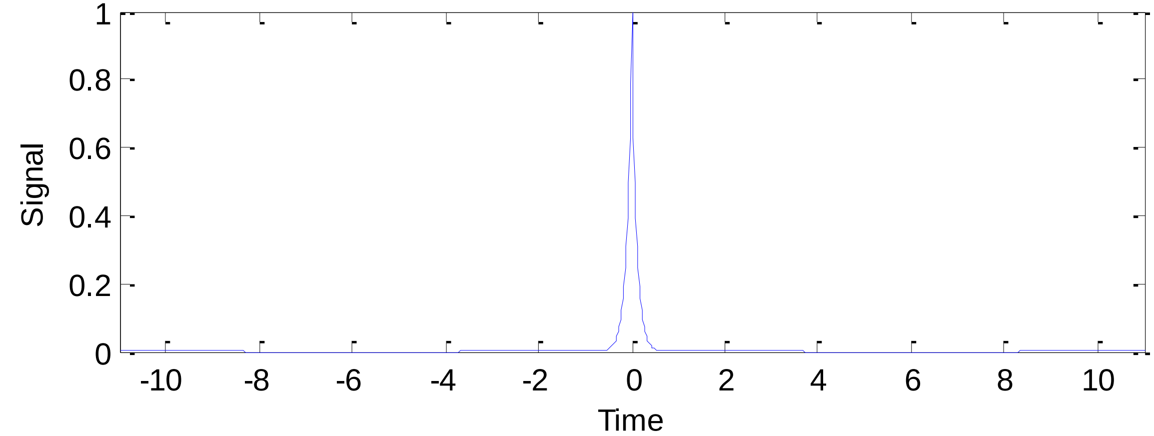


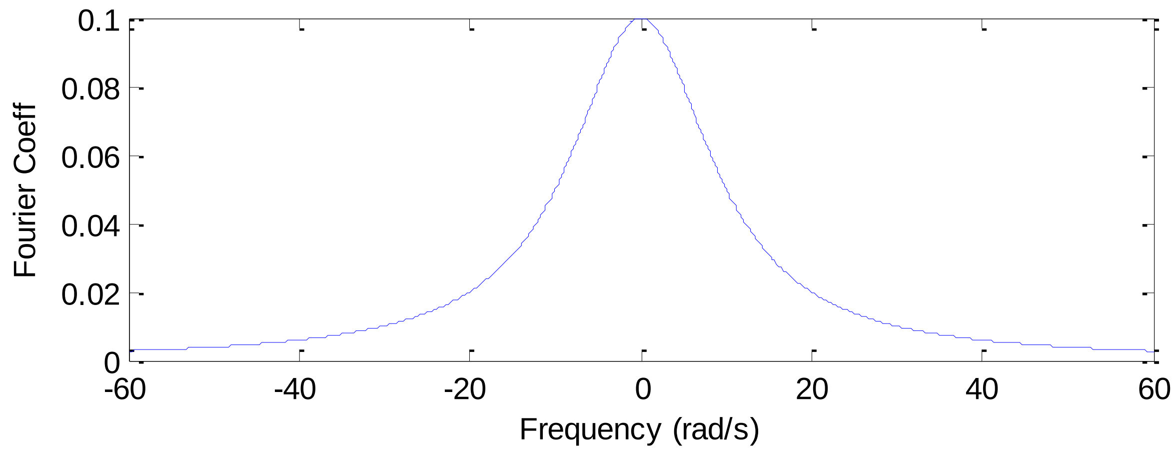
Fourier Transform

If the period is allowed to go to infinity we end up with a single pulse.



The interval between harmonics has become zero - the harmonics now form a continuum.

And the scaled amplitude envelope, is determined by the pulse shape alone.



Fourier Transform

- For non-periodic signals
- Obtain from Fourier series as period$\rightarrow\infty$
- Compute forward Fourier transform from x(t):

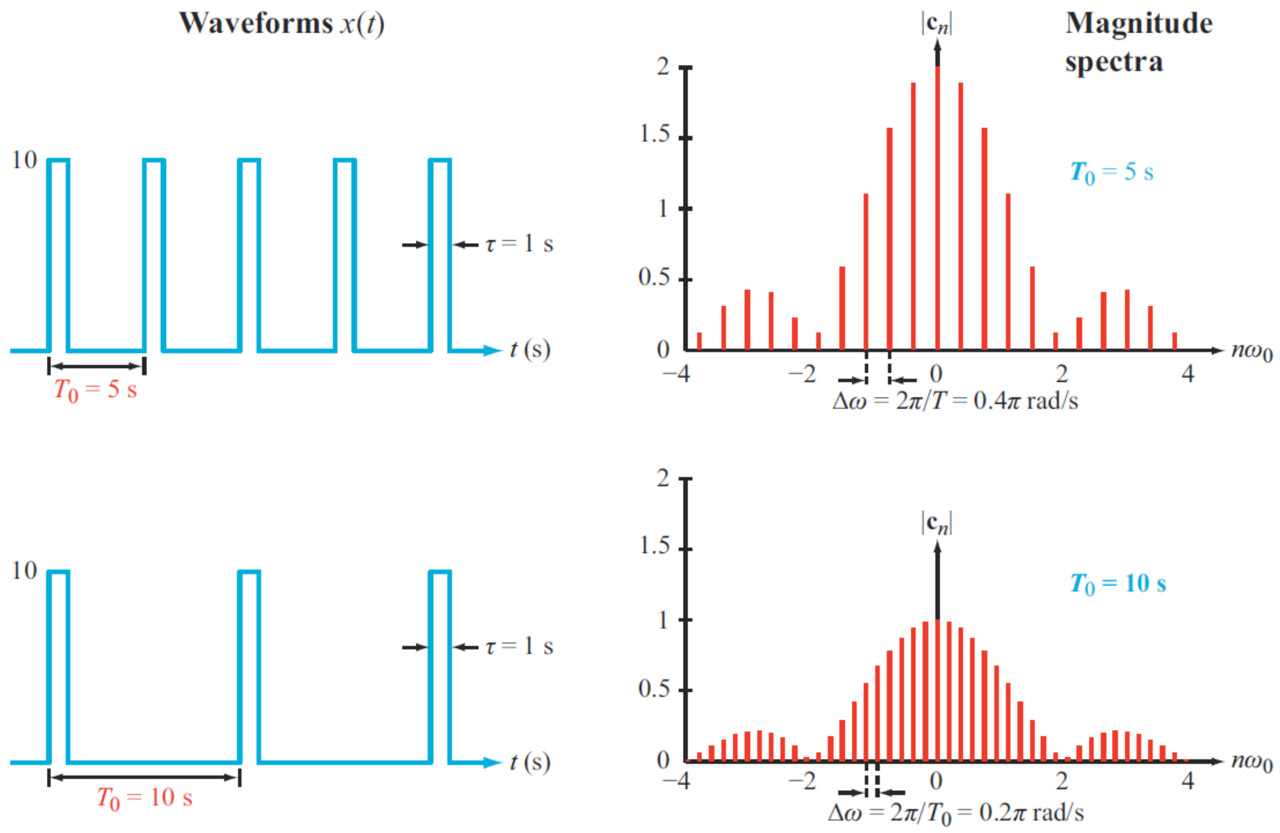
$$\mathbf{X}(\omega) = \mathfrak{F}[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

- Compute inverse Fourier transform: recover x(t):

$$x(t) = \mathfrak{F}^{-1}[\mathbf{X}(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{X}(\omega)\, e^{j\omega t}\, d\omega$$

Fourier Series→Fourier Transform

- Consider a periodic pulse train. Let its period→∞. Then its two-sided magnitude line spectrum looks like:



Fourier Series→Fourier Transform

- Exponential Fourier series: $x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t}$
- Coefficients computed using: $\mathbf{x}_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t')\, e^{-jn\omega_0 t'}\, dt'$
- Spacing between harmonics: $\Delta\omega = (n+1)\omega_0 - n\omega_0 = \omega_0 = \frac{2\pi}{T_0}$
- Insert coefficient formula into exponential series:

$$x(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-T_0/2}^{T_0/2} x(t')\, e^{-jn\omega_0 t'}\, dt' \right] e^{jn\omega_0 t}\, \Delta\omega.$$

- Let the period→∞. Bracketed quantity is $\hat{\mathbf{X}}(\omega)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t')\, e^{-j\omega t'}\, dt' \right] e^{j\omega t}\, d\omega.$$

Important Fourier Transform Pairs

- Impulses in time and frequency:

$$\delta(t) \leftrightarrow 1. \qquad 1 \leftrightarrow 2\pi\,\delta(\omega).$$

- Causal exponential signals:

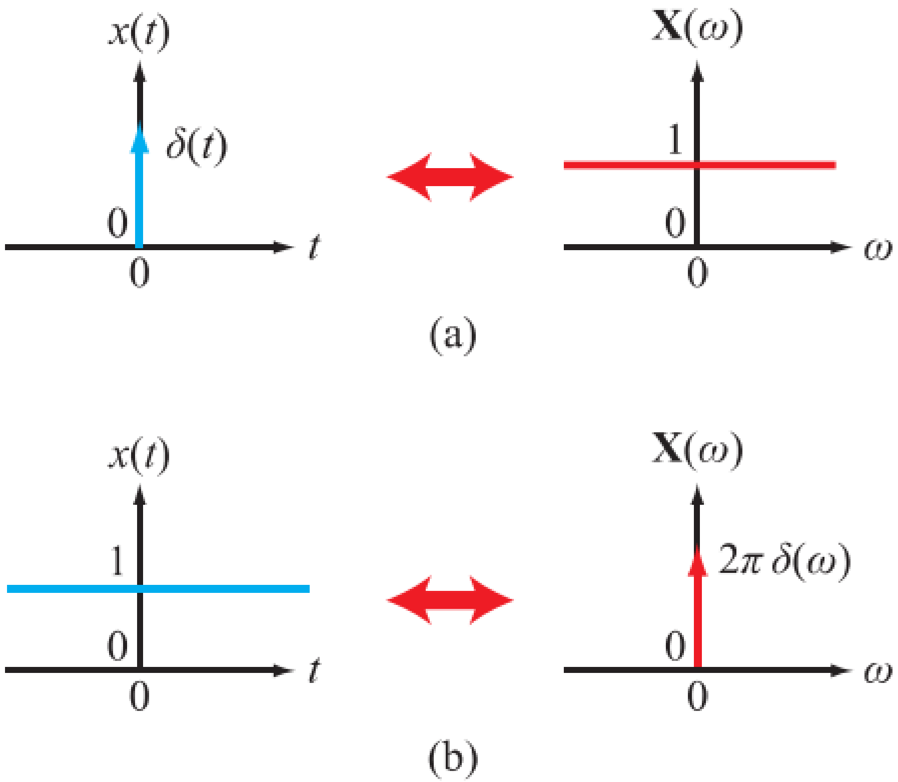
$$Ae^{-at}\,u(t) \leftrightarrow \frac{A}{a+j\omega} \quad \text{for } a>0.$$

- Eternal sinusoidal signals:

$$\cos\omega_0 t \leftrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)],$$

$$\sin\omega_0 t \leftrightarrow j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)].$$

Important Fourier Transform Pairs



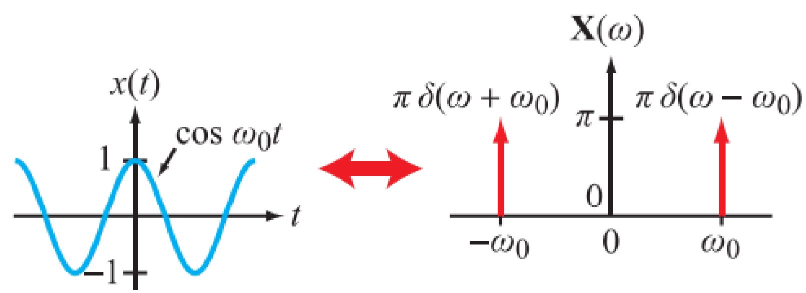


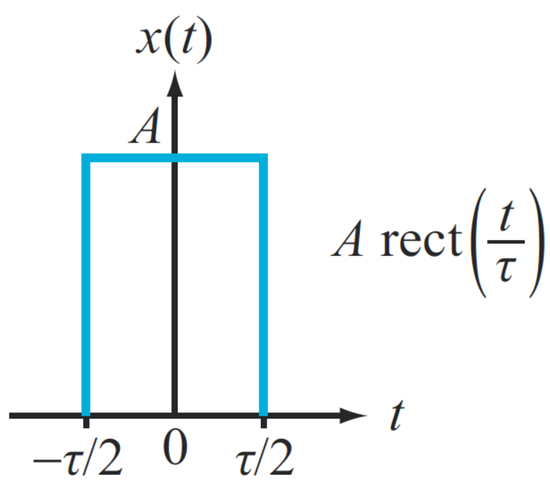
Figure 5-15: The Fourier transform of $\delta(t)$ is 1 and the Fourier transform of 1 is $2\pi\ \delta(\omega)$.

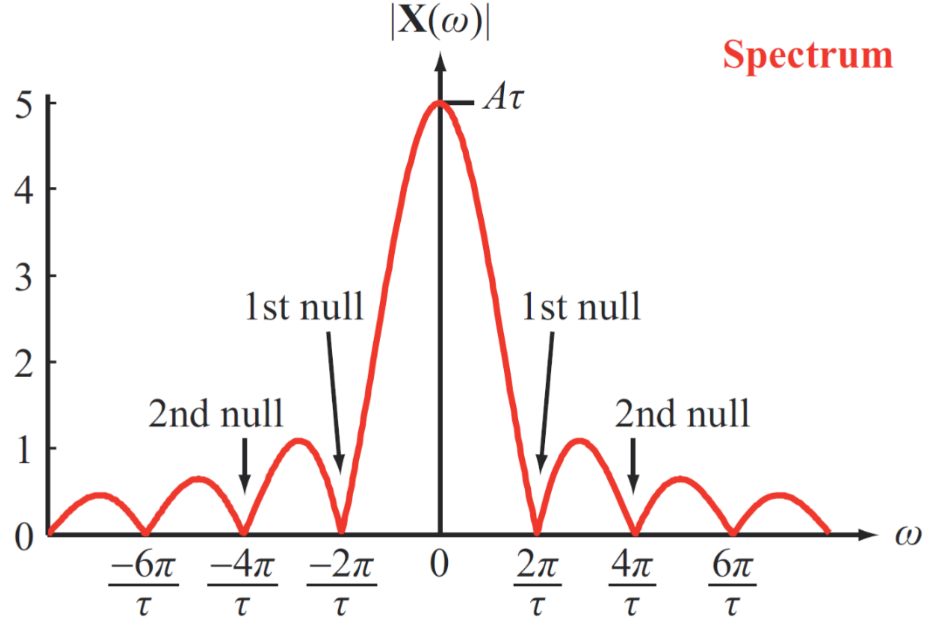
Fourier Transform of a Pulse

$x(t) = A\,\mathrm{rect}(t/\tau)$ leads to $\mathbf{X}(\omega) = A\tau\,\dfrac{\sin(\omega\tau/2)}{(\omega\tau/2)} = A\tau\,\mathrm{sinc}\left(\dfrac{\omega\tau}{2}\right)$

The sinc function is defined as $\mathrm{sinc}(\theta) = \dfrac{\sin\theta}{\theta}$ $\quad \mathrm{sinc}(0) = \dfrac{\sin(\theta)}{\theta}\Big|_{\theta=0} = 1$

Signal





Demo: derivation of a pulse

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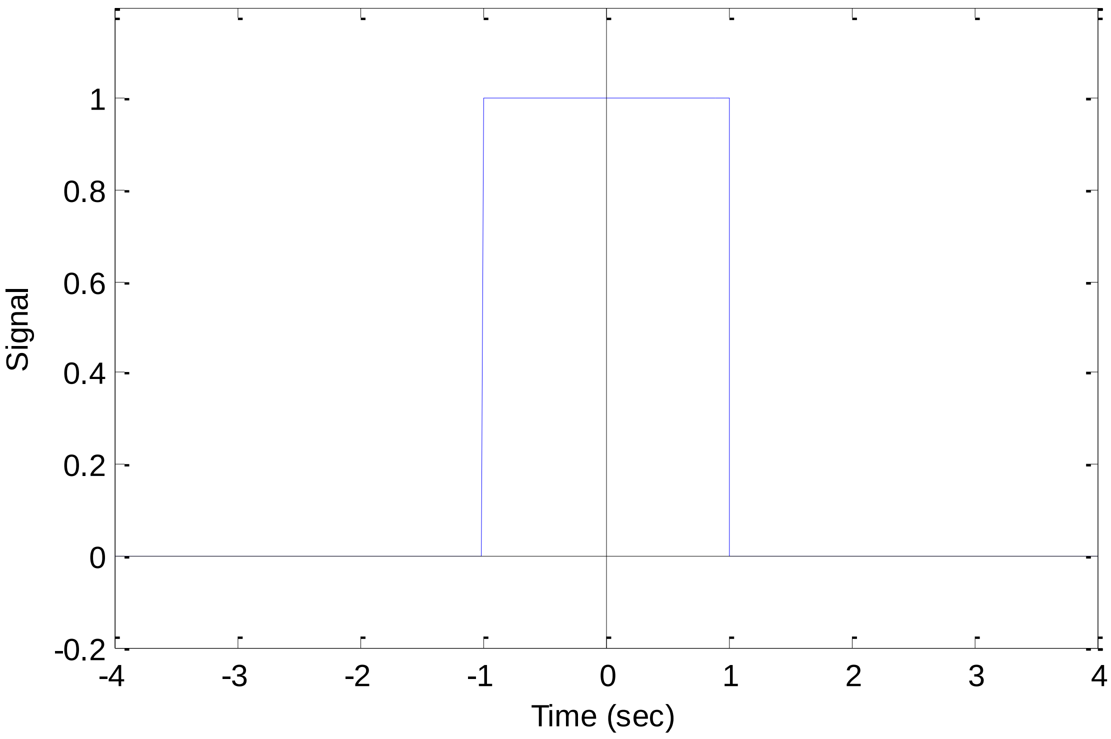
Fourier Transform - Examples - Rectangular pulse

The most commonly encountered pulse is the ideal rectangular pulse.

$$f(t) = P_T(t)$$



Fourier Transform - Examples - Rectangular pulse

The Fourier transform is readily calculated

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Substitute in definition

$$= \int_{-\infty}^{\infty} P_T(t) e^{-j\omega t} dt$$

Definition of pulse

$$F(\omega) = \int_{-T}^{T} e^{-j\omega t} dt$$

Integral of exponential

$$= \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-T}^{T}$$

Euler's Theorem

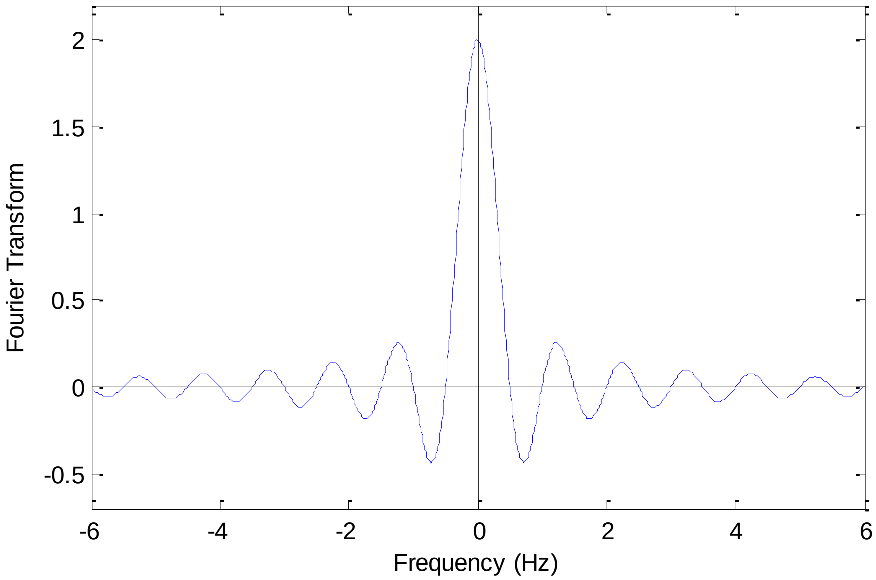
$$= 2\frac{\sin(\omega T)}{\omega}$$

Fourier Transform - Examples - Rectangular pulse

This is very similar to the Fourier coefficients of the rectangular pulse train, and it is also convenient to write this in terms of the *sinc* function

$$P_T(t) \leftrightarrow 2T\,\mathrm{sinc}(\omega T)$$

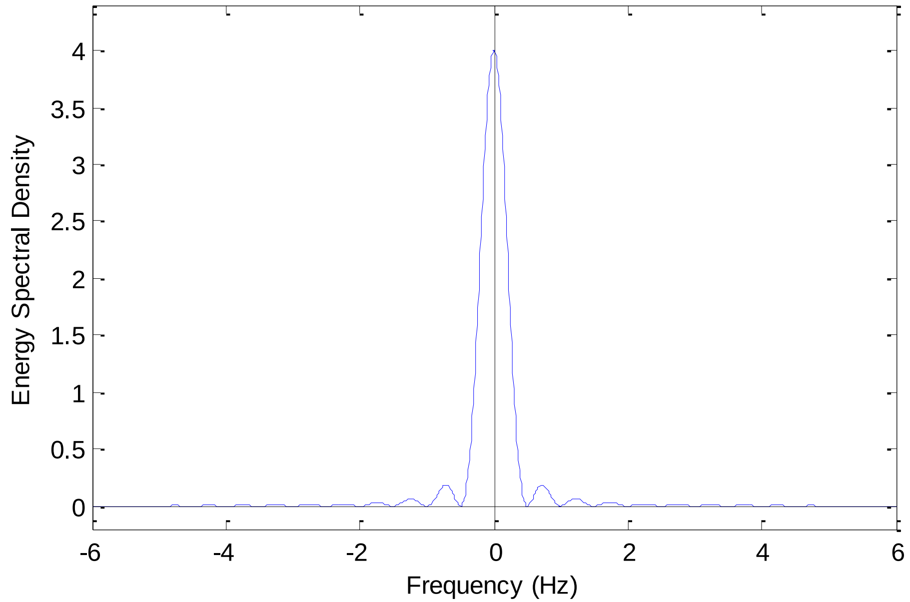
The double headed arrow notation means that the function on the right is the Fourier transform of the function on the left.



Fourier Transform - Examples - Rectangular pulse

Since, in this case, the transform is real, the energy spectral density is just the square of the Fourier transform:

$$S(\omega) = 4T^2 \operatorname{sinc}^2(\omega T)$$

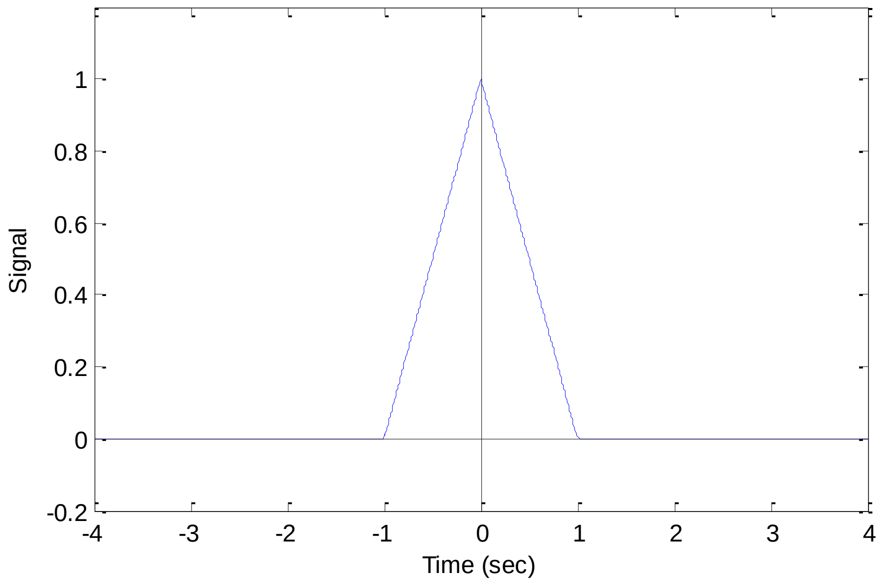


As with the rectangular pulse train, an appreciable fraction of the energy lies in the high frequency side lobes.
In general high frequency components are associated with rapid changes in the signal (in this case the edges of the pulse).

Fourier Transform - Example - Triangular pulse

Another common ideal pulse is the unit height triangular pulse:

$$Q_T(t) = \begin{cases} \left(1 - \dfrac{|t|}{T}\right) & : \quad |t| \le T \\ 0 & : \quad |t| > T \end{cases}$$



Demo: derivation of a triangular pulse

Fourier Transform - Example - Triangular pulse

The Fourier transform is fairly readily calculated

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Substitute in definition

$$= \int_{-\infty}^{\infty} Q_T(t) e^{-j\omega t} dt$$

Definition of pulse

$$\mathrm{F}(\omega) = \int_{-T}^{T} (1 - |t| / T) e^{-j\omega t} dt$$

Rectangular pulse result

$$= 2T \operatorname{sinc}(\omega T) - \frac{2}{T} \int_{0}^{T} t \cos(\omega t) dt$$

Fourier Transform - Example - Triangular pulse

The second term on the RHS can be of integrated by parts:

$$\frac{2}{T}\int_0^T t\cos(\omega t)\,dt = 2T\,\mathrm{sinc}(\omega T) - \frac{2(1-\cos(\omega T))}{\omega^2 T}$$

Substitute in previous line

$$F(\omega) = \frac{2(1-\cos(\omega T))}{\omega^2 T}$$

Half angle formula

$$F(\omega) = \frac{4\sin^2\left(\tfrac{1}{2}\omega T\right)}{\omega^2 T}$$

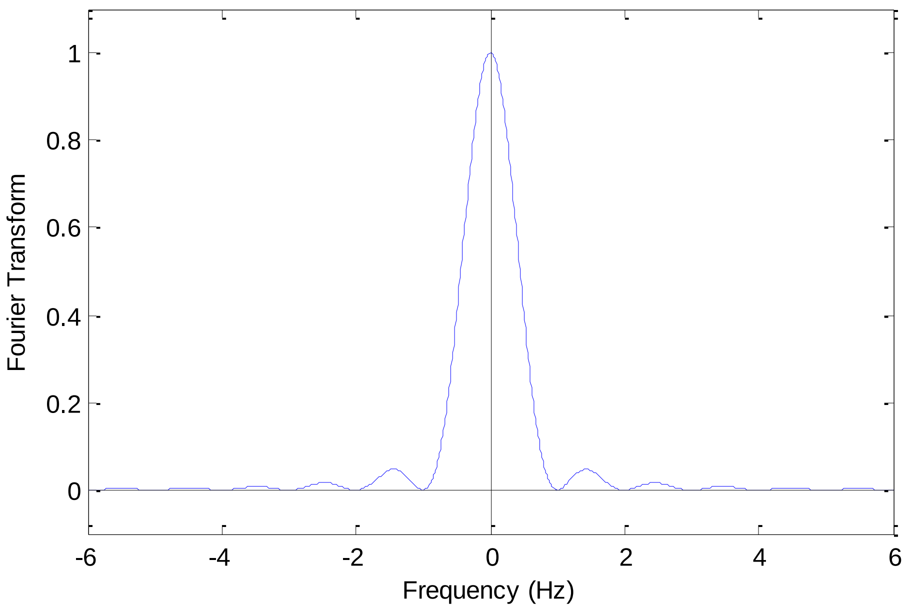
Tidy up

$$F(\omega) = \frac{T\sin^2\left(\tfrac{1}{2}\omega T\right)}{\left(\tfrac{1}{2}\omega T\right)^2}$$

Fourier Transform - Example - Triangular pulse

This result can also be written in terms of the *sinc* function.

$$Q_T(t) \leftrightarrow T\,\mathrm{sinc}^2(\omega T/2)$$

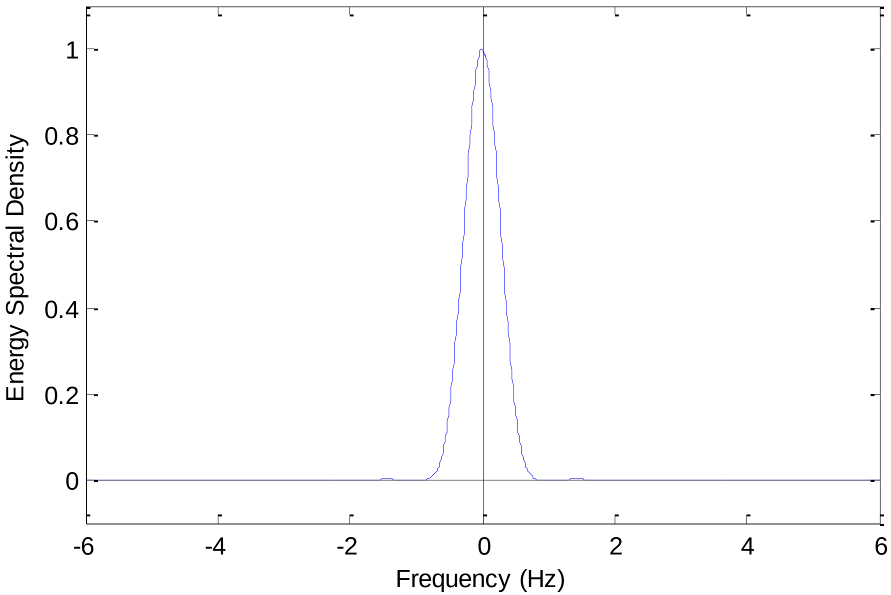


The central lobe of this transform is wider than for the rectangular pulse, indicating that more of the signal energy is concentrated at low frequencies.
This is because the triangular pulse lacks the rapid transitions seen in the rectangular pulse.

Fourier Transform - Example - Triangular pulse

In this case, the transform is also real, so the energy spectral density (ESD) is also just the square of the Fourier transform:

$$Q_T(t) \leftrightarrow T^2 \,\mathrm{sinc}^4(\omega T/2)$$

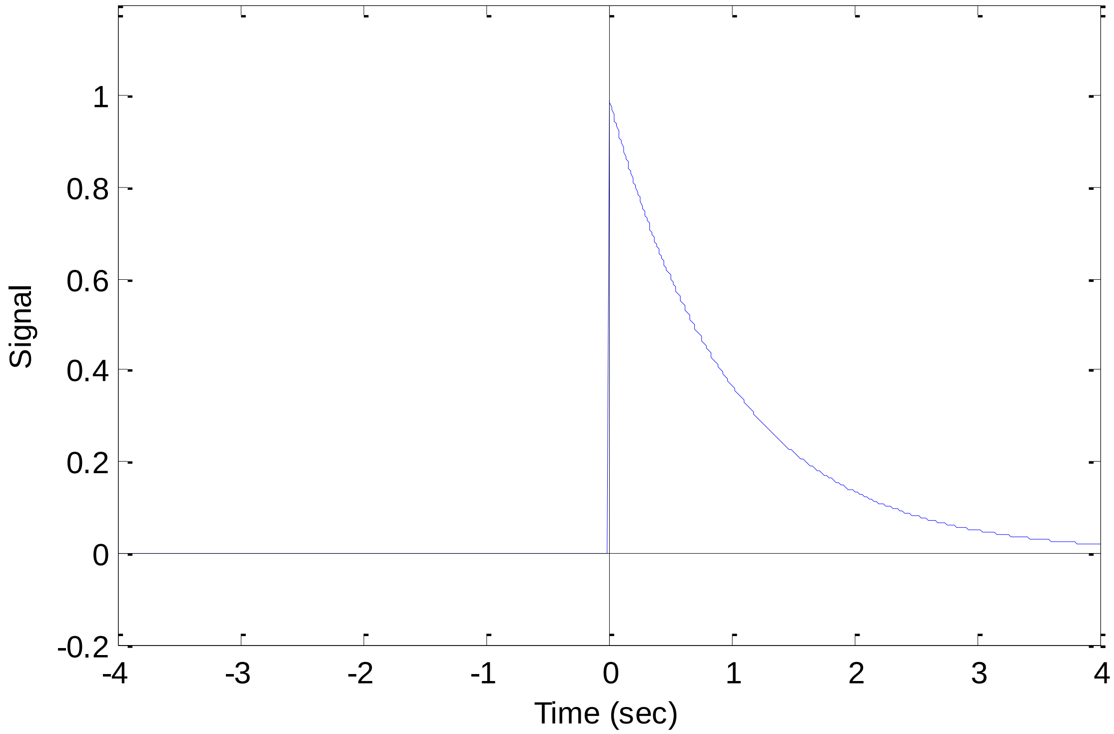


The ESD of the triangular pulse shows much less high frequency energy than the rectangular pulse of the same width.

Fourier Transform - Example – Causal Exponential

Decaying exponential waveforms are common in many physical systems, including electronic circuits. Use the idealised form:

$$f(t) = U(t)e^{-t/T} = \begin{cases} 0 & : \ t < 0 \\ e^{-t/T} & : \ t \geq 0 \end{cases}$$



Fourier Transform - Example – Causal Exponential

The Fourier transform is easily calculated

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Substitute in definition

$$= \int_{-\infty}^{\infty} U(t) e^{-\frac{t}{T}} e^{-j\omega t} dt$$

Definition of U(t) and product of exponentials

$$\mathrm{F}(\omega) = \int_{0}^{\infty} e^{-(\frac{1}{T}+j\omega)t} dt$$

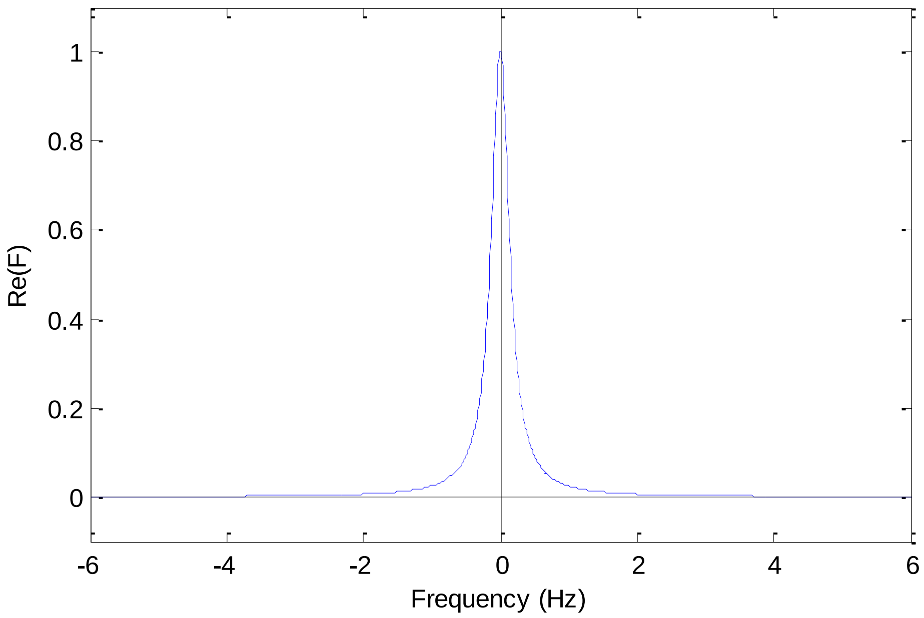
Integral of exponential

$$= \left[\frac{e^{-(\frac{1}{T}+j\omega)t}}{-(\frac{1}{T}+j\omega)} \right]_0^{\infty}$$

Fourier Transform - Example – Causal Exponential

The result is a complex Fourier transform

$$U(t)e^{-t/T} \leftrightarrow \frac{T}{1+j\omega T}$$



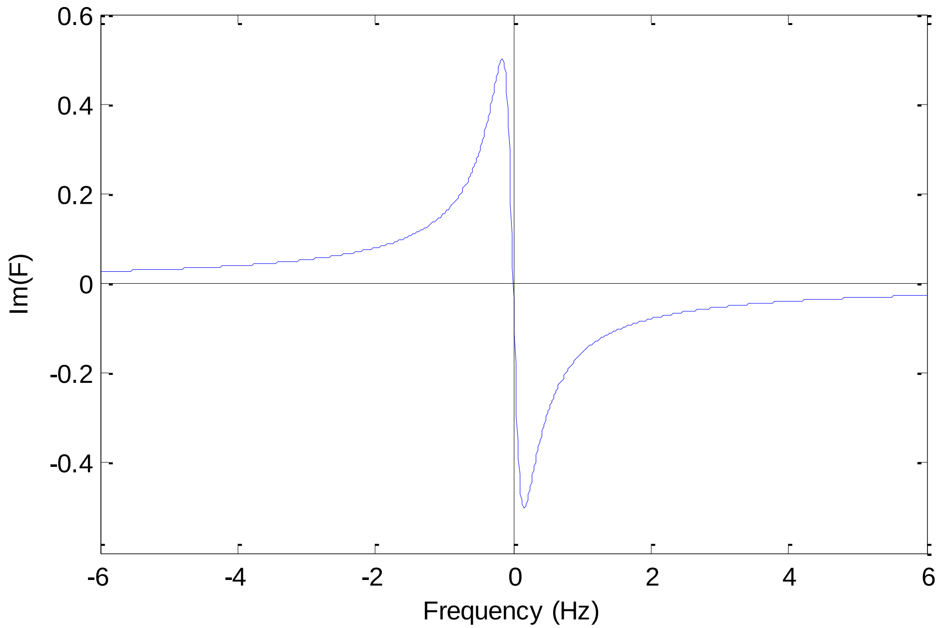
The real part of the transform is an even function of frequency.

$$\mathrm{Re}\,F(\omega) = \frac{T}{1+(\omega T)^2}$$

Fourier Transform - Example – Causal Exponential

The result is a complex Fourier transform

$$U(t)e^{-t/T} \leftrightarrow \frac{T}{1+j\omega T}$$



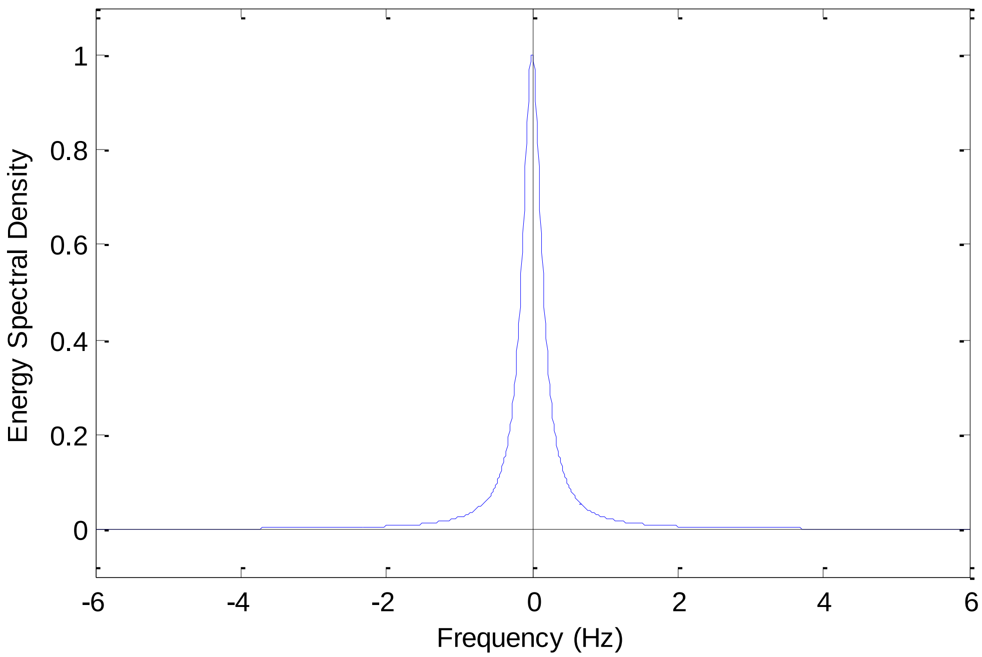
The imaginary part of the transform is an odd function of frequency.

$$\operatorname{Im} F(\omega) = -\frac{\omega T^2}{1+(\omega T)^2}$$

Fourier Transform - Example – Causal Exponential

The ESD is the square of the magnitude of the complex Fourier transform.

$$S(\omega) = F^*(\omega)F(\omega) = \frac{T^2}{1+(\omega T)^2}$$

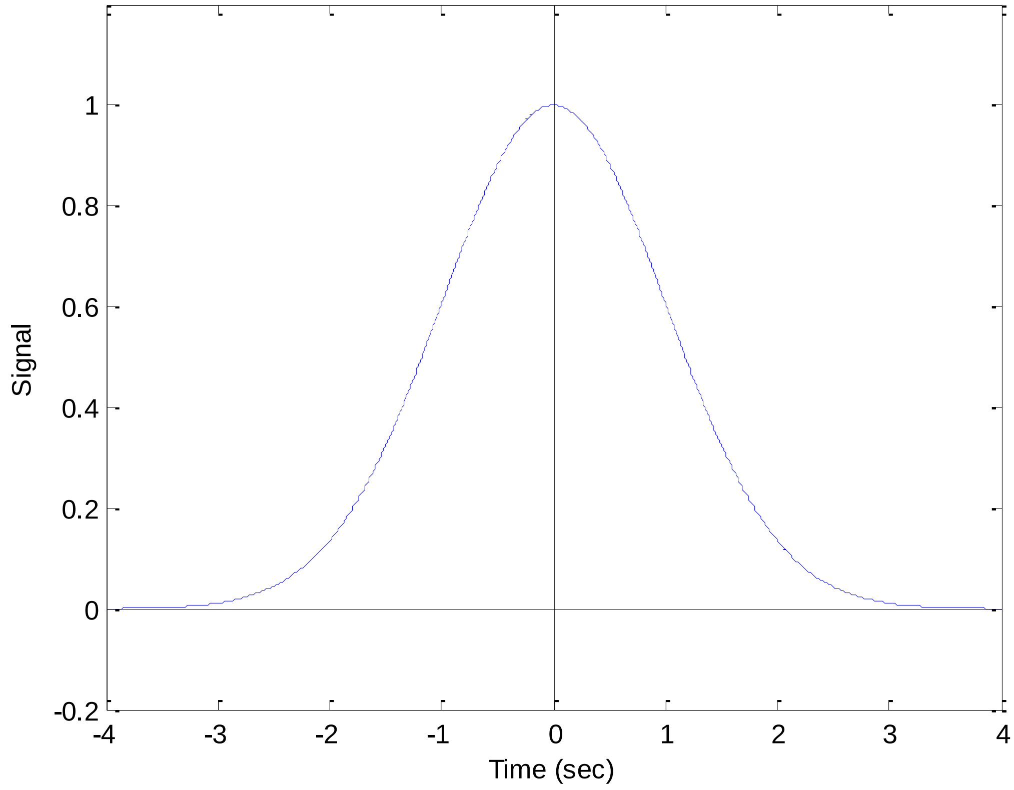


The rapid transition in the exponential pulse at t = 0 is responsible for the high frequency part of the ESD - it falls of at much the same rate as does the ESD of the rectangular pulse.

Fourier Transform - Example – Gaussian

The Gaussian pulse is the limiting form for pulses degraded by transmission effects, for spectral lines from high density gasses etc.

$$f(t) = e^{-\frac{1}{2}(t/T)^2}$$



Fourier Transform - Example – Gaussian

The Fourier transform can be calculated

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Substitute in definition

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{t}{T}\right)^2} e^{-j\omega t} dt$$

Product of exponentials

$$\mathrm{F}(\omega) = \int_{0}^{\infty} e^{-\frac{1}{2}\left[\left(\frac{t}{T}\right)^2 + 2j\omega t\right]} dt$$

Complete the square

$$\mathrm{F}(\omega) = \int_{0}^{\infty} e^{-\frac{1}{2}\left[\left(\frac{t}{T}\right)^2 + 2j\omega t + (j\omega T)^2 - (j\omega T)^2\right]} dt$$

Fourier Transform - Example – Gaussian

Simplify the argument of the exponential to get

$$F(\omega) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{t}{T} + j\omega T\right)^2} e^{-\frac{1}{2}(\omega T)^2}\, dt$$

Shift time independent term outside integral

$$F(\omega) = e^{-\frac{1}{2}(\omega T)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{t}{T} + j\omega T\right)^2}\, dt$$

Change variable of integration

$$F(\omega) = e^{-\frac{1}{2}(\omega T)^2}\, T \int_{-\infty}^{\infty} e^{-\frac{1}{2}(s)^2}\, ds$$

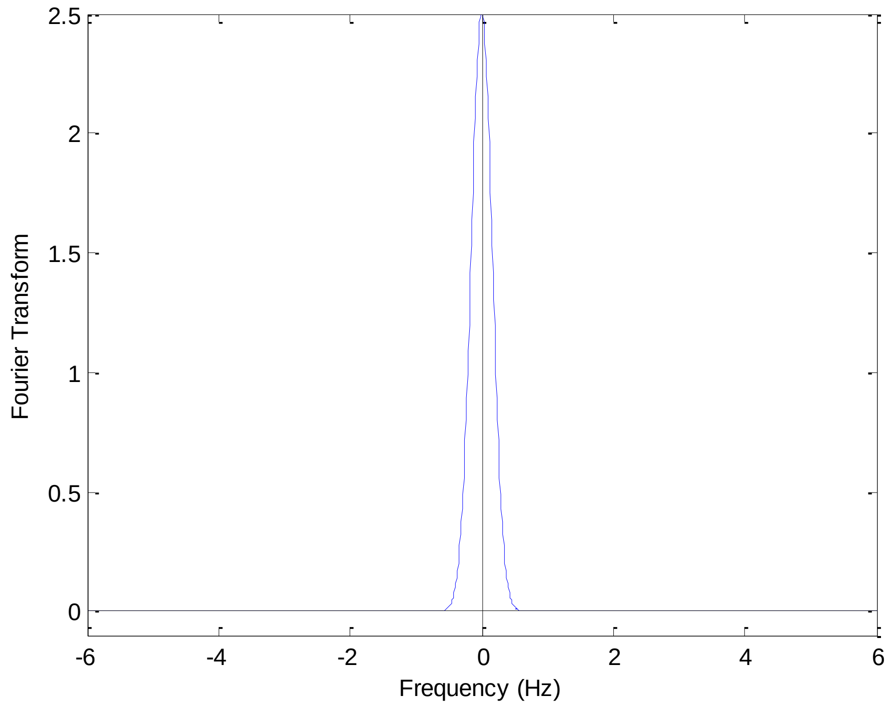
Use known definite integral for Gaussian

$$F(\omega) = e^{-\frac{1}{2}(\omega T)^2}\, T \sqrt{2\pi}$$

Fourier Transform - Example – Gaussian

Thus the Fourier transform of a Gaussian is a Gaussian.

$$e^{-\frac{1}{2}(t/T)^2} \leftrightarrow \sqrt{2\pi}Te^{-\frac{1}{2}(\omega T)^2}$$



The width of the Gaussian in the frequency domain is the inverse of the width of the Gaussian pulse in the time domain.

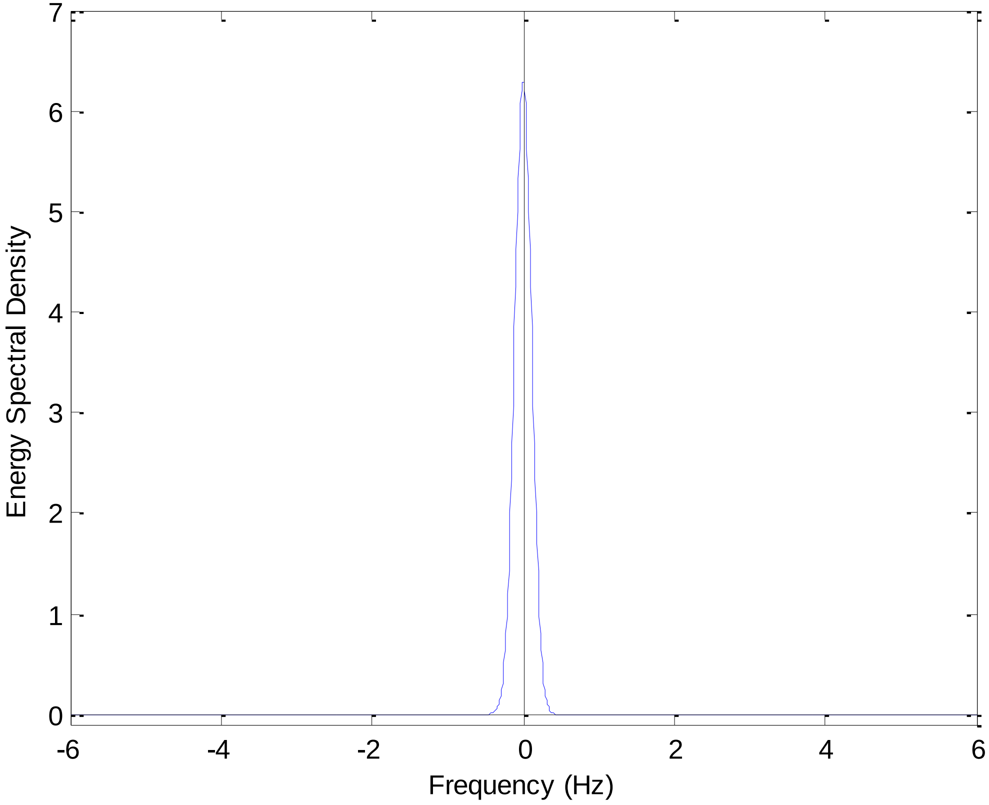
This is a common property of transform pairs, sometimes stated in the form of an uncertainty relation:

$$\Delta\omega\Delta t \sim 1$$

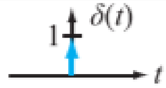
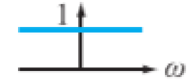
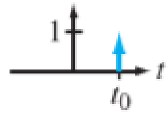
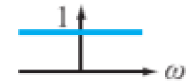
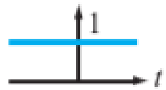
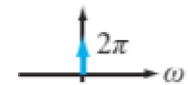
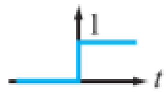
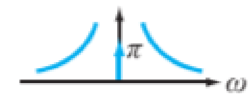
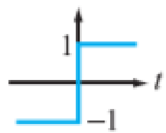
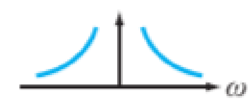
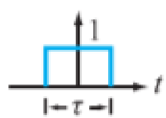
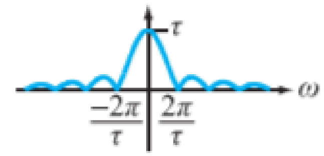
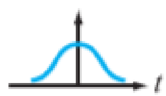
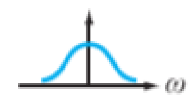
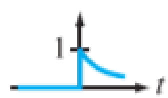
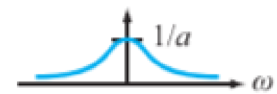
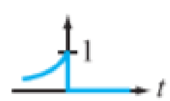
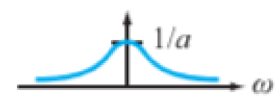
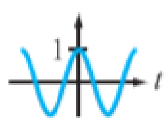
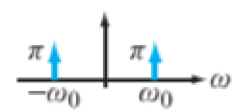
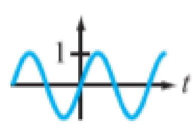
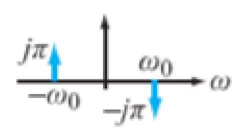
Fourier Transform - Example – Gaussian

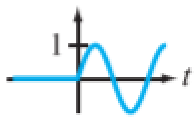
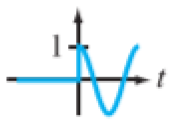
The transform is real so the energy spectral density is its square

$$S(\omega) = 2\pi T^2 e^{-(\omega T)^2}$$



The ESD of a Gaussian is also a Gaussian.

	$x(t)$			$\mathbf{X}(\omega) = \mathcal{F}[x(t)]$	$\lvert\mathbf{X}(\omega)\rvert$
	BASIC FUNCTIONS				
1.		$\delta(t)$	↔	1	
1a.		$\delta(t-t_0)$	↔	$e^{-j\omega t_0}$	
2.		1	↔	$2\pi\ \delta(\omega)$	
3.		$u(t)$	↔	$\pi\ \delta(\omega) + 1/j\omega$	
4.		$\mathrm{sgn}(t)$	↔	$2/j\omega$	
5.		$\mathrm{rect}(t/\tau)$	↔	$\tau\ \mathrm{sinc}(\omega\tau/2)$	
6.		$\dfrac{e^{-t^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}}$	↔	$e^{-\omega^2\sigma^2/2}$	
7a.		$e^{-at}\ u(t)$	↔	$1/(a+j\omega)$	
7b.		$e^{at}\ u(-t)$	↔	$1/(a-j\omega)$	
8.		$\cos\omega_0 t$	↔	$\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$	
9.		$\sin\omega_0 t$	↔	$j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$	

ADDITIONAL FUNCTIONS				
10.		$e^{j\omega_0 t}$	$\Leftrightarrow$	$2\pi\ \delta(\omega - \omega_0)$
11.		$te^{-at}\ u(t)$	$\Leftrightarrow$	$1/(a + j\omega)^2$
12a.		$[e^{-at} \sin \omega_0 t]\ u(t)$	$\Leftrightarrow$	$\omega_0/[(a + j\omega)^2 + \omega_0^2]$
12b.		$[\sin \omega_0 t]\ u(t)$	$\Leftrightarrow$	$(\pi/2j)[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + [\omega_0^2/(\omega_0^2 - \omega^2)]$
13a.		$[e^{-at} \cos \omega_0 t]\ u(t)$	$\Leftrightarrow$	$(a + j\omega)/[(a + j\omega)^2 + \omega_0^2]$
13b.		$[\cos \omega_0 t]\ u(t)$	$\Leftrightarrow$	$(\pi/2)[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + [j\omega/(\omega_0^2 - \omega^2)]$

Important Fourier Transform Properties

$$K_1\, x_1(t) + K_2\, x_2(t) \quad \longleftrightarrow \quad K_1\, \mathbf{X}_1(\omega) + K_2\, \mathbf{X}_2(\omega),$$

(linearity property) (5.90)

$$x'(t) \quad \longleftrightarrow \quad j\omega\, \mathbf{X}(\omega).$$

(derivative property)

$$x(at) \quad \longleftrightarrow \quad \frac{1}{|a|}\mathbf{X}\left(\frac{\omega}{a}\right), \qquad \text{for any } a.$$

(scaling property)

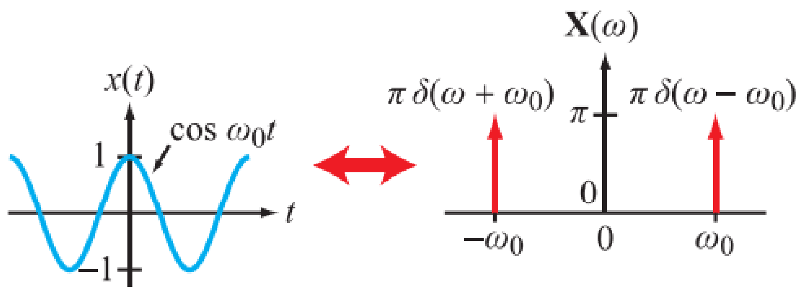
For real-valued x(t), we have:

$$\mathbf{X}(-\omega) = \mathbf{X}^*(\omega).$$

(reversal property)

$$x(t)\cos(\omega_0 t) \quad \longleftrightarrow \quad \frac{1}{2}[\mathbf{X}(\omega - \omega_0) + \mathbf{X}(\omega + \omega_0)].$$

(modulation property) (5.109)



Example of Time Scaling Property

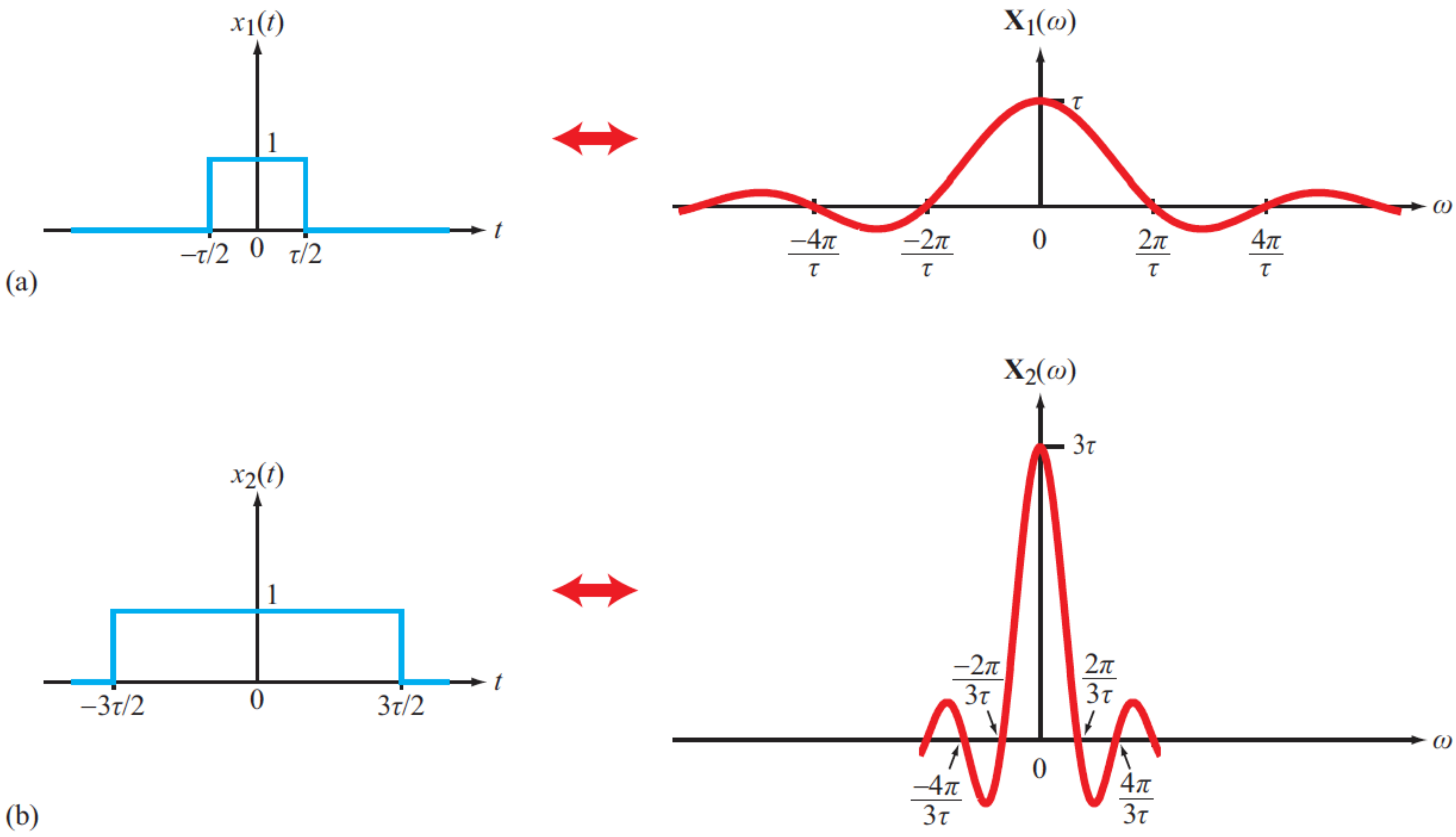


Figure 5-14: Stretching $x_1(t)$ to get $x_2(t)$ entails stretching t by a factor of 3. The corresponding spectrum $\mathbf{X}_2(\omega)$ is a 3-times compressed version of $\mathbf{X}_1(\omega)$.

Important Fourier Transform Properties

- The following two properties are duals of each other:

$$e^{j\omega_0 t}\, x(t) \quad \longleftrightarrow \quad \mathbf{X}(\omega - \omega_0),$$

(frequency-shift property)

$$x(t - t_0) \quad \longleftrightarrow \quad e^{-j\omega t_0}\, \mathbf{X}(\omega).$$

(time-shift property)

- Parseval's theorem for the Fourier transform: ENERGY is the same when computed in time or in frequency:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathbf{X}(\omega)|^2\, d\omega.$$

(Parseval's theorem)

Example of Duality

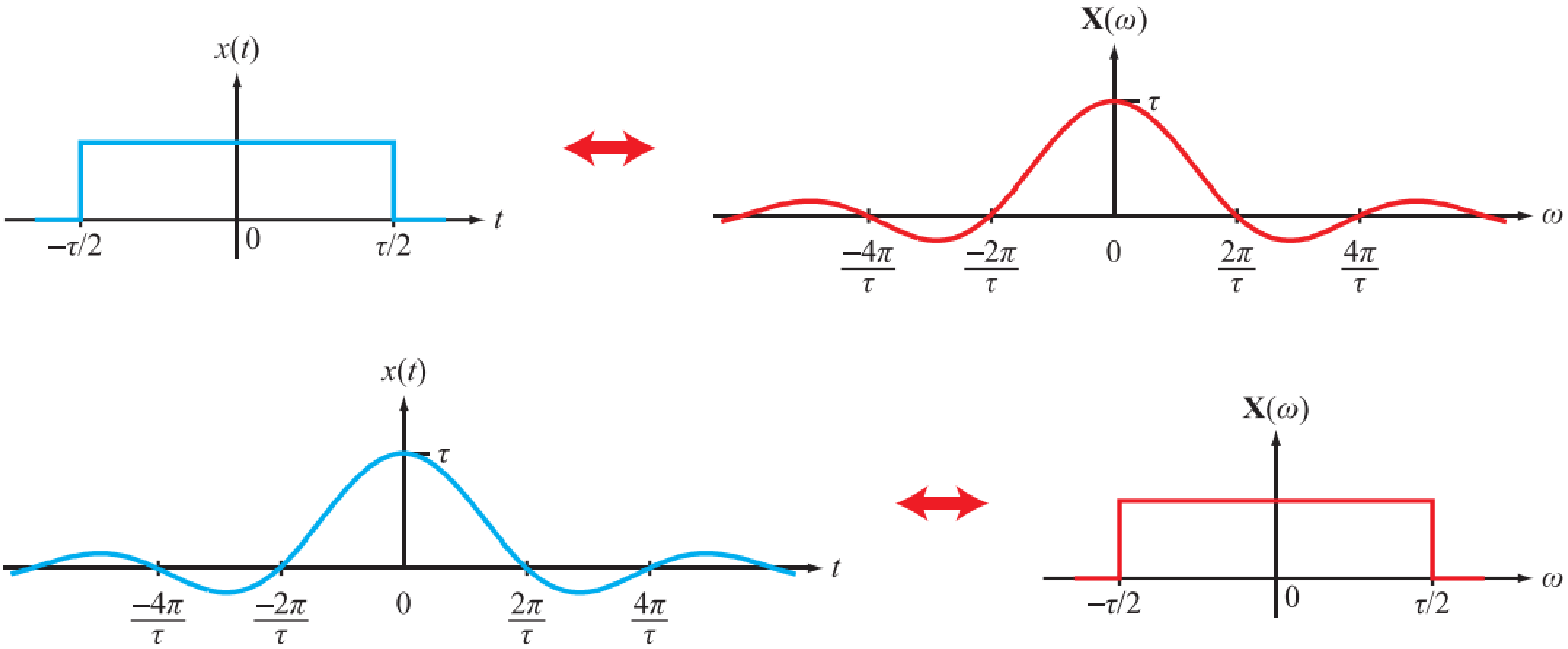


Figure 5-16: Time-frequency duality: a rectangular pulse generates a sinc spectrum and, conversely, a sinc-pulse generates a rectangular spectrum.

Conjugate Symmetry Implications

- <u>If x(t) is real-valued</u>, then $\mathbf{X}(\omega)$ has conjugate symmetry:

$$\mathbf{X}(-\omega) = \mathbf{X}^*(\omega).$$

(reversal property)

Consequently, for a *real-valued* $x(t)$, its Fourier transform $\mathbf{X}(\omega)$ exhibits the following symmetry properties:

(a) $|\mathbf{X}(\omega)|$ is an even function ($|\mathbf{X}(\omega)| = |\mathbf{X}(-\omega)|$).

(b) $\arg[\mathbf{X}(\omega)]$ is an odd function ($\arg[\mathbf{X}(\omega)] = -\arg[\mathbf{X}(-\omega)]$).

(c) If $x(t)$ is real and an even function of time, $\mathbf{X}(\omega)$ will be purely real and an even function.

(d) If $x(t)$ is real and an odd function of time, $\mathbf{X}(\omega)$ will be purely imaginary and an odd function.

Property	$x(t)$		$\mathbf{X}(\omega)=\mathcal{F}[x(t)]=\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$
1. Multiplication by a constant	$K\, x(t)$	$\Leftrightarrow$	$K\, \mathbf{X}(\omega)$
2. Linearity	$K_1\, x_1(t)+K_2\, x_2(t)$	$\Leftrightarrow$	$K_1\, \mathbf{X}_1(\omega)+K_2\, \mathbf{X}_2(\omega)$
3. Time scaling	$x(at)$	$\Leftrightarrow$	$\frac{1}{\lvert a\rvert}\, \mathbf{X}\left(\frac{\omega}{a}\right)$
4. Time shift	$x(t-t_0)$	$\Leftrightarrow$	$e^{-j\omega t_0}\, \mathbf{X}(\omega)$
5. Frequency shift	$e^{j\omega_0 t}\, x(t)$	$\Leftrightarrow$	$\mathbf{X}(\omega-\omega_0)$
6. Time 1st derivative	$x'=\frac{dx}{dt}$	$\Leftrightarrow$	$j\omega\, \mathbf{X}(\omega)$
7. Time nth derivative	$\frac{d^n x}{dt^n}$	$\Leftrightarrow$	$(j\omega)^n\, \mathbf{X}(\omega)$
8. Time integral	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\Leftrightarrow$	$\frac{\mathbf{X}(\omega)}{j\omega}+\pi\, \mathbf{X}(0)\, \delta(\omega)$, where $\mathbf{X}(0)=\int_{-\infty}^{\infty} x(t)\, dt$
9. Frequency derivative	$t^n\, x(t)$	$\Leftrightarrow$	$(j)^n \frac{d^n \mathbf{X}(\omega)}{d\omega^n}$
10. Modulation	$x(t)\, \cos\omega_0 t$	$\Leftrightarrow$	$\frac{1}{2}[\mathbf{X}(\omega-\omega_0)+\mathbf{X}(\omega+\omega_0)]$
11. Convolution in t	$x_1(t) * x_2(t)$	$\Leftrightarrow$	$\mathbf{X}_1(\omega)\, \mathbf{X}_2(\omega)$
12. Convolution in ω	$x_1(t)\, x_2(t)$	$\Leftrightarrow$	$\frac{1}{2\pi}\, \mathbf{X}_1(\omega) * \mathbf{X}_2(\omega)$

13. Conjugate symmetry $\quad \mathbf{X}(-\omega)=\mathbf{X}^*(\omega)$

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega - \theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

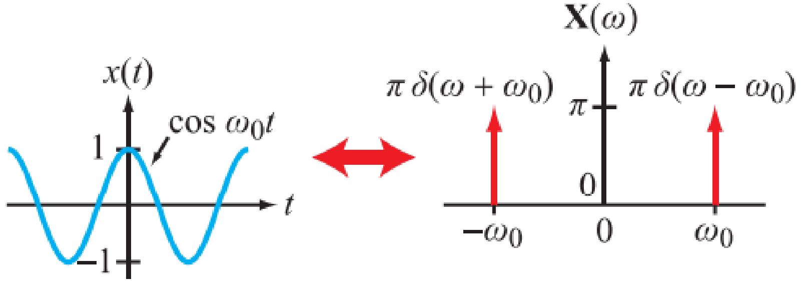
$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Example: Modulation Property of FT

Or frequency shift property 4.3.6 in Table 4.1

$$x(t)\cos(\omega_0 t) \longleftrightarrow \frac{1}{2}[\mathbf{X}(\omega-\omega_0)+\mathbf{X}(\omega+\omega_0)]. \quad (5.109)$$

(modulation property)



Important in communication engineering!

Example: Convolution & Multiplication Properties of FT

- 4.4 + 4.5 in Table 4.1
- Triangular wave is convolution of rectangular wave with itself in time domain;
- Its FT can be found by getting the FT of rectangular wave multiplied with itself.
- Convolution in time = multiplication in frequency
- Convolution in frequency = multiplication in time

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Example of Parseval's Theorem

- Confirm Parseval's theorem holds for $x(t) = e^{-at}\, u(t)$
- Parseval's theorem: $E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{\mathbf{X}}(\omega)|^2\, d\omega$
- Fourier transform pair: $Ae^{-at}\, u(t) \leftrightarrow \frac{A}{a + j\omega}$ for $a > 0$

Energy in time domain:

$$\int_0^{\infty} |e^{-at}|^2\, dt = \int_0^{\infty} e^{-2at}\, dt = \frac{1}{2a}\,.$$

Energy in frequency domain:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{1}{a + j\omega} \right|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{a^2 + \omega^2}\, d\omega = \frac{1}{2a}$$