LECTURE BLOCK 4B: FOURIER TRANSFORM (OF PERIODIC SIGNALS)

Dr. Yau Hee Kho

yauhee.kho@vuw.ac.nz

Fourier Transform

We will learn:

- Compute Fourier Transform of nonperiodic signals,
- Compute Fourier Transform of periodic signals.
- Discrete time Fourier Transform.

Fourier Transform of Impulses and Periodic Signals

Strictly speaking periodic signals do not have Fourier transforms in the simple sense.

If the integral used to define the transform is interpreted in such a way as to allow results which are delta functions, then it is possible to define the Fourier transform of a periodic signal.

To this end we start with the Fourier transform of a delta function and work backwards.

Fourier Transform of a Delta Function

For convenience of notation we introduce a second shorthand notation for Fourier transforms :

$$F(\omega) = \mathbf{F}_{\omega}[f(t)]$$

or more usually:

$$F(\omega) = \mathbf{F}[f(t)]$$

The Fourier transform of a delta function follows directly from the defining property of a delta function applied to the complex exponential:

$$\mathbf{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega 0} = 1$$

Fourier Transform of a Delta Function

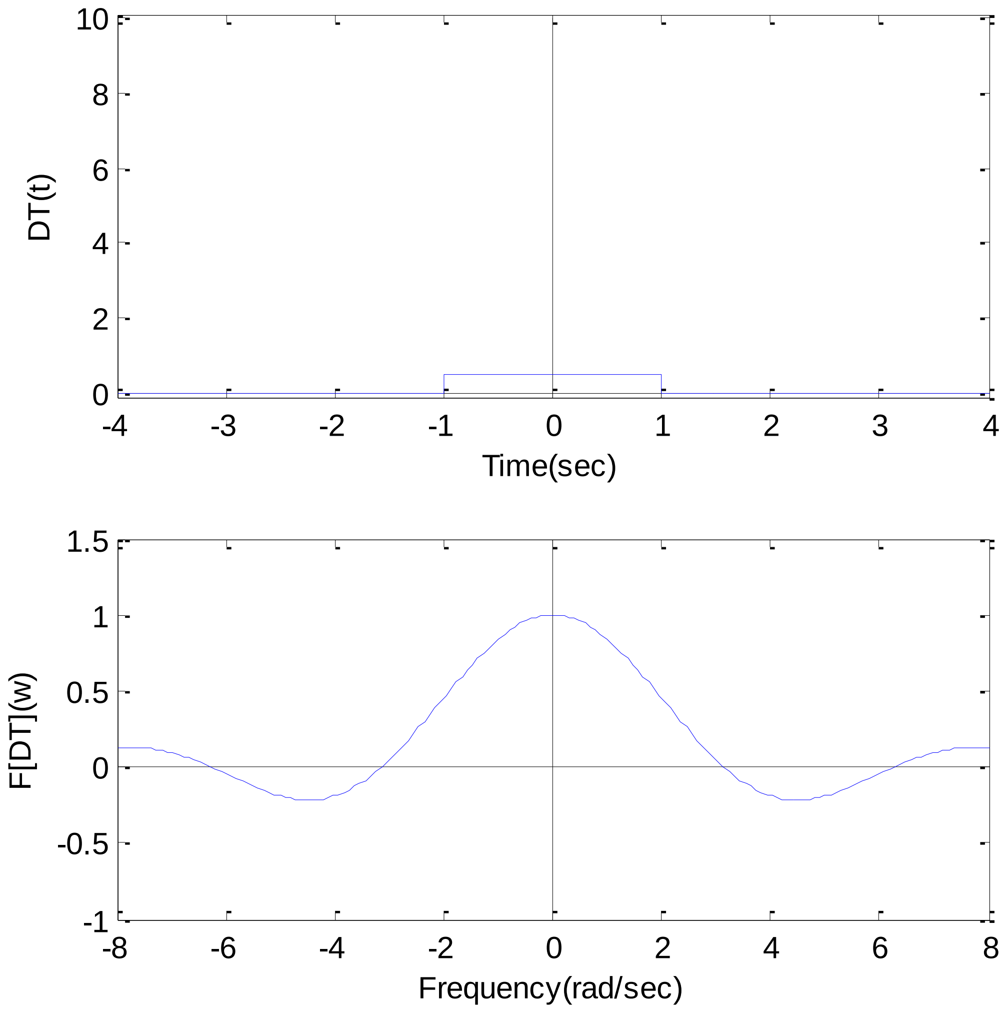
Thus the Fourier transform of a delta function at the origin is a constant function of frequency.

This result can also be seen if we start with a finite width unit-area pulse:

$$\Delta_T(t) = \frac{1}{2T} P_T(t)$$

And calculate its Fourier transform:

$$\mathbf{F}\left[\Delta_T(t)\right] = \mathrm{sinc}(\omega T)$$

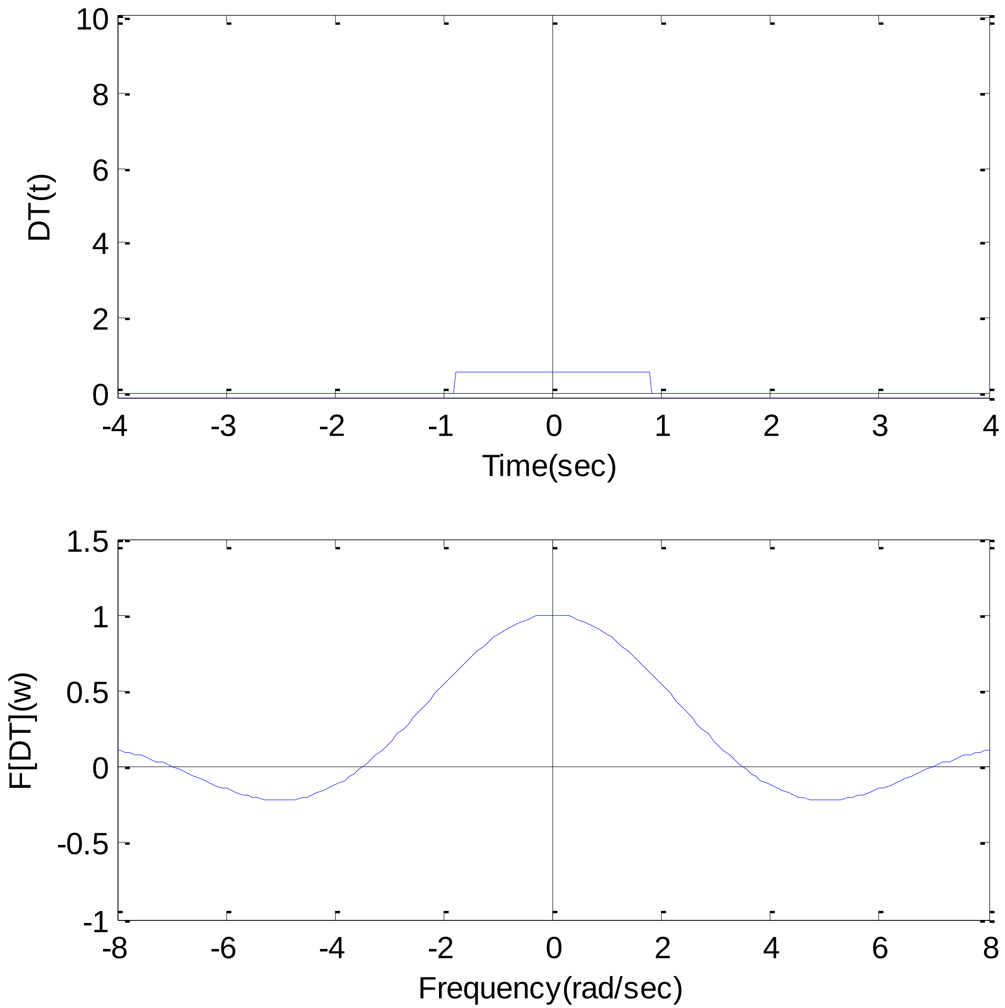


Fourier Transform of a Delta Function

As the pulse width (T) reduces the width of the first central lobe of the transform increases in proportion to 1/T.

$$\Delta_T(t) = \frac{1}{2T} P_T(t)$$

$$\mathbf{F}\left[\Delta_T(t)\right] = \mathrm{sinc}(\omega T)$$

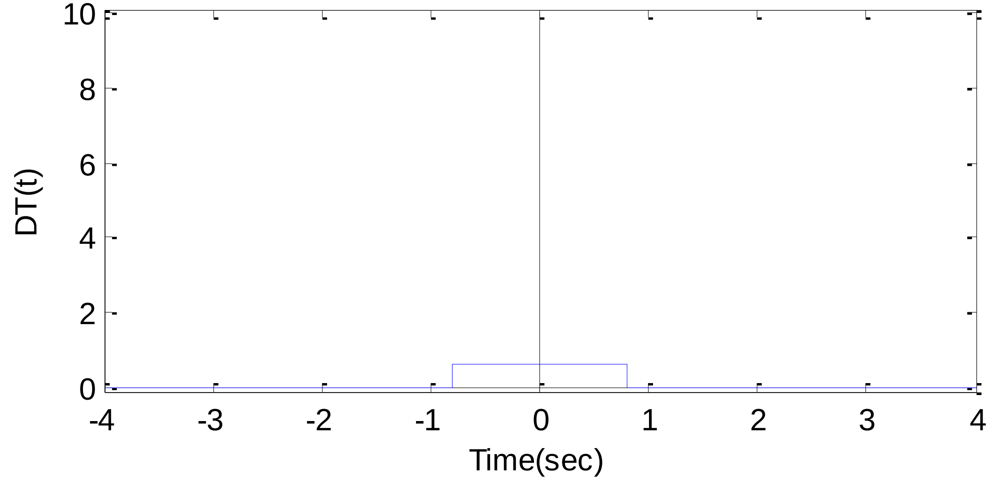


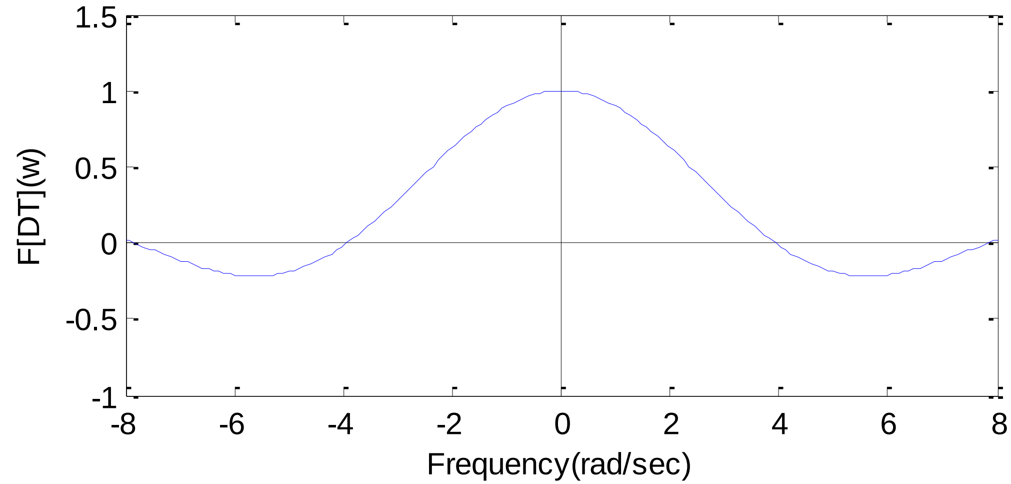
Fourier Transform of a Delta Function

As the pulse width (T) reduces the width of the first central lobe of the transform increases in proportion to 1/T.

$$\Delta_T(t) = \frac{1}{2T} P_T(t)$$

$$\mathbf{F}\left[\Delta_T(t)\right] = \mathrm{sinc}(\omega T)$$



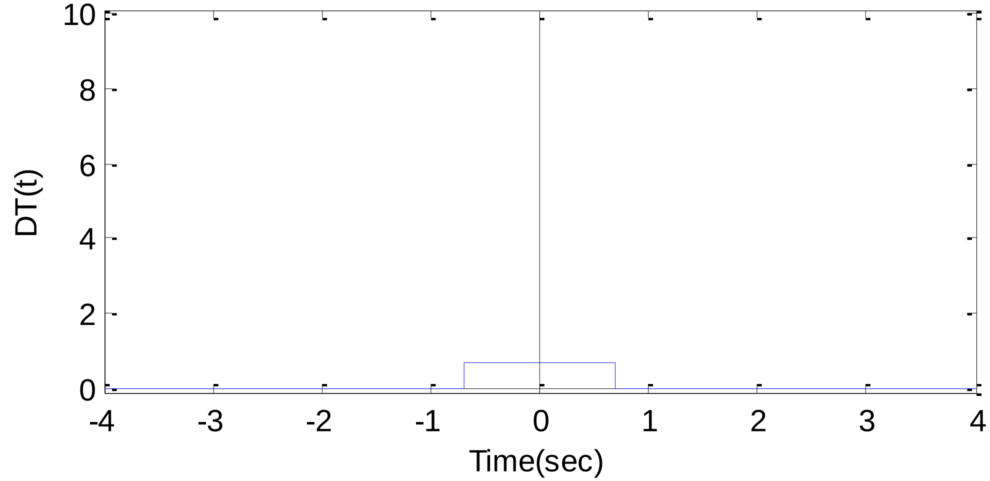


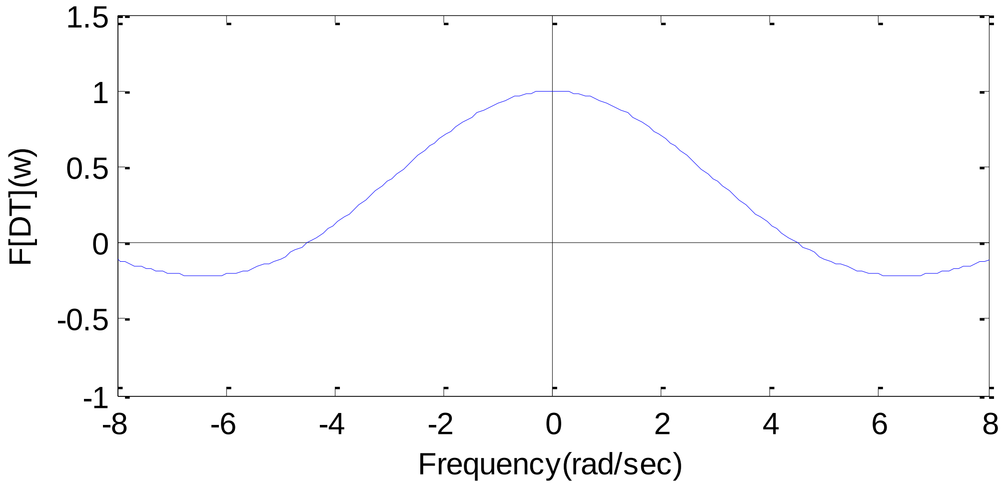
Fourier Transform of a Delta Function

As the pulse width (T) reduces the width of the first central lobe of the transform increases in proportion to 1/T.

$$\Delta_T(t) = \frac{1}{2T} P_T(t)$$

$$\mathbf{F}\left[\Delta_T(t)\right] = \mathrm{sinc}(\omega T)$$



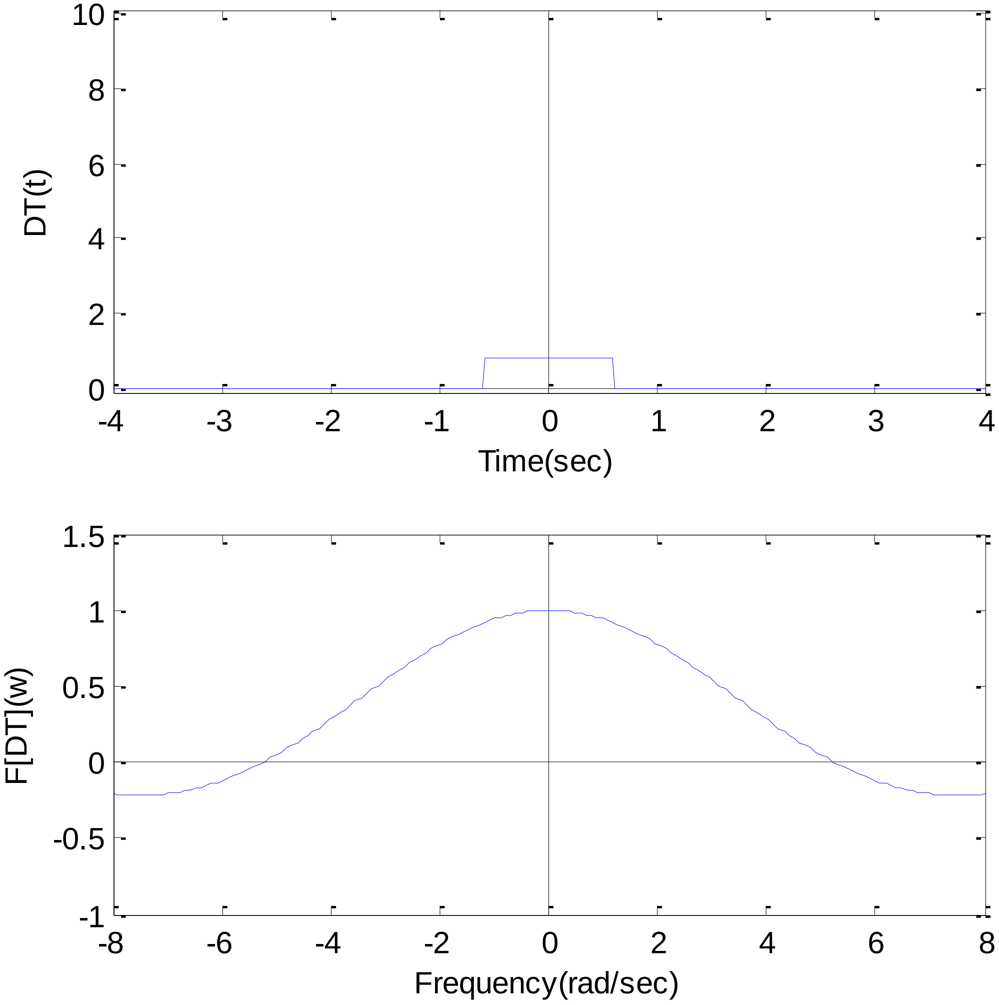


Fourier Transform of a Delta Function

As the pulse width (T) reduces the width of the first central lobe of the transform increases in proportion to 1/T.

$$\Delta_T(t) = \frac{1}{2T} P_T(t)$$

$$\mathbf{F}\left[\Delta_T(t)\right] = \mathrm{sinc}(\omega T)$$

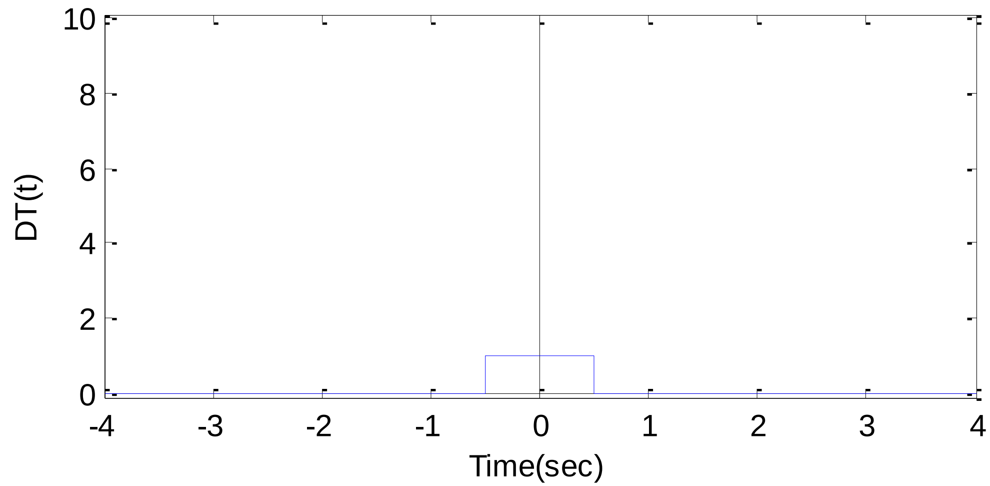


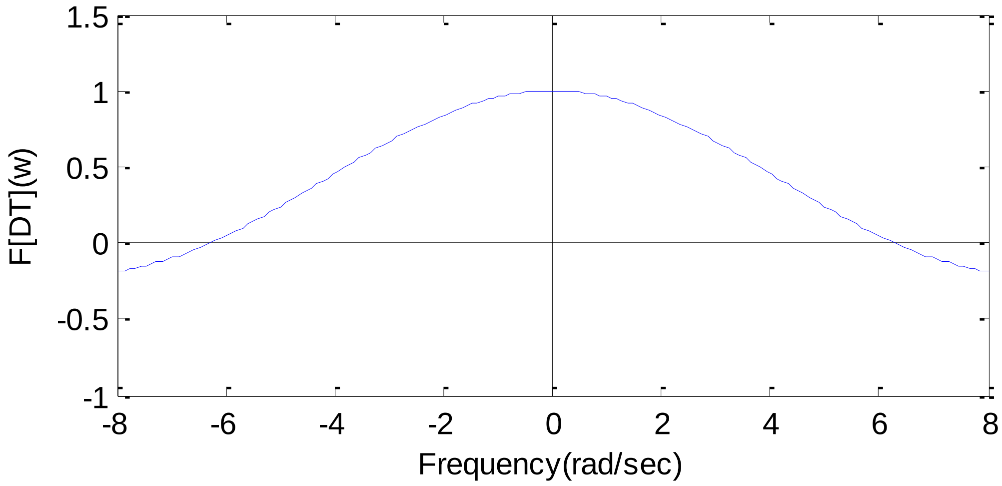
Fourier Transform of a Delta Function

As the pulse width (T) reduces the width of the first central lobe of the transform increases in proportion to 1/T.

$$\Delta_T(t) = \frac{1}{2T} P_T(t)$$

$$\mathbf{F}\left[\Delta_T(t)\right] = \mathrm{sinc}(\omega T)$$



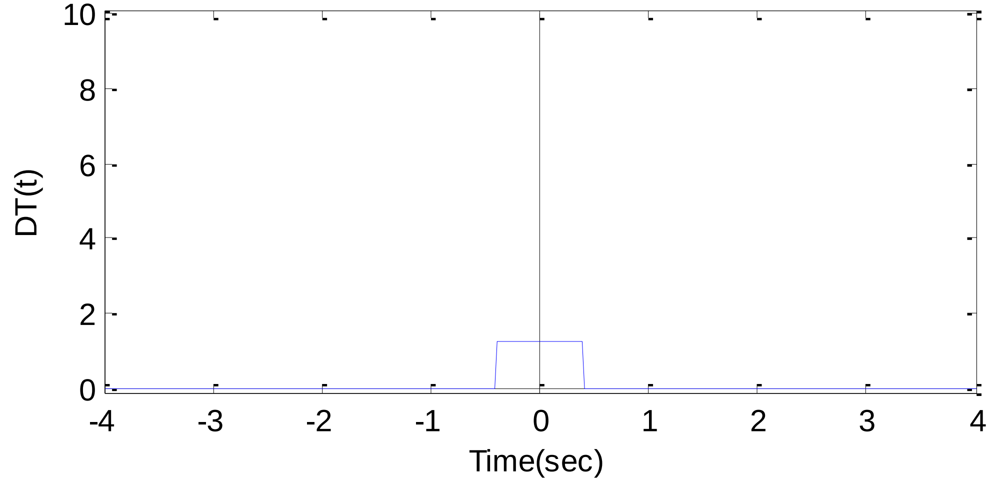


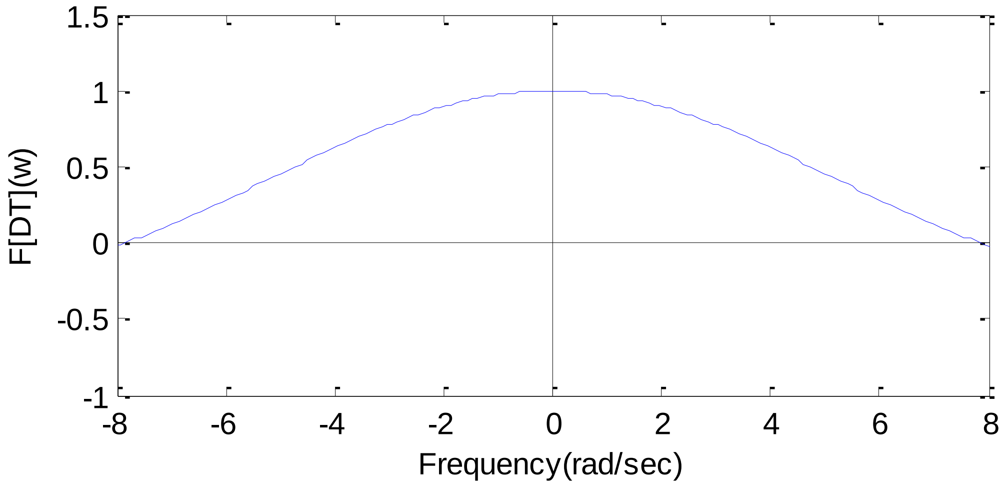
Fourier Transform of a Delta Function

By taking the width small enough we can see that for a finite, but arbitrarily large range of frequencies the transform remains substantially constant.

$$\Delta_T(t) = \frac{1}{2T} P_T(t)$$

$$\mathbf{F}\left[\Delta_T(t)\right] = \mathrm{sinc}(\omega T)$$



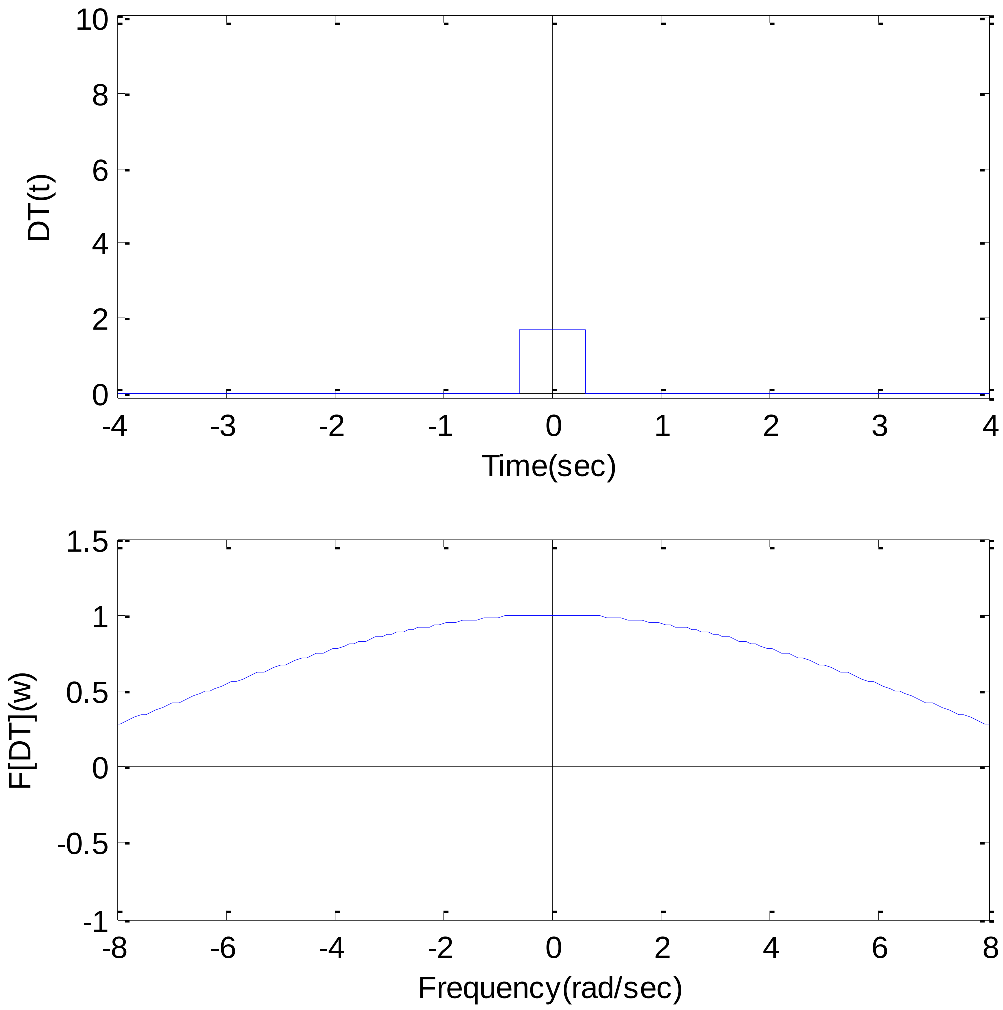


Fourier Transform of a Delta Function

By taking the width small enough we can see that for a finite, but arbitrarily large range of frequencies the transform remains substantially constant.

$$\Delta_T(t) = \frac{1}{2T} P_T(t)$$

$$\mathbf{F}\left[\Delta_T(t)\right] = \operatorname{sinc}(\omega T)$$

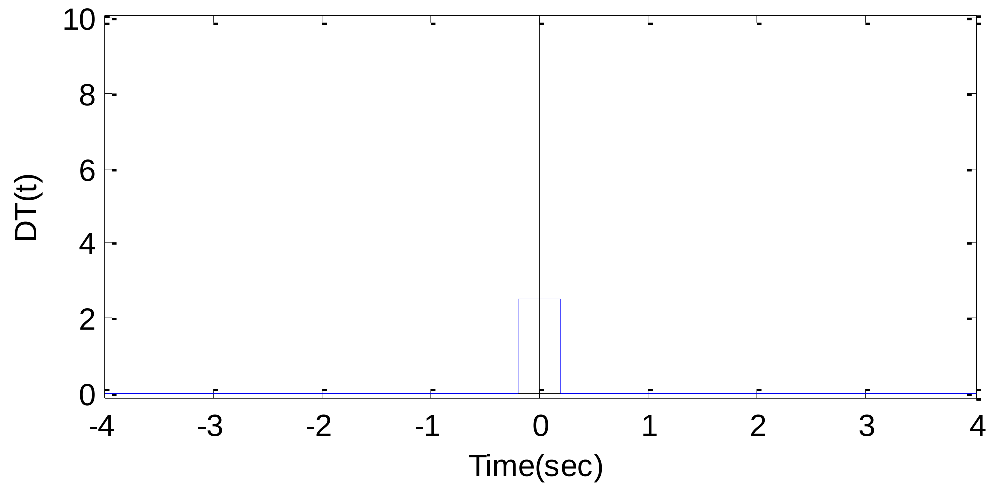


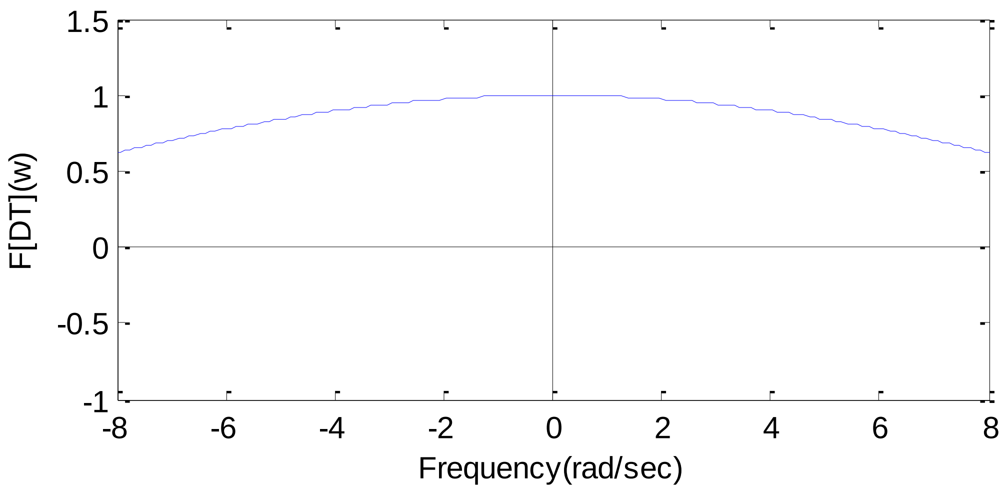
Fourier Transform of a Delta Function

By taking the width small enough we can see that for a finite, but arbitrarily large range of frequencies the transform remains substantially constant.

$$\Delta_T(t) = \frac{1}{2T} P_T(t)$$

$$\mathbf{F}\left[\Delta_T(t)\right] = \mathrm{sinc}(\omega T)$$



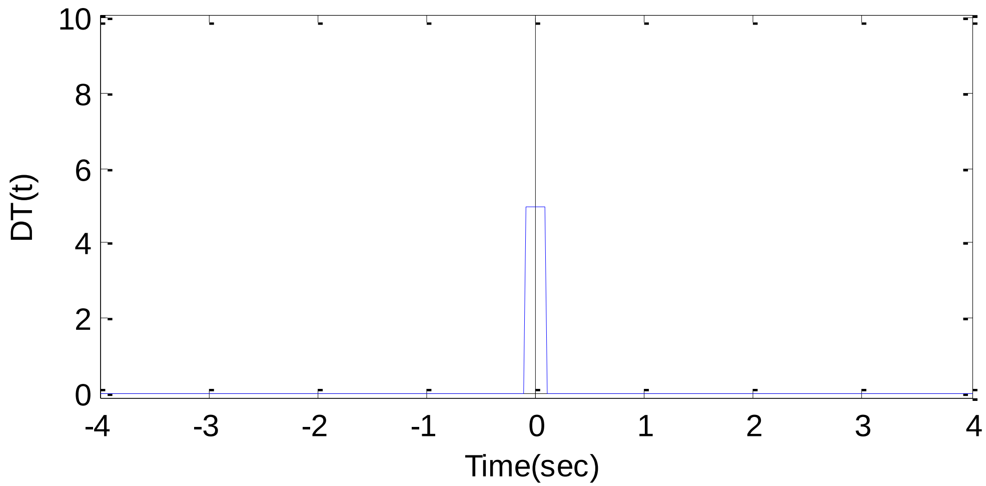


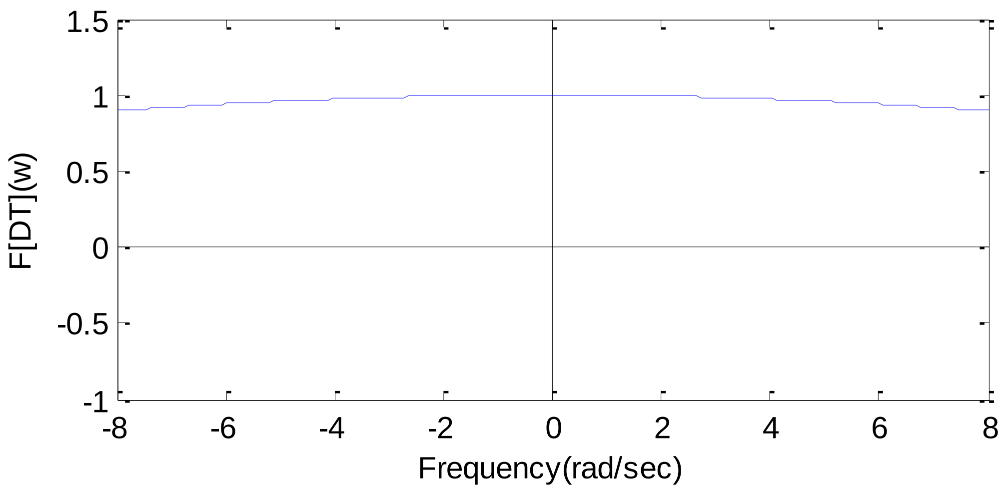
Fourier Transform of a Delta Function

By taking the width small enough we can see that for a finite, but arbitrarily large range of frequencies the transform remains substantially constant.

$$\Delta_T(t) = \frac{1}{2T} P_T(t)$$

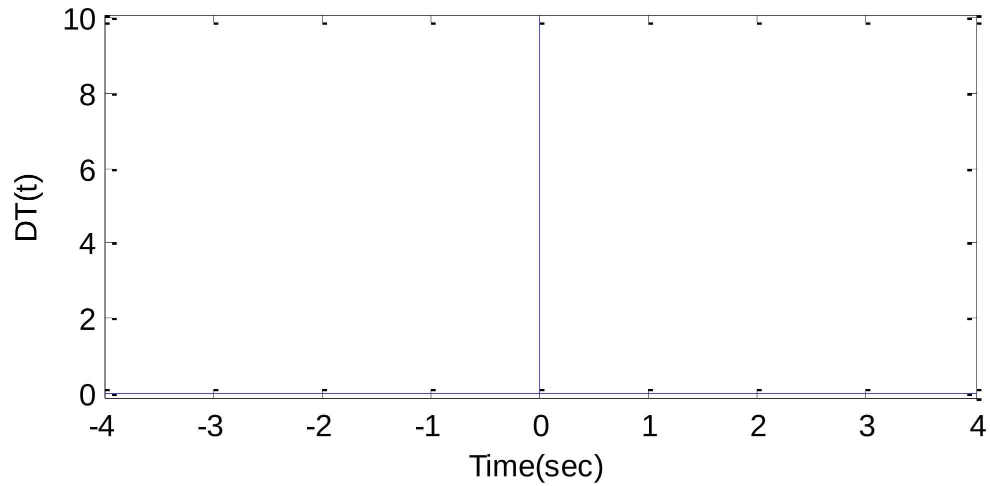
$$\mathbf{F}\left[\Delta_T(t)\right] = \operatorname{sinc}(\omega T)$$



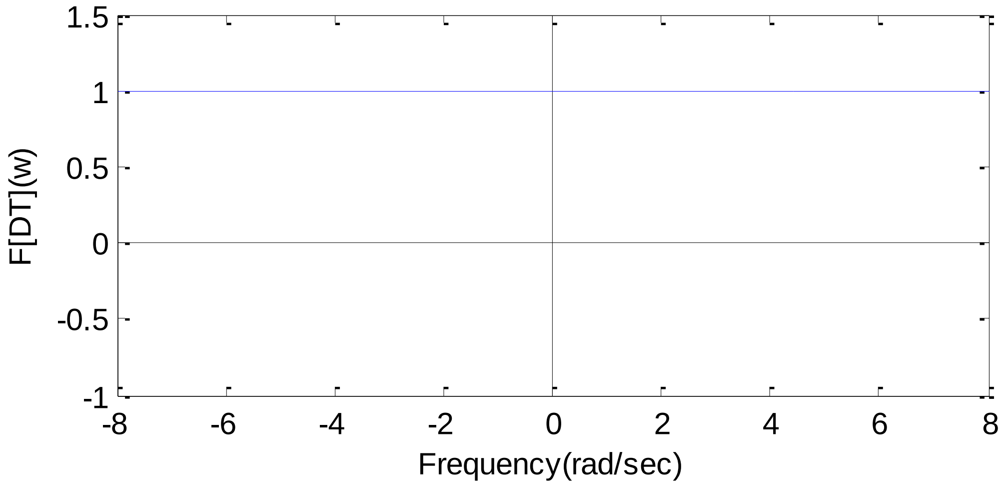


Fourier Transform of a Delta Function

Eventually as T→0 the pulse becomes a delta function and the transform becomes the constant function.



$$\Delta_T(t) = \frac{1}{2T} P_T(t) \to \delta(t)$$



$$\mathbf{F}\left[\Delta_T(t)\right] = \mathrm{sinc}(\omega T) \to 1$$

Fourier Transform of a Delta Function

If the delta function is shifted away from the origin a similar calculation yields

$$\mathbf{F}[\delta(t-T)] = \int_{-\infty}^{\infty} \delta(t-T) e^{-j\omega t} dt = e^{-j\omega T}$$

In general, the Fourier transform of a delta function is a complex exponential function of frequency.

This result can also be derived using the shifting property of the Fourier transform:

$$\mathbf{F}[\delta(t-T)] = e^{-j\omega T}\mathbf{F}[\delta(t)] = e^{-j\omega T}$$

Fourier Transform of a Constant Function

A constant function does not satisfy the conditions for the existence of the Fourier integral unless some limiting procedure is agreed upon to handle the improper integral.

This is, in effect, what is done when we use the Duality property of the transform to calculate the transform of a constant function:

$$f(t) \leftrightarrow F(\omega) \Rightarrow F(t) \leftrightarrow 2\pi \ f\ (-\omega)$$

$$\delta\ (t) \leftrightarrow 1(\omega)\ \Rightarrow\ \ 1(t) \leftrightarrow 2\pi\delta\ (-\omega)$$

Fourier Transform of a Constant Function

The symmetry property of the Fourier transform allows the final expression above to be simplified

$$\delta(-\omega) = \delta(\omega)$$

so that

$$1(t) \leftrightarrow 2\pi\delta\ (\omega)$$

Physically, this can be interpreted to mean that all the energy in a constant (ie DC signal) is contained in the zero frequency component.

Fourier Transform of a Complex Harmonic Signal

The Fourier transform of harmonic signals follows a similar argument: use duality on the transform of a shifted delta function

$$\delta(t-T) \leftrightarrow e^{-j\omega T}$$

Duality

$$\Rightarrow e^{-jtT} \leftrightarrow 2\pi\delta(-\omega-T)$$

Symmetry

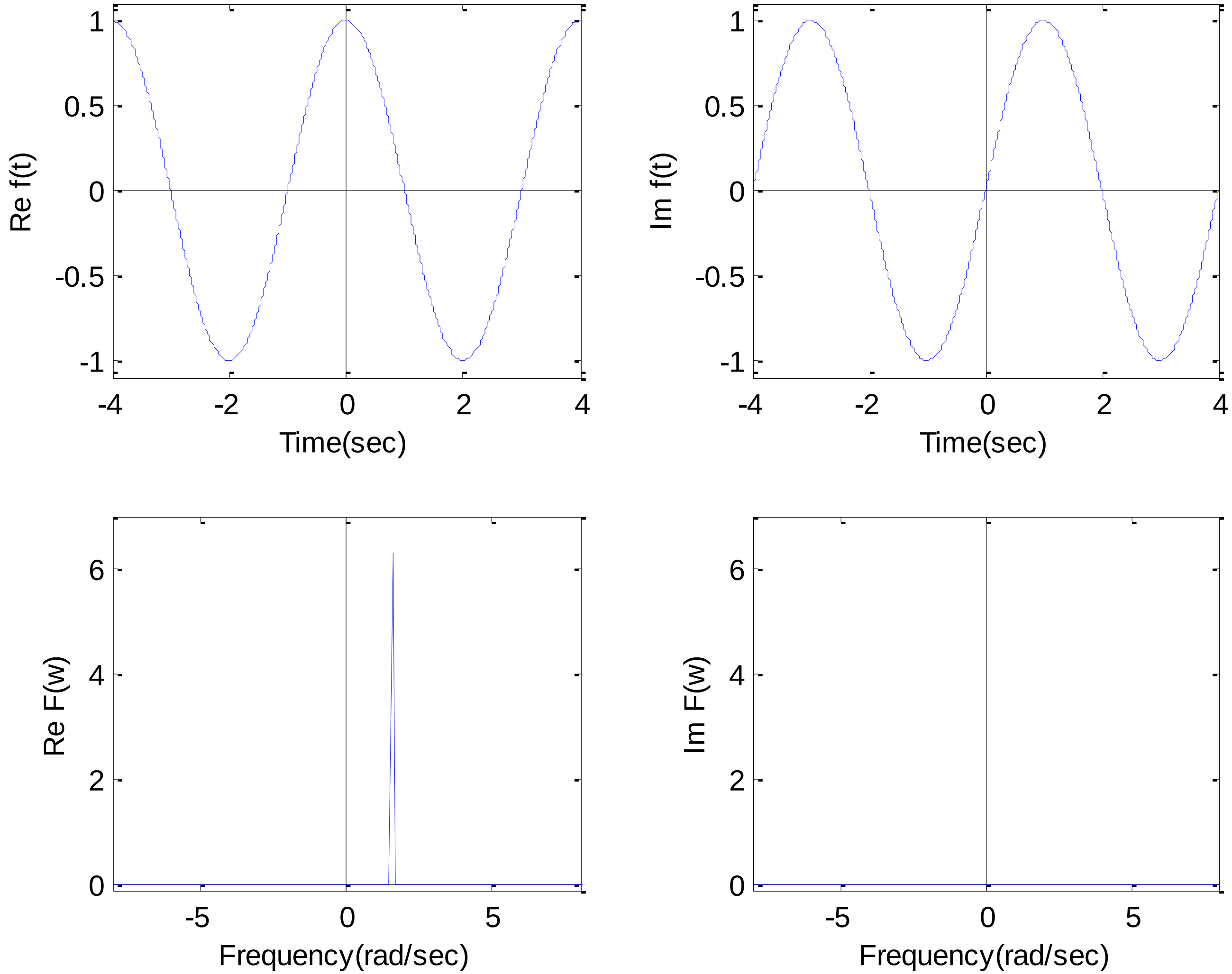
$$\Rightarrow e^{-jtT} \leftrightarrow 2\pi\delta(\omega+T)$$

so that, on changing notation slightly (put $T = -\Omega$)

$$e^{j\Omega t} \leftrightarrow 2\pi\delta(\omega-\Omega)$$

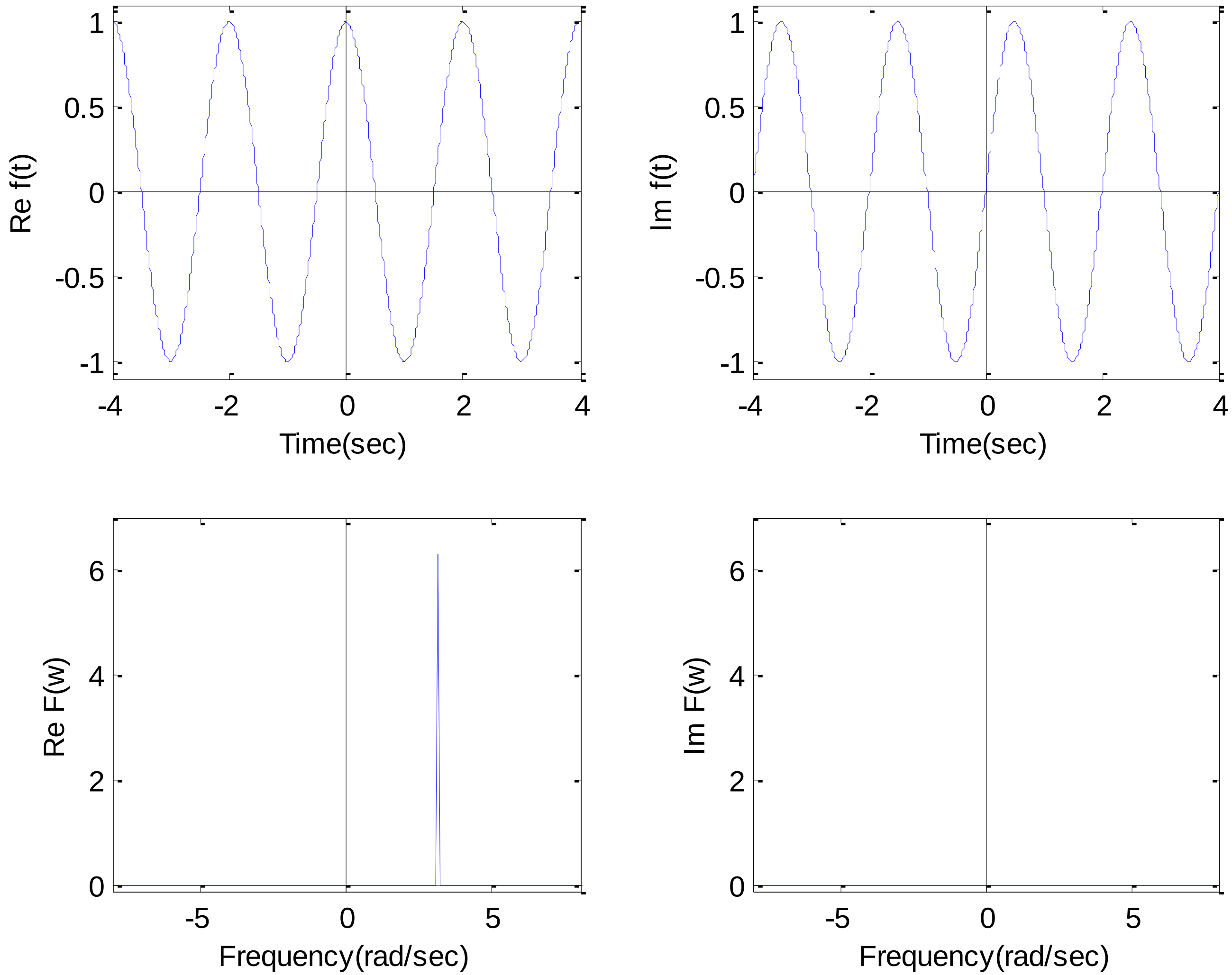
Fourier Transform of a Complex Harmonic Signal

Thus all the energy in a single (complex exponential) harmonic term is concentrated at one frequency.



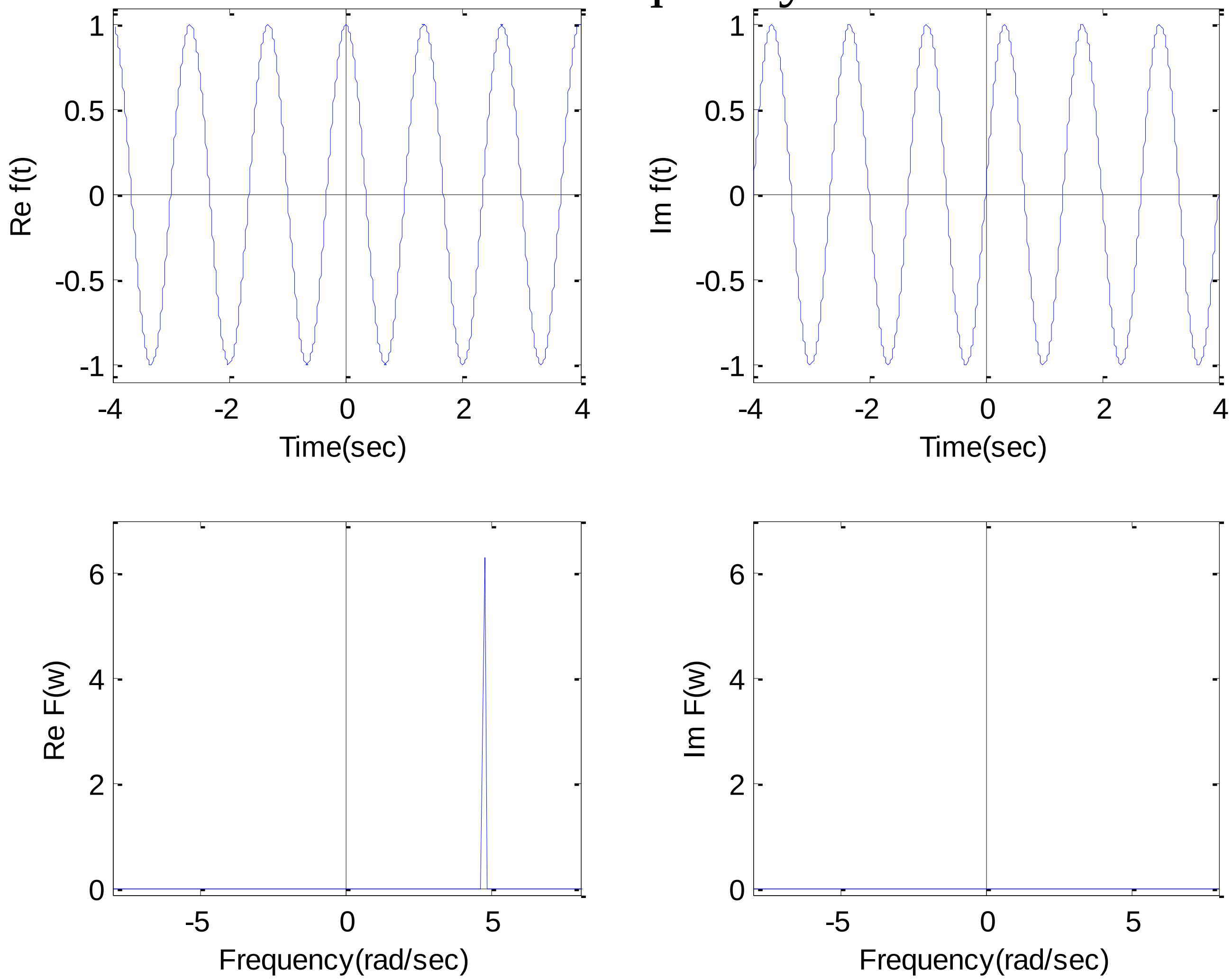
Fourier Transform of a Complex Harmonic Signal

Thus all the energy in a single (complex exponential) harmonic term is concentrated at one frequency.



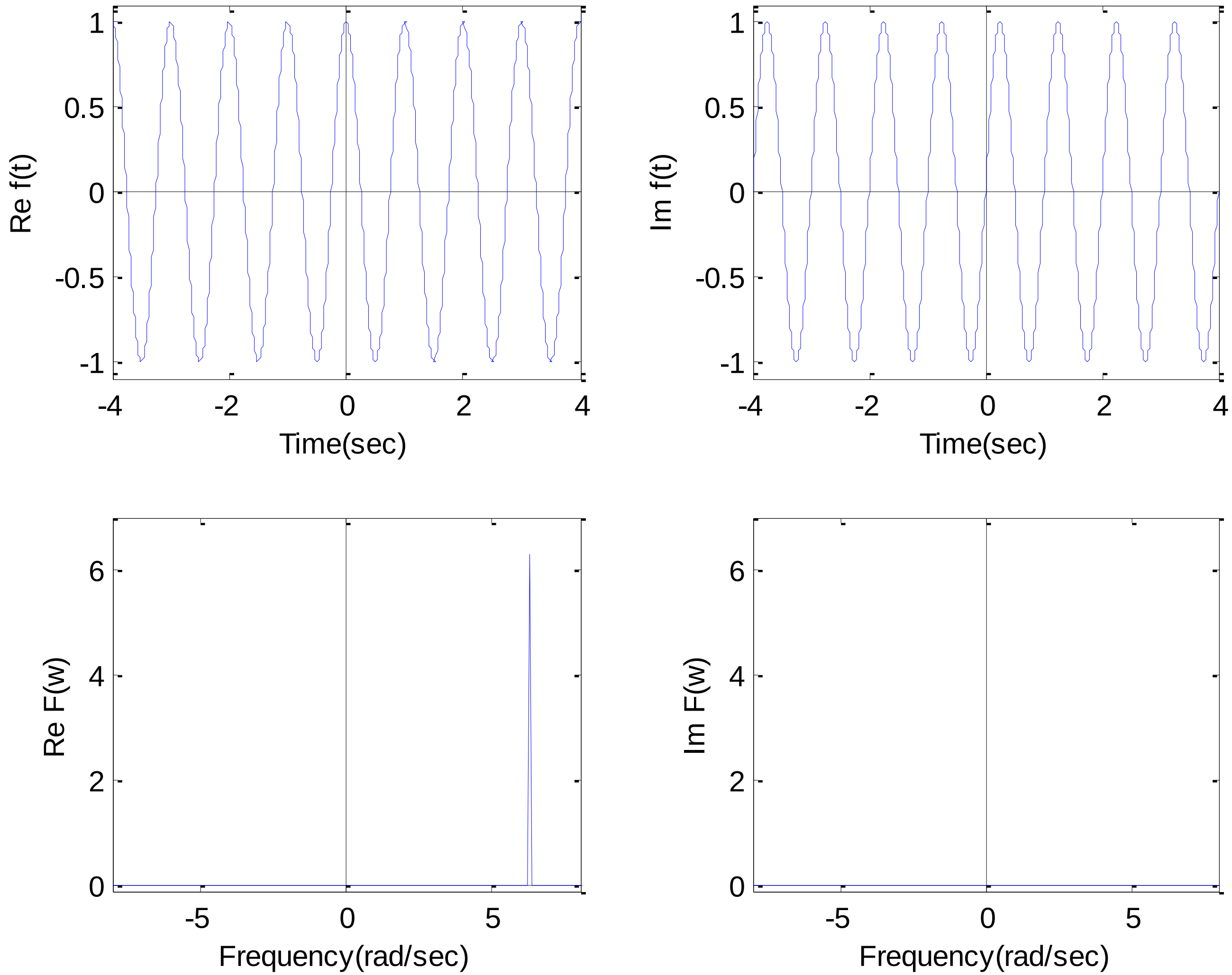
Fourier Transform of a Complex Harmonic Signal

Thus all the energy in a single (complex exponential) harmonic term is concentrated at one frequency.



Fourier Transform of a Complex Harmonic Signal

Thus all the energy in a single (complex exponential) harmonic term is concentrated at one frequency.



Fourier Transform of a Real Harmonic Signal

The general harmonic function can be transformed by using the usual relations between sines and cosines and complex exponentials. The result is :

$$f(t) = \cos(\Omega t + \phi)$$

Expand

$$= \frac{1}{2}\left(e^{j(\Omega t+\phi)} + e^{-j(\Omega t+\phi)}\right)$$

Product of Exponentials

$$= \frac{1}{2}\left(e^{j\phi}e^{j\Omega t} + e^{-j\phi}e^{-j\Omega t}\right)$$

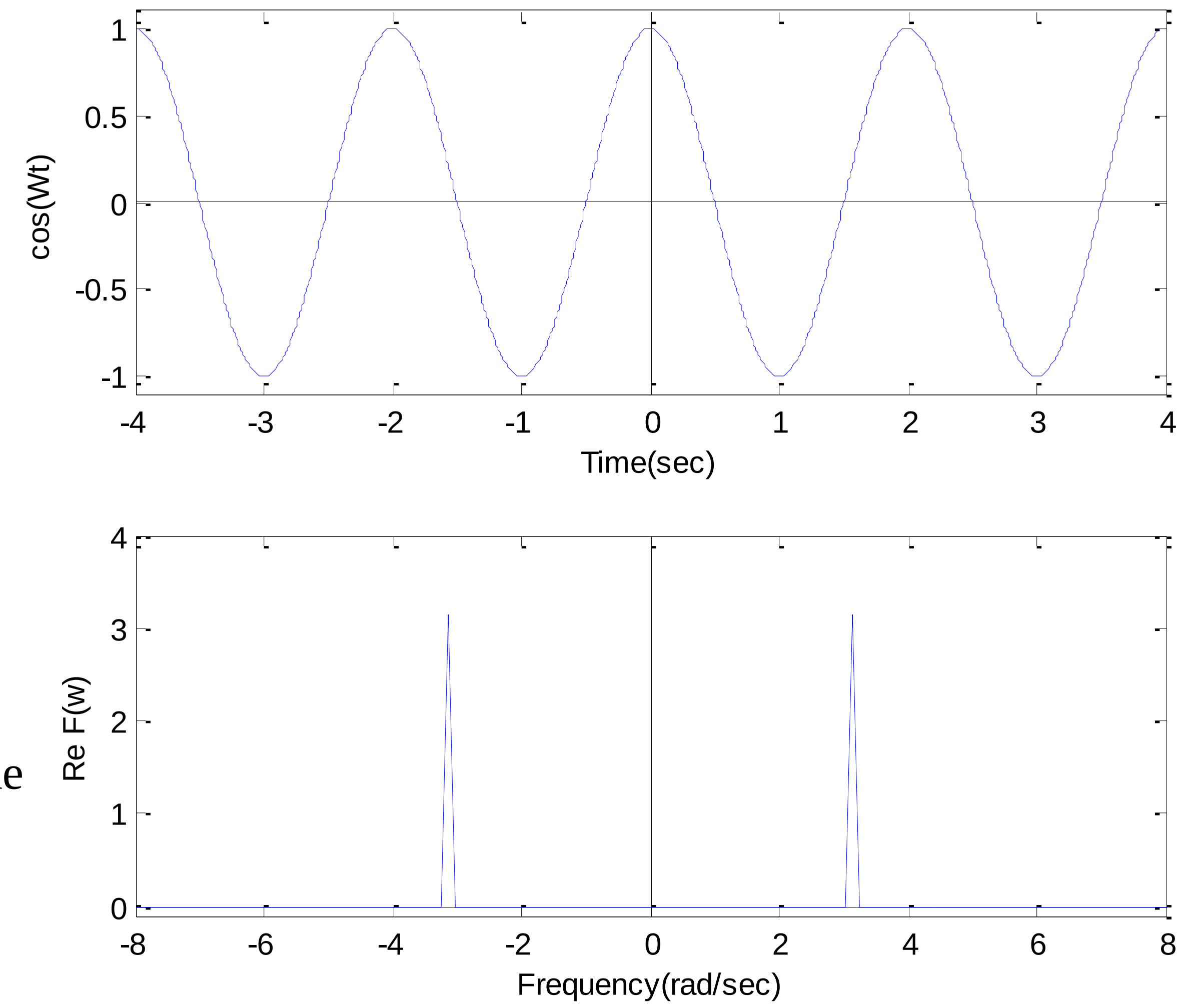
Transforms of Complex Harmonics

$$f(t) \leftrightarrow \pi\left(e^{j\phi}\delta(\omega - \Omega) + e^{-j\phi}\delta(\omega + \Omega)\right)$$

A general harmonic signal has both +ve and -ve frequencies and in general, real and imaginary parts.

Fourier Transform of a Real Harmonic Signal

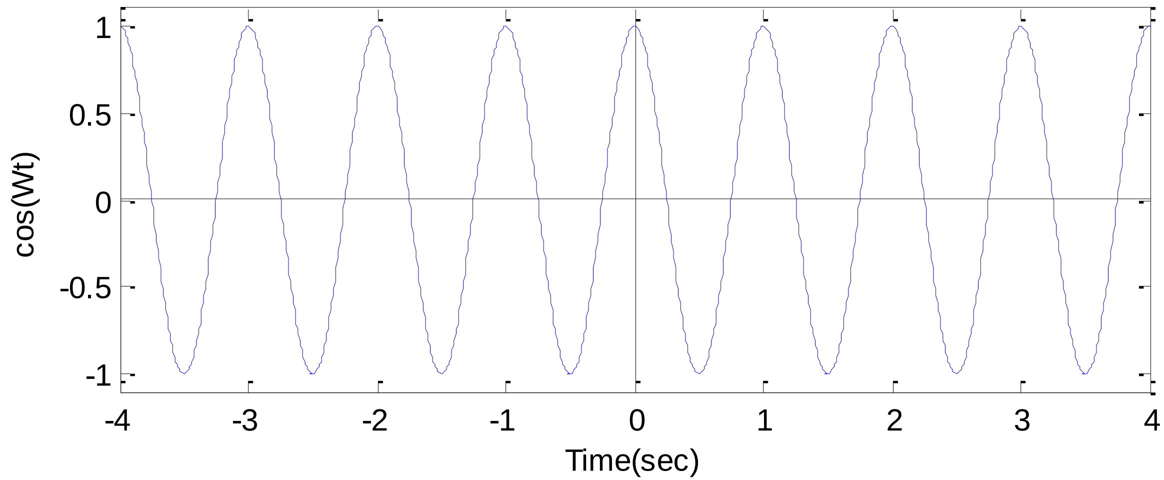
By taking $\phi = 0$ we find : $\cos(\Omega t) \leftrightarrow \pi\left(\delta(\omega - \Omega) + \delta(\omega + \Omega)\right)$

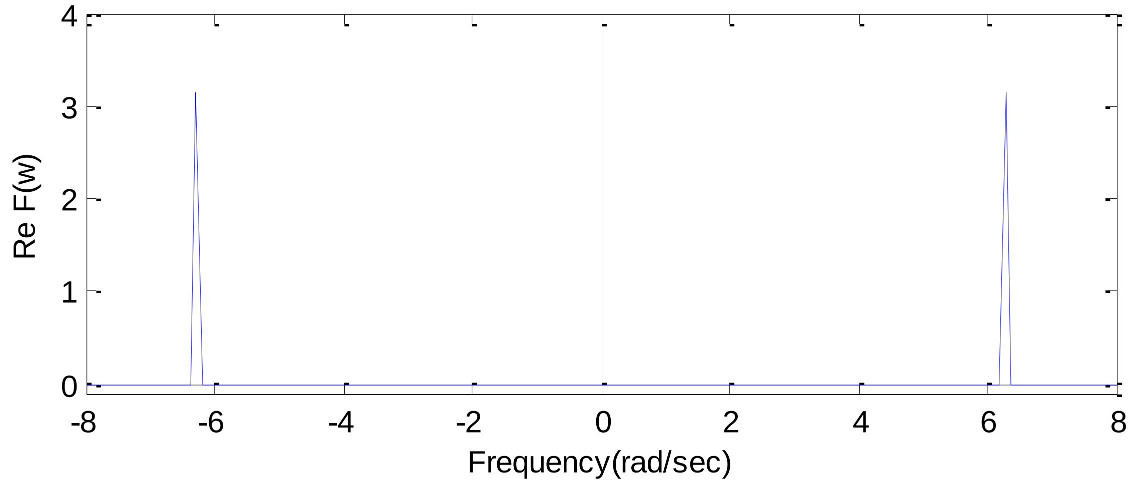


Note that the transform of the cosine signal is real and an even function of frequency.

Fourier Transform of a Real Harmonic Signal

And by taking $\phi = 0$ we find : $\cos(\Omega t) \leftrightarrow \pi\left(\delta(\omega - \Omega) + \delta(\omega + \Omega)\right)$

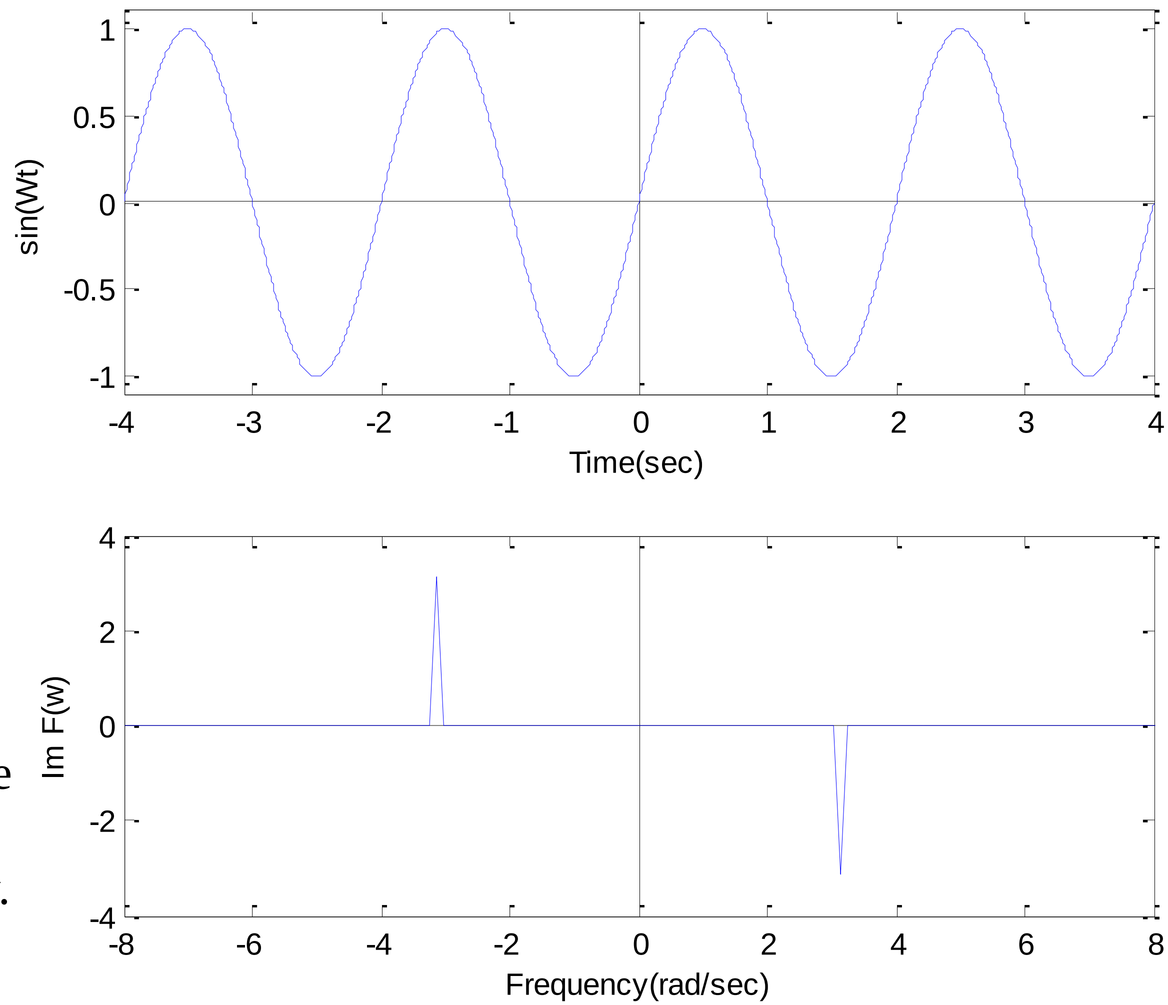




Note that the transform of the cosine signal is real and an even function of frequency.

Fourier Transform of a Real Harmonic Signal

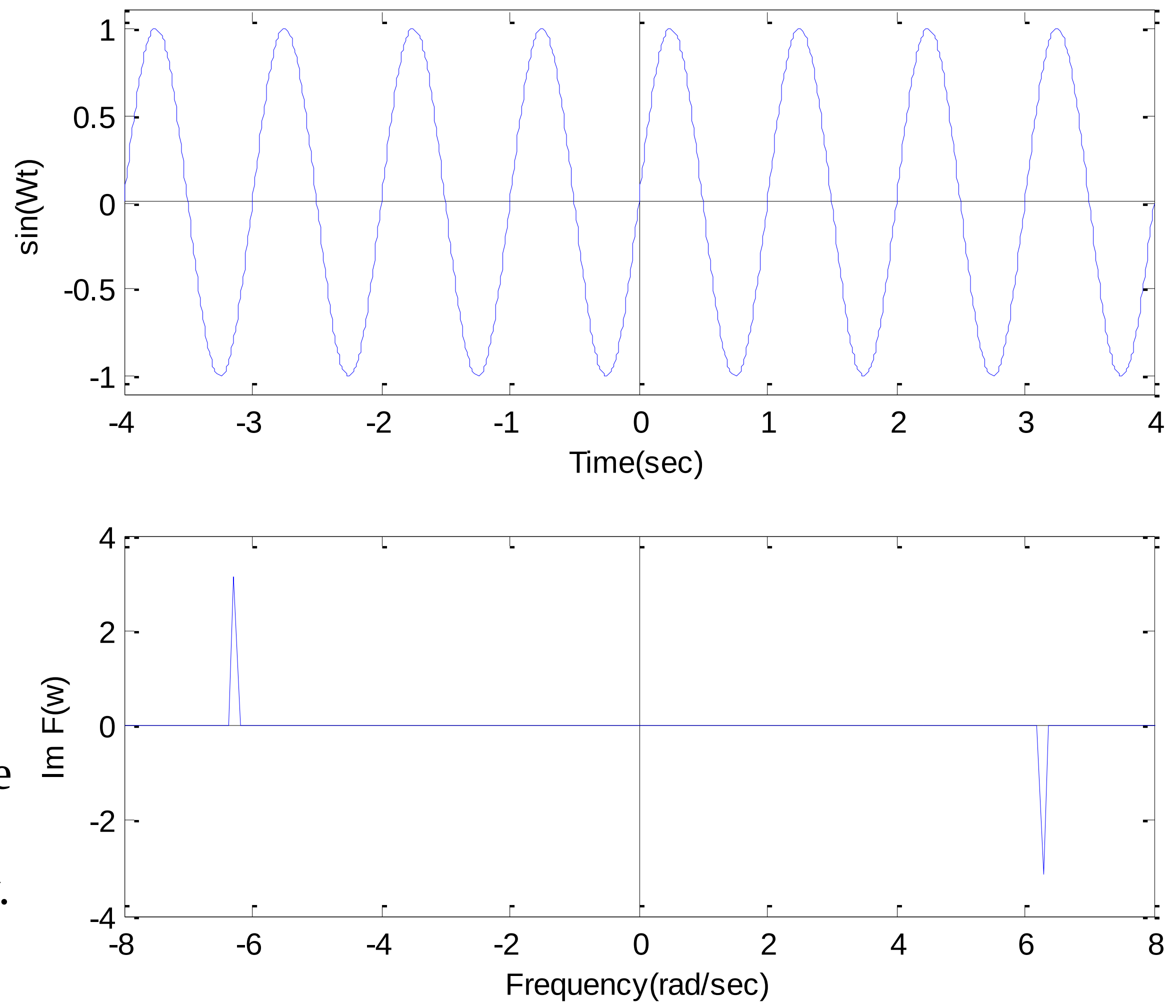
And by taking $\phi = -\pi/2$ we find : $\sin(\Omega\, t) \leftrightarrow -j\pi\big(\delta\,(\omega-\Omega) - \delta\,(\omega+\Omega)\big)$



Note that the transform of the sine signal is imaginary and an odd function of frequency.

Fourier Transform of a Real Harmonic Signal

And by taking $\phi = -\pi/2$ we find : $\sin(\Omega\, t) \leftrightarrow -j\pi\big(\delta\,(\omega-\Omega) - \delta\,(\omega+\Omega)\big)$



Note that the transform of the sine signal is imaginary and an odd function of frequency.

Fourier Transform of a Periodic Signal

The results above can be used to find expressions, in terms of delta functions, for the Fourier transform of a general periodic functions.

Not surprisingly it resembles the Fourier series.

In fact it is easiest to start with a Fourier series expansion and transform it.

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{j(2n\pi/T)t}$$

Transform

$$\mathbf{F}[f(t)] = 2\pi \sum_{n=-\infty}^{\infty} F_n \mathbf{F}\left[e^{j(2n\pi/T)t}\right]$$

Transform of harmonics

$$= 2\pi \sum_{n=-\infty}^{\infty} F_n \delta\left(\omega - \frac{2n\pi}{T}\right)$$

Fourier Transform of a Periodic Signal

Thus

$$f(t) \leftrightarrow 2\pi \sum_{n=-\infty}^{\infty} F_n \delta\left(\omega - \frac{2n\pi}{T}\right)$$

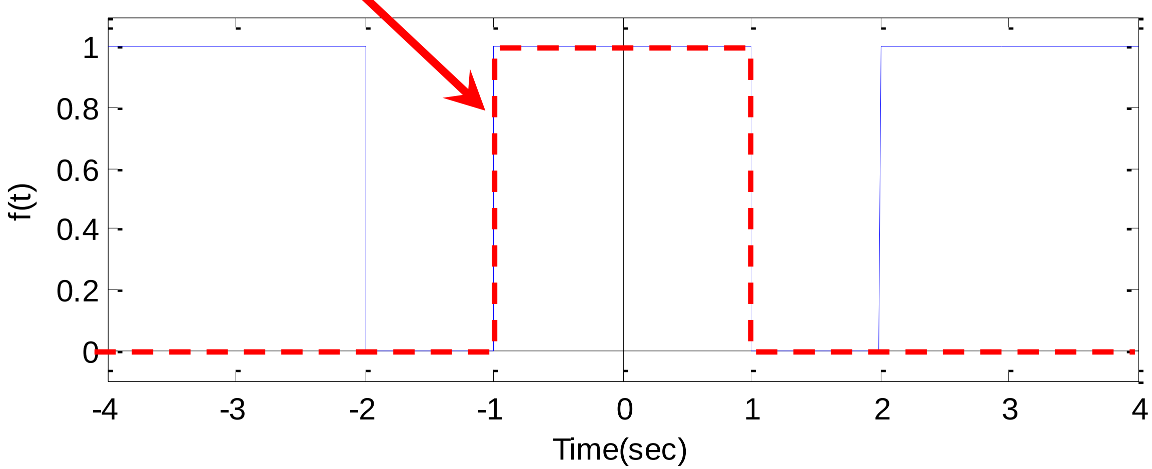
The Fourier coefficients are given by the usual expression:

$$F_n = \frac{1}{T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} f(t) e^{-j(2n\pi/T)} dt$$

Fourier Transform of a Periodic Signal

It is convenient to define a new function, which is just one cycle of the periodic function :

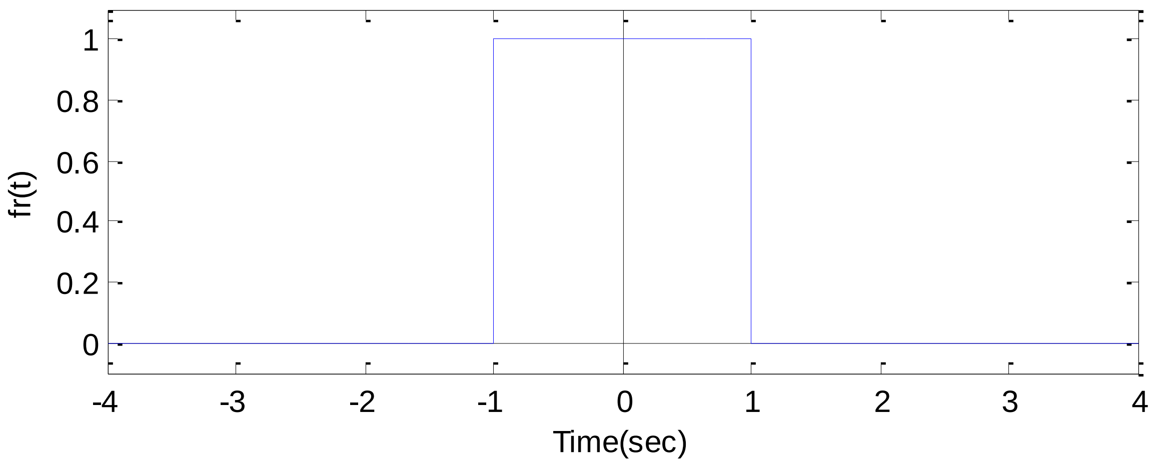
$$f_r(t) = \begin{cases} f(t) & : \quad -\tfrac{1}{2}T \le t < \tfrac{1}{2}T \\ 0 & : \; t < -\tfrac{1}{2}T \;\text{ or }\; t \ge \tfrac{1}{2}T \end{cases}$$



Fourier Transform of a Periodic Signal

Note that this function is defined for all time, but is non-zero only over one cycle of the periodic signal.

$$f_r(t) = \begin{cases} f(t) & : \quad -\frac{1}{2}T \le t < \frac{1}{2}T \\ 0 & : \; t < -\frac{1}{2}T \ \text{ or } \ \text{t} \ge \frac{1}{2}T \end{cases}$$



Fourier Transform of a Periodic Signal

The advantage of using $f_r(t)$ is that the Fourier coefficients can be written easily in terms of the Fourier transform of $f_r(t)$.

$$F_n = \frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} f(t)e^{-j(2n\pi/T)}dt$$

Definition of $f_r(t)$

$$= \frac{1}{T}\int_{-\infty}^{\infty} f_r(t)e^{-j(2n\pi/T)}dt$$

Definition of Fourier Transform

$$= \frac{1}{T}F_r(2n\pi/T)$$

where $f_r(t) \leftrightarrow F_r(\omega)$

Fourier Transform of a Periodic Signal

Substitute the expression for the Fourier coefficients into the expression for the Fourier transform of *f(t)*

$$f(t) \leftrightarrow \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} F_r(2n\pi/T)\delta\left(\omega - \frac{2n\pi}{T}\right)$$

The multiplication property of the delta function allows us to write:

$$F_r(2n\pi/T)\delta\left(\omega - \frac{2n\pi}{T}\right) = F_r(\omega)\delta\left(\omega - \frac{2n\pi}{T}\right)$$

Fourier Transform of a Periodic Signal

So that

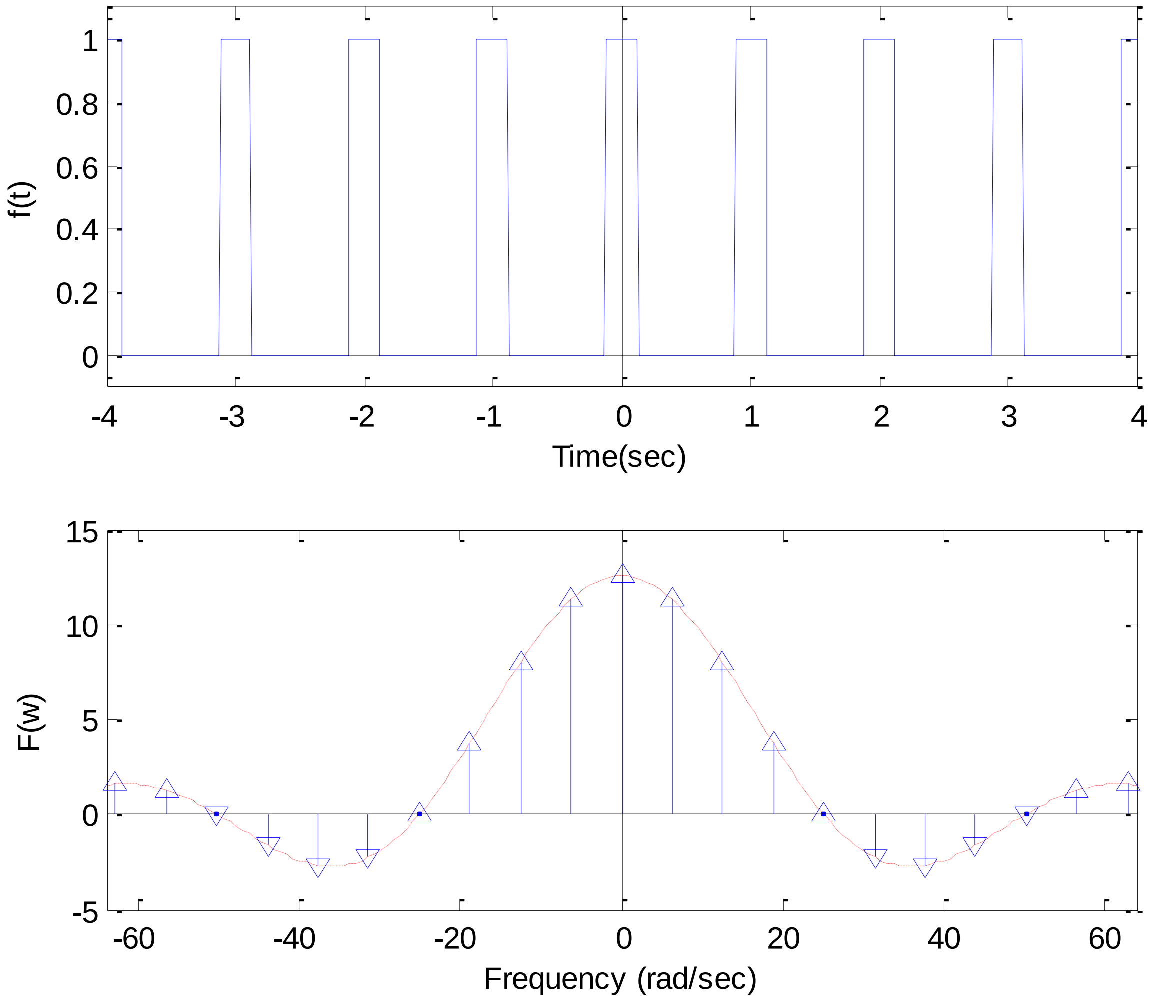
$$f(t) \leftrightarrow \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} F_r(\omega)\delta\left(\omega - \frac{2n\pi}{T}\right)$$

$F_r(\omega)$ is independent of n

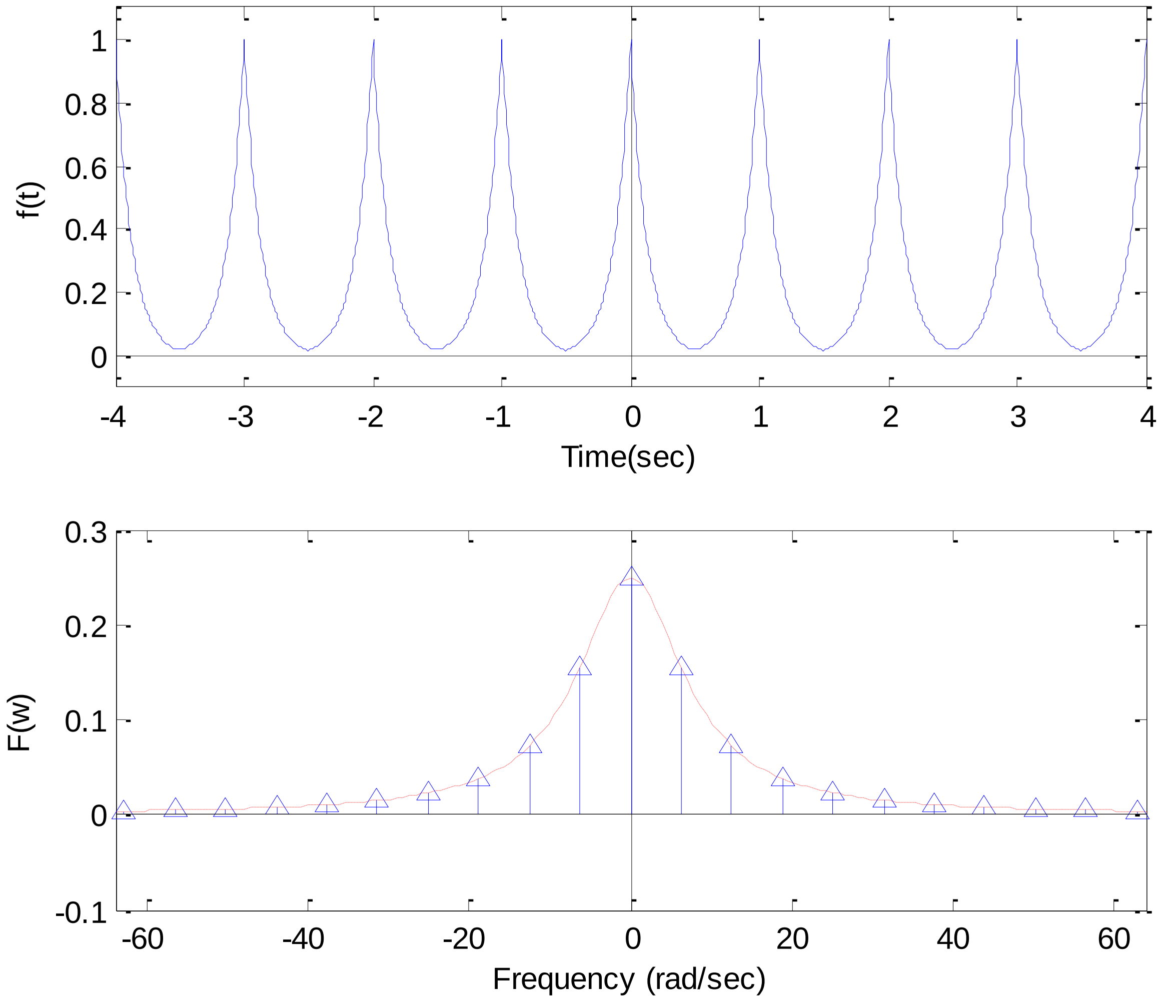
$$f(t) \leftrightarrow F_r(\omega) \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2n\pi}{T}\right)$$

Thus the Fourier transform of a periodic signal is a sequence of delta functions located at the harmonics of the fundamental frequency modulated by the Fourier transform of the "restricted function" $f_r(t)$.

Fourier Transform of a Periodic Signal



Fourier Transform of a Periodic Signal

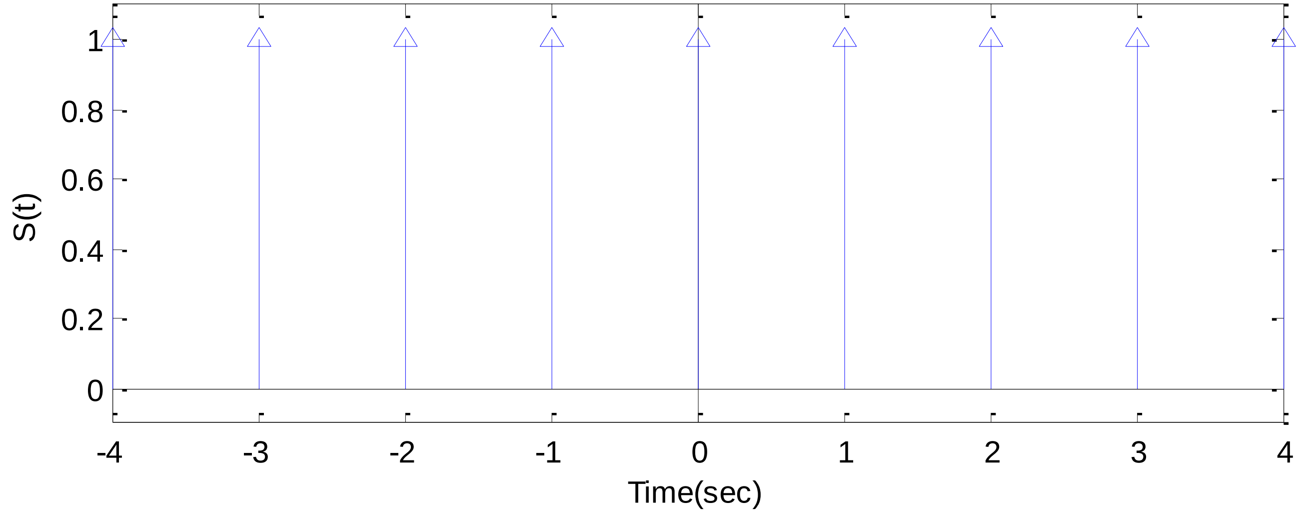


Fourier Transform of a Dirac Comb

The result for the Fourier transform of a periodic signal can be concisely written in terms of a special function, the "Dirac Comb", which is a net of uniformly spaced delta functions of uniform impulse.

$$S_X(x) = \sum_{n=-\infty}^{\infty} \delta(x - nX)$$

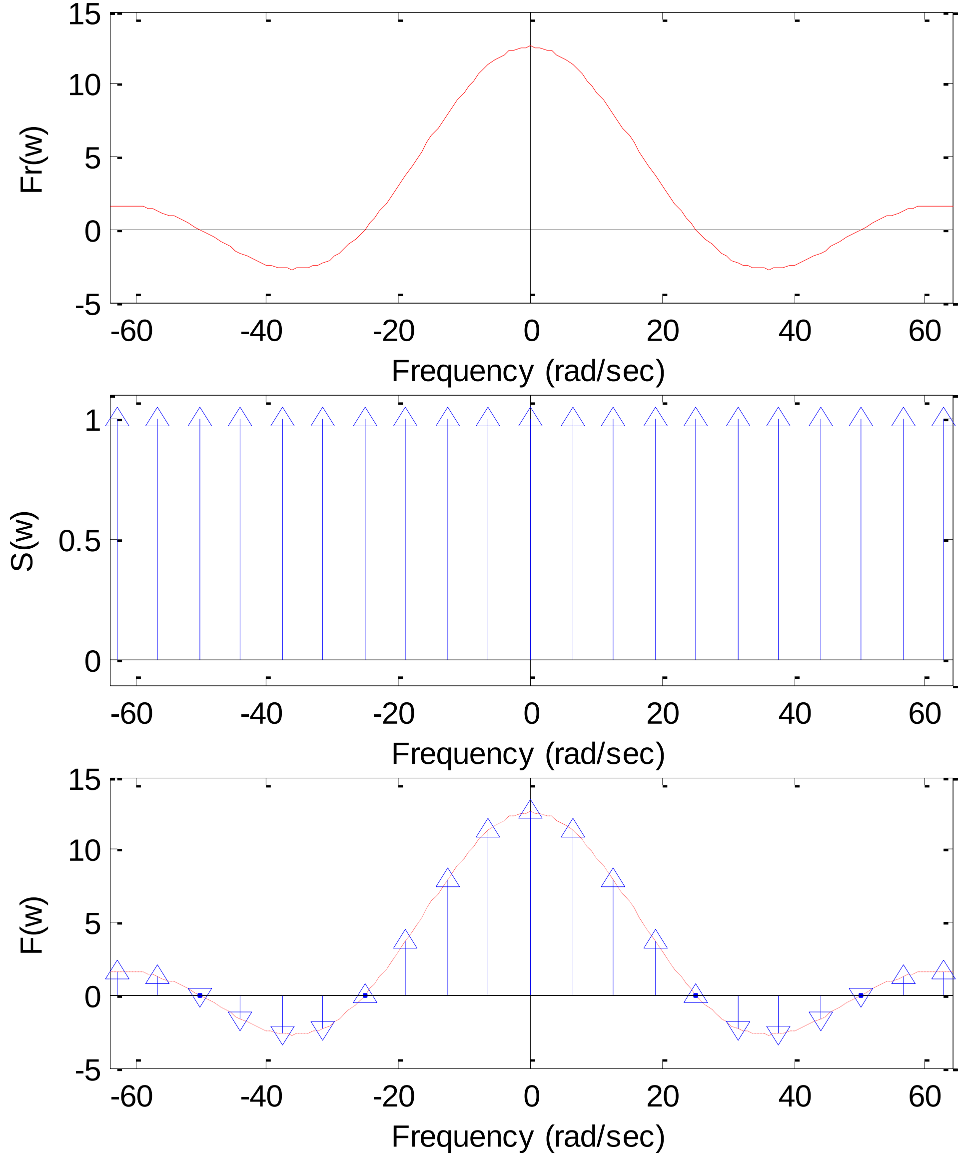
This functions is the idealization of an infinite sequence of uniform short duration unit-impulse pulses.



Fourier Transform of a Dirac Comb

We can use the Dirac Comb (as a function of frequency) to tidy the result for the Fourier transform of a periodic signal.

$$f(t) \leftrightarrow F_r(\omega)\frac{2\pi}{T}S_{2\pi/T}(\omega)$$



Fourier Transform of a Dirac Comb

The Fourier transform of a Dirac comb can be found by an instructive trick.
Using the convolution property of delta functions, we can write, for a periodic function *f(t)*

$$f(t) = \sum_{n=-\infty}^{\infty} f_r(t - nT)$$

Convolution property of delta function

$$= \sum_{n=-\infty}^{\infty} f_r(t) * \delta(t - nT)$$

Convolution is distributive

$$= f_r(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Definition of Dirac Comb

$$= f_r(t) * S_T(t)$$

Fourier Transform of a Dirac Comb

The Convolution theorem can then be used to transform this

$$f(t) \leftrightarrow F_r(\omega)\mathbf{F}[S_T(t)]$$

Compare this with the other result for the Fourier transform of a periodic signal:

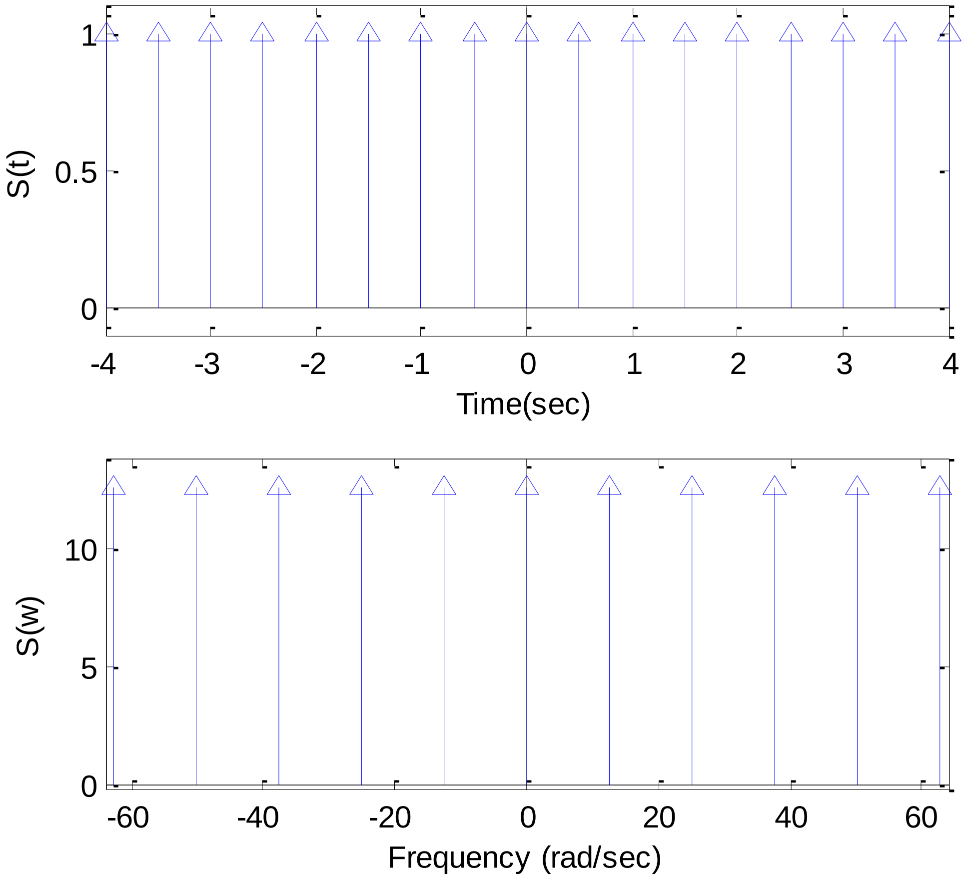
$$f(t) \leftrightarrow F_r(\omega)\frac{2\pi}{T}S_{2\pi/T}(\omega)$$

The conclusion is that the Fourier transform of the Dirac comb also has the form of a Dirac comb.

$$S_T(t) \leftrightarrow \frac{2\pi}{T}S_{2\pi/T}(\omega)$$

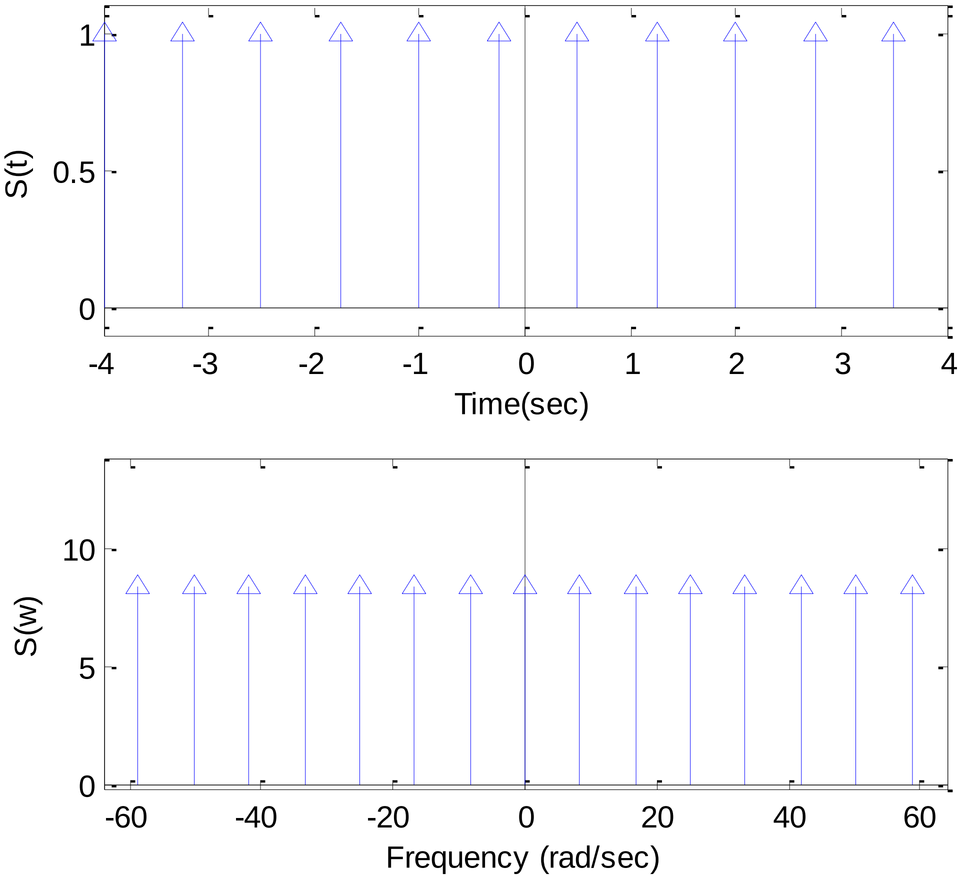
Fourier Transform of a Dirac Comb

Thus the Fourier transform of a Dirac comb is a Dirac comb. The spacing between the delta functions in the transform is proportional to the reciprocal of the spacing in the time domain.



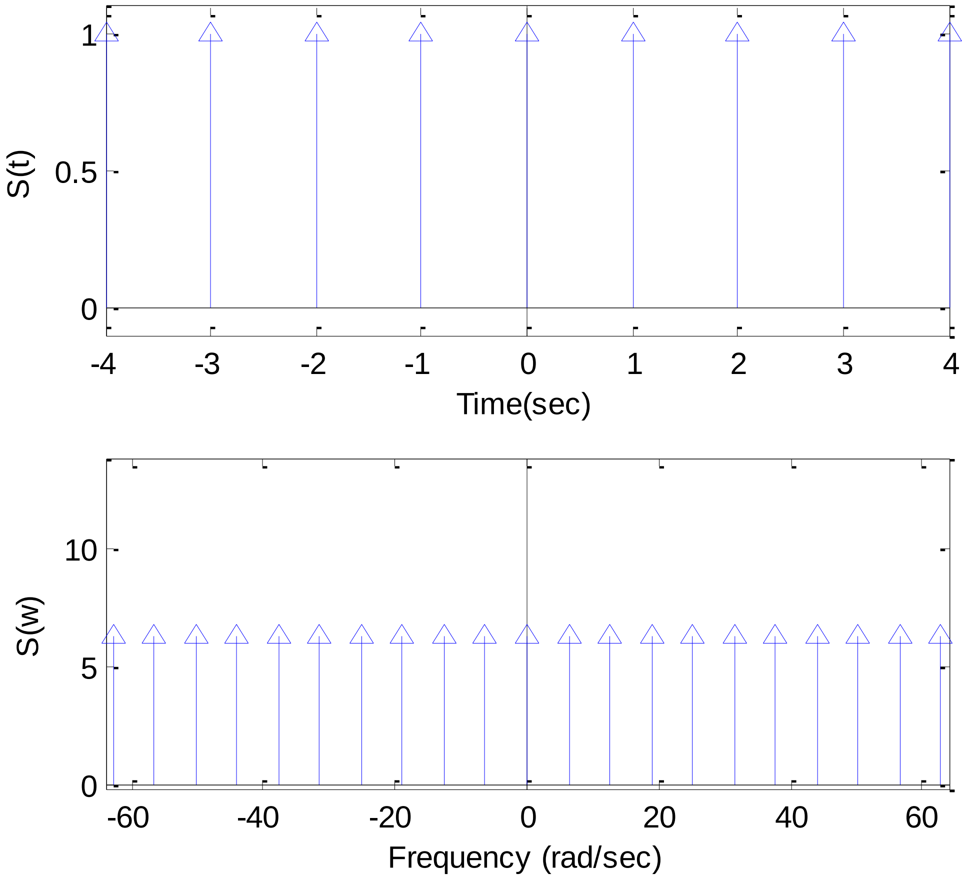
Fourier Transform of a Dirac Comb

This result is physically familiar from the theory of the diffraction grating, the spacing of lines in the spectrum is inversely proportional to the spacing of the lines in the grating.



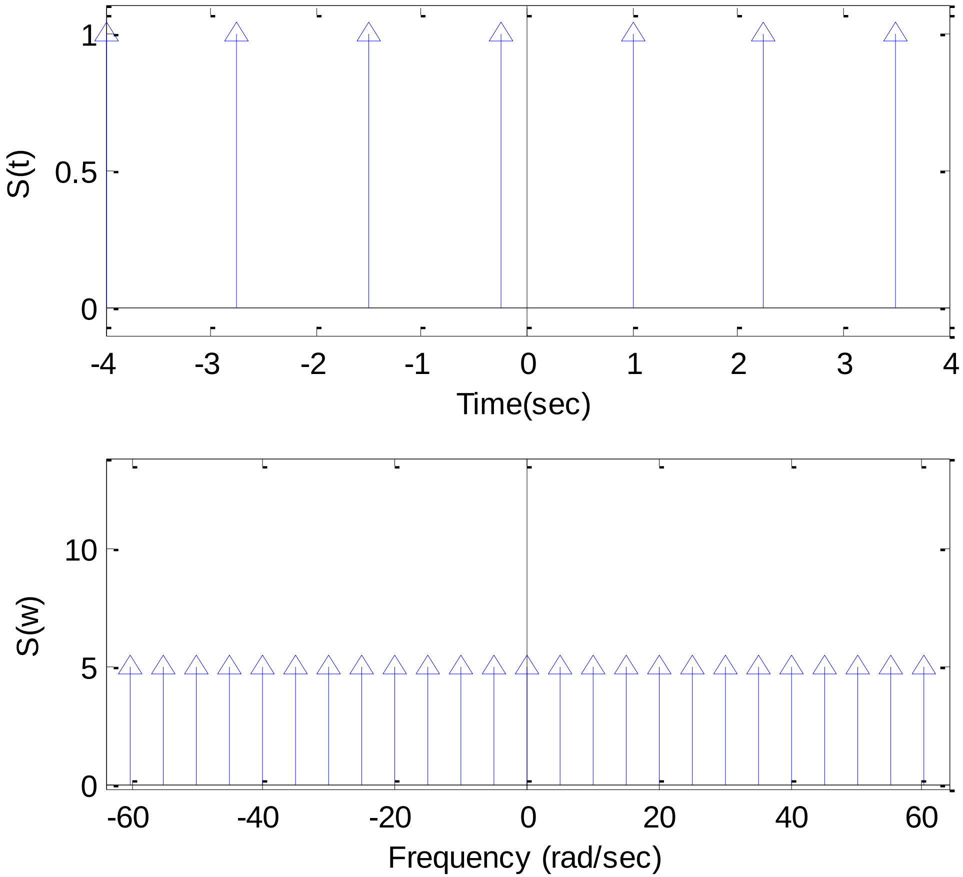
Fourier Transform of a Dirac Comb

In fact the pattern produced at large distances when coherent light is passed through an aperture of any shape (the Fraunhofer diffraction pattern) is proportional to the Fourier transform of the aperture.



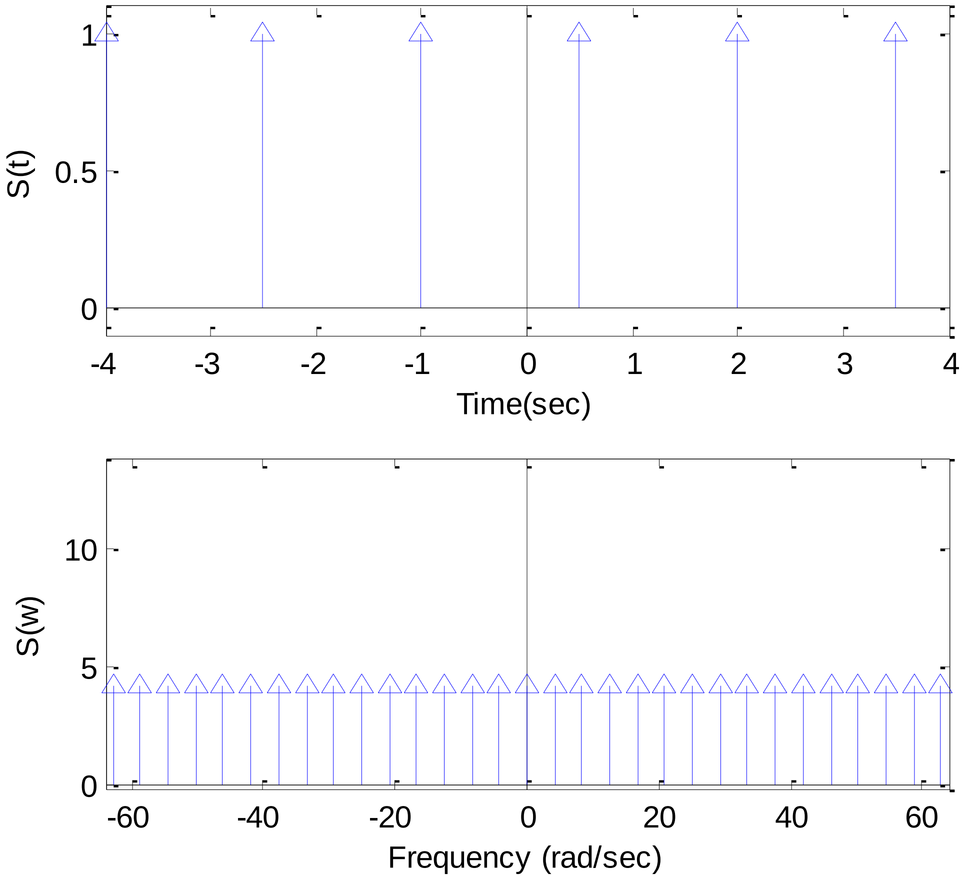
Fourier Transform of a Dirac Comb

This result has actually been used in the past to evaluate the Fourier transforms.



Fourier Transform of a Dirac Comb

In diffraction physics the Fourier transform domain is often called "reciprocal space" due to the reciprocal character of scales in the two domains.



Fourier Transform of Periodic Signal

- The Fourier transform of a periodic signal is found by Fourier transforming its Fourier series expansion:

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t}.$$

- The Fourier transform of a complex exponential is

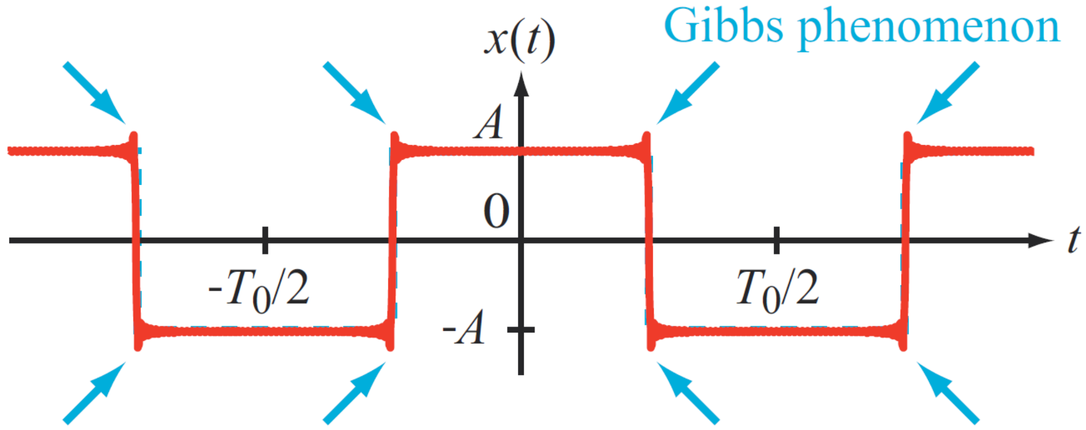
$$\mathcal{F}\left\{e^{j\omega_0 t}\right\} = 2\pi\ \delta(\omega - \omega_0)$$

- So the Fourier transform of the periodic signal is

$$\mathcal{F}\{x(t)\} = \sum_{n=-\infty}^{\infty} \mathbf{x}_n 2\pi\ \delta(\omega - n\omega_0).$$

Gibbs's Phenomenon

- If x(t) is not continuous at t=t_i then the Fourier series expansion of x(t) converges to the midpoint of the discontinuity, which is $\frac{1}{2}[x(t_i^+) + x(t_i^-)]$
- There is an overshoot, called Gibbs phenomenon
- This does not disappear if more terms are used



(b) Fourier series with 100 terms