LECTURE BLOCK 4C: FOURIER TRANSFORM (DTFT)

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Fourier Transform

We will learn:

- Compute Fourier Transform of nonperiodic signals,
- Compute Fourier Transform of periodic signals.
- Discrete time Fourier Transform.

Discrete vs. Cont. Time: Fourier Transform

Discrete-Time

$$\mathbf{X}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{\Omega_1}^{\Omega_1+2\pi} \mathbf{X}(e^{j\Omega})\, e^{j\Omega n}\, d\Omega,$$

Continuous-Time

$$\mathbf{X}(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{X}(\omega)\, e^{j\omega t}\, d\omega$$

If x[n] and x(t) are causal signals, then:

$$\mathbf{X}(e^{j\Omega}) = \mathbf{X}(\mathbf{z})\Big|_{\mathbf{z}=e^{j\Omega}}$$

$$\mathbf{X}(\omega) = \mathbf{X}(\mathbf{s})\Big|_{\mathbf{s}=j\omega}$$

Discrete-Time Fourier Transform (DTFT)

The analogy between the DTFT $\mathbf{X}(e^{j\Omega})$ and the continuous-time Fourier transform $\mathbf{X}(\omega)$ can be demonstrated further by expressing the discrete-time signal $x[n]$ in the form of the continuous-time signal $x(t)$ given by

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\,\delta(t-n).$$

The continuous-time Fourier transform of $x(t)$ is

$$\mathbf{X}(\omega) = \sum_{n=-\infty}^{\infty} x[n]\,\mathcal{F}[\delta(t-n)] = \sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega n} = \mathbf{X}(e^{j\Omega}),$$

▶ The DTFT is the continuous-time Fourier transform of the continuous-time sum-of-impulses

$$\sum_{n=-\infty}^{\infty} x[n]\,\delta(t-n).$$

Most Properties of the CTFT also hold for DTFT

Property	$x[n]$		$\mathbf{X}(e^{j\Omega})$
1. Linearity	$k_1\, x_1[n] + k_2\, x_2[n]$	$\longleftrightarrow$	$k_1\, \mathbf{X}_1(e^{j\Omega}) + k_2\, \mathbf{X}_2(e^{j\Omega})$
2. Time shift	$x[n-n_0]$	$\longleftrightarrow$	$\mathbf{X}(e^{j\Omega})\, e^{-jn_0\Omega}$
3. Frequency shift	$x[n]\, e^{j\Omega_0 n}$	$\longleftrightarrow$	$\mathbf{X}(e^{j(\Omega-\Omega_0)})$
4. Multiplication by n (frequency differentiation)	$n\, x[n]$	$\longleftrightarrow$	$j\dfrac{d\mathbf{X}(e^{j\Omega})}{d\Omega}$
5. Time Reversal	$x[-n]$	$\longleftrightarrow$	$\mathbf{X}(e^{-j\Omega})$
6. Time convolution	$x_1[n] * x_2[n]$	$\longleftrightarrow$	$\mathbf{X}_1(e^{j\Omega})\, \mathbf{X}_2(e^{j\Omega})$
7. Frequency convolution	$x_1[n]\, x_2[n]$	$\longleftrightarrow$	$\dfrac{1}{2\pi}\, \mathbf{X}_1(e^{j\Omega}) * \mathbf{X}_2(e^{j\Omega})$
8. Conjugation	$x^*[n]$	$\longleftrightarrow$	$\mathbf{X}^*(e^{-j\Omega})$
9. Parseval's theorem	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2$	$=$	$\dfrac{1}{2\pi} \int_{\Omega_1}^{\Omega_1+2\pi} \lvert\mathbf{X}(e^{j\Omega})\rvert^2\, d\Omega$
10. Conjugate symmetry	$\mathbf{X}^*(e^{j\Omega}) = \mathbf{X}(e^{-j\Omega})$		

► In particular, the following conjugate symmetry properties extend directly from the continuous-time Fourier transform to the DTFT:

$x[n]$	$\mathbf{X}(e^{j\Omega})$
Real and even	Real and even
Real and odd	Imaginary and odd
Imaginary and even	Imaginary and even
Imaginary and odd	Real and odd

An important additional property of the DTFT that does not hold for the continuous-time Fourier transform is:

► The DTFT $\mathbf{X}(e^{j\Omega})$ is periodic with period 2π. ◄

- The DTFT can be viewed as a continuous-time Fourier series expansion of the periodic function $\mathbf{X}(e^{j\Omega})$. So that the inverse DTFT given by $$x[n] = \frac{1}{2\pi} \int_{\Omega_1}^{\Omega_1+2\pi} \mathbf{X}(e^{j\Omega})\, e^{j\Omega n}\, d\Omega,$$ is simply the formula for computing the continuous-time Fourier series coefficients of a periodic signal with period $T_0 = 2\pi$.
- The relation between the DTFT and the z-transform is analogous to the relation between continuous-time Fourier transform and the Laplace transform.

The relations between periodicity and discreteness in time and frequency can be summarized as follows:

Time Domain	Frequency Domain	Relation
Periodic	Discrete	Fourier Series
Discrete	Periodic	DTFT
Discrete and Periodic	Discrete and Periodic	DTFS

The Discrete-Time Fourier Transform (DTFT)

(a) For a discrete-time signal $x[n]$, its DTFT is its spectrum $\mathbf{X}(e^{j\Omega})$.

(b) For a discrete-time system, the DTFT of its impulse response $h[n]$ is the system's frequency response $\mathbf{H}(e^{j\Omega})$.

Examples: Computing DTFT

Compute the DTFT of

(a) $x_1[n] = \{3, 1, \underline{4}, 2, 5\}$,

(b) $x_2[n] = \left(\frac{1}{2}\right)^n \; u[n]$,

(c) $x_3[n] = 4\sin(0.3n)$,

Note that all three DTFTs are periodic with period 2π.

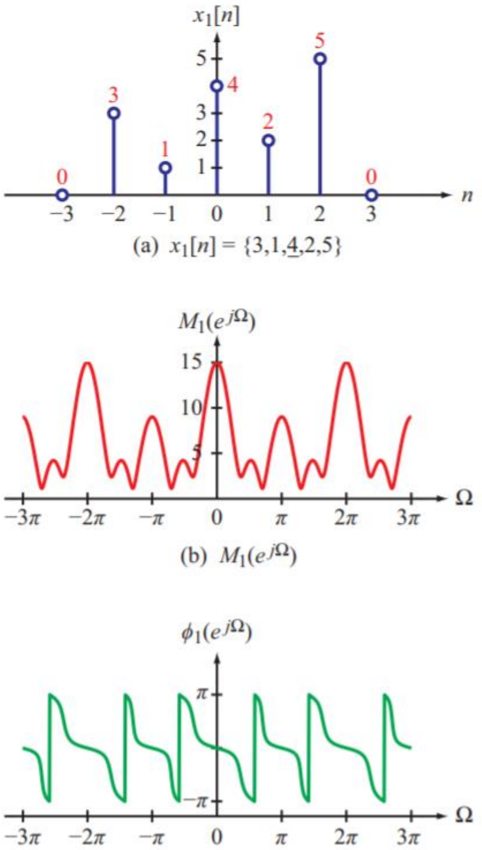
Example 1: Computing DTFT

Compute the DTFT of

(a) $x_1[n] = \{3, 1, \underline{4}, 2, 5\}$,

Answer: $\mathbf{X}_1(e^{j\Omega}) = 3e^{j2\Omega} + 1e^{j\Omega} + 4 + 2e^{-j\Omega} + 5e^{-j2\Omega}$.

Steps:



(a) $x_1[n] = \{3,1,\underline{4},2,5\}$

(b) $M_1(e^{j\Omega})$

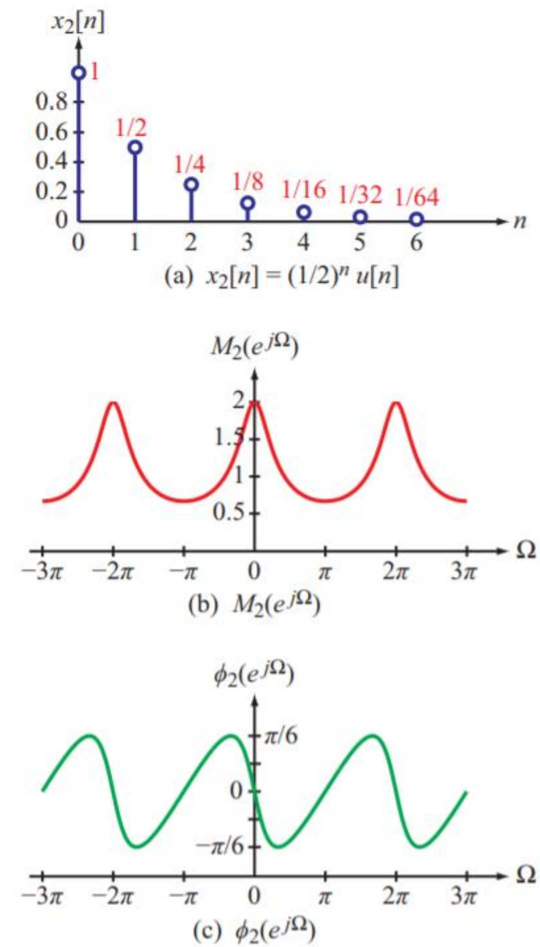
(c) $\phi_1(e^{j\Omega})$

Example 2: Computing DTFT

Compute the DTFT of (b) $x_2[n] = \left(\frac{1}{2}\right)^n u[n]$,

Answer: $$\mathbf{X}_2(e^{j\Omega}) = \frac{1}{1 - \dfrac{1}{2e^{j\Omega}}} = \frac{e^{j\Omega}}{e^{j\Omega} - \dfrac{1}{2}}.$$

Steps:



(a) $x_2[n] = (1/2)^n u[n]$

(b) $M_2(e^{j\Omega})$

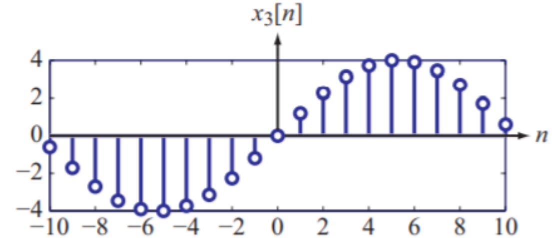
(c) $\phi_2(e^{j\Omega})$

Example 3: Computing DTFT

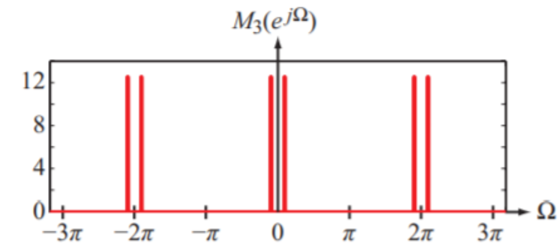
Compute the DTFT of (c) $x_3[n] = 4\sin(0.3n)$,

Answer: $$\mathbf{X}_3(e^{j\Omega}) = \frac{4\pi}{j} \sum_{k=-\infty}^{\infty} [\delta(\Omega - 0.3 - 2\pi k) - \delta(\Omega + 0.3 - 2\pi k)].$$

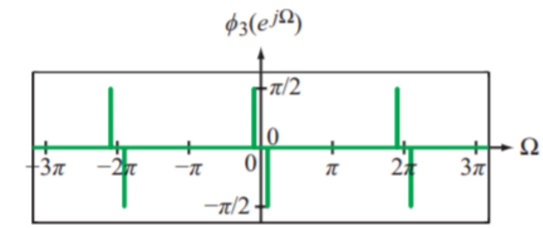
Steps:



(a) $x_3[n] = 4\sin(0.3n)$



(b) $M_3(e^{j\Omega})$

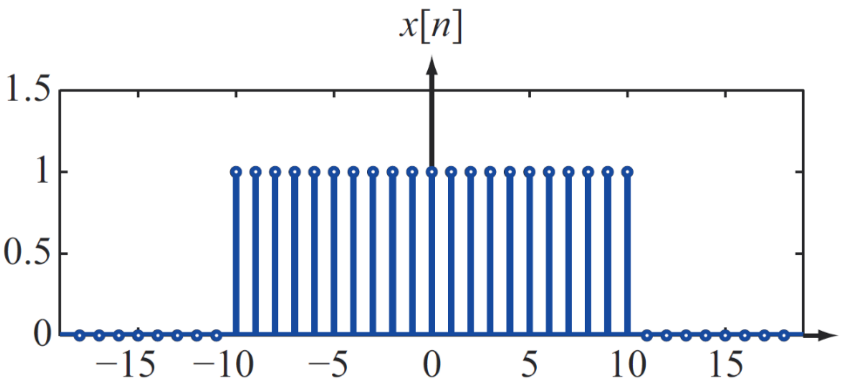


(c) $\phi_3(e^{j\Omega})$

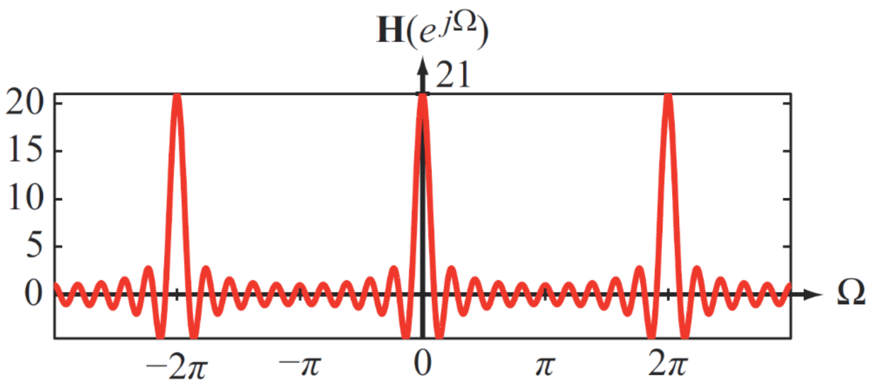
Discrete Sinc Function=DTFT[pulse]

$$x[n] = \begin{cases} 1 & \text{for } |n| \leq N, \\ 0 & \text{for } |n| > N. \end{cases}$$

$$\mathbf{X}(e^{j\Omega}) = \frac{\sin\left[\Omega\left(N + \frac{1}{2}\right)\right]}{\sin\left(\frac{\Omega}{2}\right)}$$



(a) $x[n] = \text{rect}(n/10)$

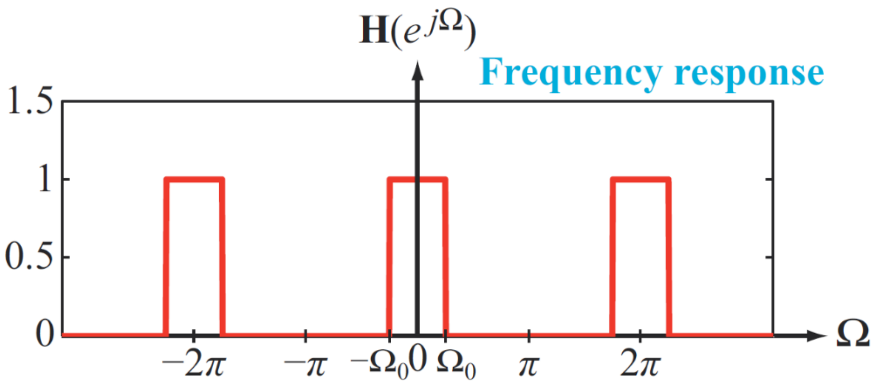


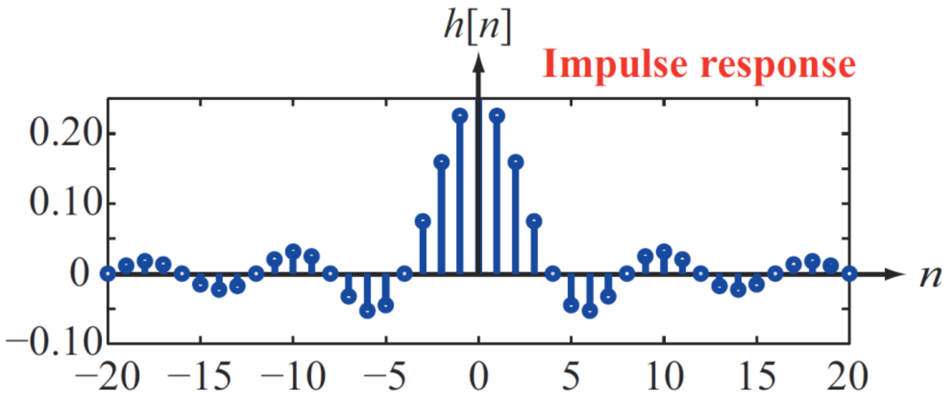
(b) $\mathbf{H}(e^{j\Omega})$

Impulse Response of Brick-Wall Lowpass Filter = Discrete-Time Sinc Function

$$\mathbf{H}(e^{j\Omega}) = \begin{cases} 1 & 0 \le |\Omega| \le \Omega_0, \\ 0 & \Omega_0 < |\Omega| \le \pi, \end{cases}$$

$$h[n] = \frac{\sin(\Omega_0 n)}{\pi n}$$





Important DTFT Pairs

	$x[n]$		$\mathbf{X}(e^{j\Omega})$	Condition
1.	$\delta[n]$	$\longleftrightarrow$	1	
1a.	$\delta[n-m]$	$\longleftrightarrow$	$e^{-jm\Omega}$	m = integer
2.	1	$\longleftrightarrow$	$2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$	
3.	$u[n]$	$\longleftrightarrow$	$\frac{e^{j\Omega}}{e^{j\Omega}-1} + \sum_{k=-\infty}^{\infty} \pi\delta(\Omega - 2\pi k)$	
3a.	$\mathbf{a}^n\, u[n]$	$\longleftrightarrow$	$\frac{e^{j\Omega}}{e^{j\Omega}-\mathbf{a}}$	$\lvert\mathbf{a}\rvert < 1$
3b.	$n\mathbf{a}^n\, u[n]$	$\longleftrightarrow$	$\frac{\mathbf{a}e^{j\Omega}}{(e^{j\Omega}-\mathbf{a})^2}$	$\lvert\mathbf{a}\rvert < 1$
4.	$e^{j\Omega_0 n}$	$\longleftrightarrow$	$2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k)$	
5.	$\mathbf{a}^{-n}\, u[-n-1]$	$\longleftrightarrow$	$\frac{\mathbf{a}e^{j\Omega}}{1-\mathbf{a}e^{j\Omega}}$	$\lvert\mathbf{a}\rvert < 1$
6.	$\cos(\Omega_0 n)$	$\longleftrightarrow$	$\pi \sum_{k=-\infty}^{\infty} [\delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k)]$	
7.	$\sin(\Omega_0 n)$	$\longleftrightarrow$	$\frac{\pi}{j} \sum_{k=-\infty}^{\infty} [\delta(\Omega - \Omega_0 - 2\pi k) - \delta(\Omega + \Omega_0 - 2\pi k)]$	
8.	$\mathbf{a}^n \cos(\Omega_0 n + \theta)\, u[n]$	$\longleftrightarrow$	$\frac{e^{j2\Omega}\cos\theta - \mathbf{a}e^{j\Omega}\cos(\Omega_0 - \theta)}{e^{j2\Omega} - 2\mathbf{a}e^{j\Omega}\cos\Omega_0 + \mathbf{a}^2}$	$\lvert\mathbf{a}\rvert < 1$
9.	$\mathrm{rect}\left[\frac{n}{N}\right] = u[n+N] - u[n-1-N]$	$\longleftrightarrow$	$\frac{\sin\left[\Omega\left(N+\frac{1}{2}\right)\right]}{\sin\left(\frac{\Omega}{2}\right)}$	
10.	$\frac{\sin[\Omega_0 n]}{\pi n}$	$\longleftrightarrow$	$\sum_{k=-\infty}^{\infty} [u(\Omega + \Omega_0 - 2\pi k) - u(\Omega - \Omega_0 - 2\pi k)]$	

DTFT of Discrete Time Convolution

By the property of Fourier transform, the convolution of 2 discrete-time sequences is equivalent to the multiplication of their Fourier transform equivalent:

$$x[n] * h[n] \quad \longleftrightarrow \quad \mathbf{X}(e^{j\Omega})\,\mathbf{H}(e^{j\Omega})$$

This means that to derive the discrete-time convolution, we can compute in time-domain directly using convolution sum (see Lecture 2 on LTI), OR

- Computing the DTFT of x[n] and h[n],
- Multiplying $\mathbf{X}(e^{j\Omega})$ by $\mathbf{H}(e^{j\Omega})$, and then
- Computing the inverse DTFT of the product.

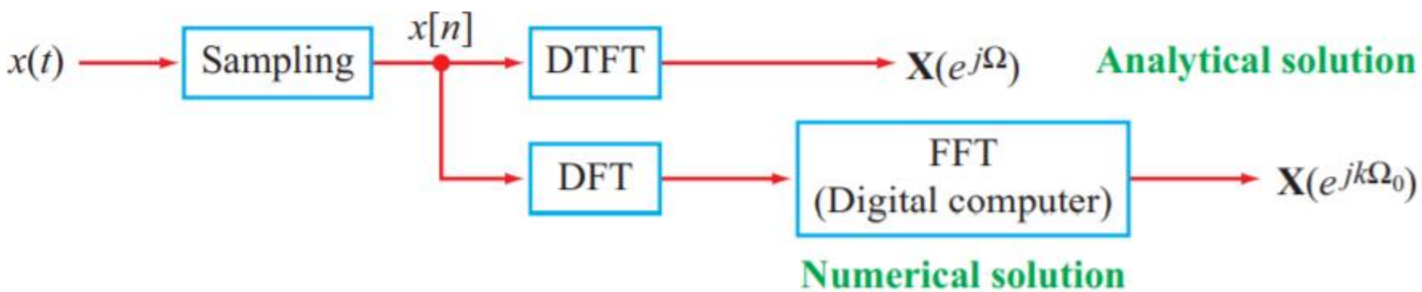
$$x[n] * h[n] \quad \longleftrightarrow \quad \mathrm{DTFT}^{-1}[\mathbf{X}(e^{j\Omega})\,\mathbf{H}(e^{j\Omega})].$$

Which method to use is dictated by computational efficiency.

Discrete Fourier Transform (DFT)

DFT is the numerical bridge between the DTFT and Fast Fourier Transform.

> ► Despite its name, the FFT is *not* a Fourier transform; it is an algorithm for computing the discrete Fourier transform (DFT). ◄



The role of the DFT is to cast the DTFT formulation of an infinite-duration signal <u>in an approximate form composed of finite summations</u> instead of infinite summations and integrals. For finite-duration signals, the DFT formulation is exact.

Discrete Fourier Transform (DFT)

For non-periodic signals:

$$\mathbf{X}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$\underbrace{x[n] = \frac{1}{2\pi} \int_{\Omega_1}^{\Omega_1+2\pi} \mathbf{X}(e^{j\Omega})\, e^{j\Omega n}\, d\Omega.}_{\text{Inverse DTFT}}$$

DFT:

$$\mathbf{X}_k = \sum_{n=0}^{N_0-1} x[n]\, e^{-jk\Omega_0 n}$$

$$x[n] = \frac{1}{N_0} \sum_{k=0}^{N_0-1} \mathbf{X}_k e^{jk\Omega_0 n},$$

$$\Omega_0 = \frac{2\pi}{N_0} \qquad\qquad \mathbf{X}_k = N_0 \mathbf{x}_k.$$

Convert integral to summation of length N_0; N_0-point DFT.

For periodic signals:

DTFS:

$$\mathbf{x}_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n]\, e^{-jk\Omega_0 n},$$

$$x[n] = \sum_{n=0}^{N_0-1} \mathbf{x}_k e^{jk\Omega_0 n}.$$

$$\underbrace{\mathbf{X}_k}_{\text{DFT}} = \underbrace{\mathbf{X}(e^{j\Omega})\Big|_{\Omega=k\Omega_0}}_{\text{DTFT}} = \underbrace{\mathbf{X}(\mathbf{z})|_{\mathbf{z}=e^{jk\Omega_0}}}_{\text{z-transform}}$$

▶ The DFT is identical to the DTFS with $\mathbf{X}[k] = N_0\mathbf{x}_k$. ◀

The preceding observations lead to two important conclusions.

(1) The DFT does not offer anything new so far as periodic signals are concerned.

(2) The DFT, in essence, *converts* a nonperiodic signal $x[n]$ into a periodic signal of period N_0 and converts the DTFT into a DTFS (within a scaling constant N_0).

Example 4: a *Periodic* Sinusoidal Signal

Compute the DFT of:

$$x[n] = A\cos(2\pi(k_0/N_0)n + \theta)$$ this is a *periodic* sinusoid.

Answer:

$$\mathbf{X}_k = N_0 \, \frac{A}{2} \, e^{j\theta} \qquad \text{if } k = k_0,$$

$$\mathbf{X}_k = N_0 \, \frac{A}{2} \, e^{-j\theta} \qquad \text{if } k = N_0 - k_0,$$

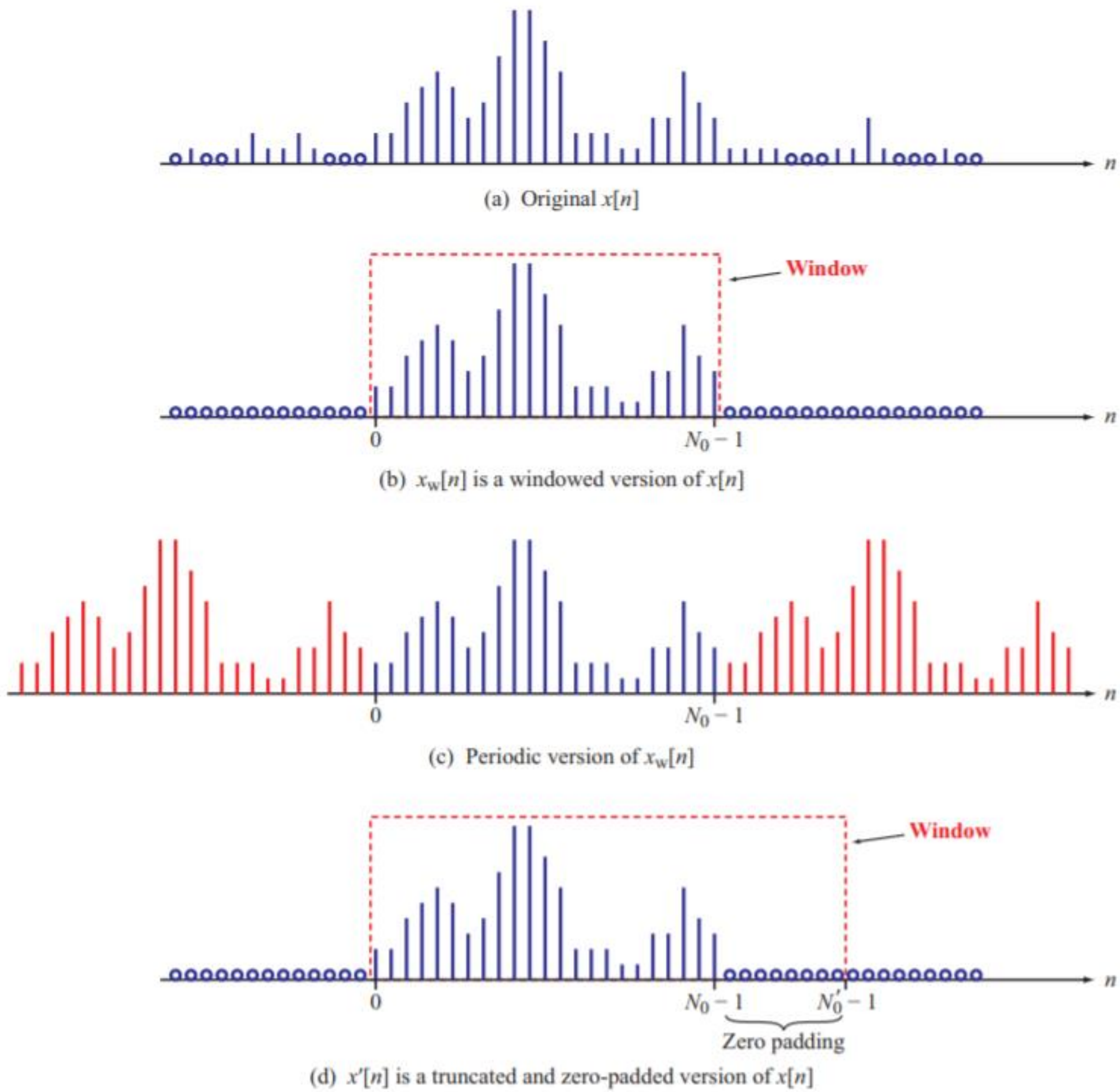
$$\mathbf{X}_k = 0 \qquad \text{for all other } k.$$

We can use the DFT to compute spectra of signals consisting of periodic sinusoids.

Steps: (Ulaby 402)

Computing Spectra of Nonperiodic Signals using DFT and Windowing

Multiply x[n] by a rectangular window that sets x[n] to 0 outside window, take the periodic extension of the result, and find its DTFS spectrum using the DFT. If length not enough, use zero-padding.



Example 5: *from example 1*

Compute the DTFT of

(a) $x_1[n] = \{3, 1, \underline{4}, 2, 5\}$,

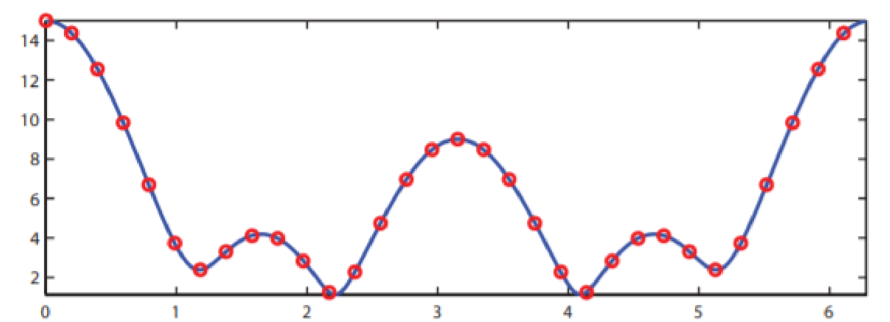
Use a 32-point DFT to compute DTFT of example 1 numerically.

Example 1 demonstrated how to compute x1[n]={ 3, 1, $\underline{4}$, 2, 5 }, and the magnitude of the DTFT was plotted there.

Since the magnitude of the DTFT is unaffected by time shifts (property #2 of DTFT Table), we compute and plot the magnitude of the 32-point DFT of { $\underline{3}$, 1, 4, 2, 5 }, zero-padded with 27 zeros, using

$$\mathbf{X}[k] = \sum_{n=0}^{N_0-1} x[n]\, e^{-jk\Omega_0 n}, \qquad k = 0, 1, \ldots, N_0 - 1$$

The results are plotted as red dots and the exact DTFT magnitude is shown in blue. Increasing 32 to a larger number would provide a much finer sampling of the DTFT.



Using DFT to Compute (linear) Convolutions

Goal: To compute the convolution of $x[n]$ and $y[n]$.
Lengths: $x[n]$:L, $y[n]$:M, $x[n]*y[n]$: $N=L+M-1$.

Procedure: Zero-pad $x[n]$ and $y[n]$ to lengths N by appending N-L zeros to $x[n]$ and N-M to $y[n]$.

DFT: $x[n]*y[n] = \mathrm{IDFT}\{\mathrm{DFT}\{x[n]\}\mathrm{DFT}\{y[n]\}\}$ where all DFTs and inverse DFTs (IDFTs) have lengths N.

Example 6: Using DFTs to Compute Convolutions

Example: Compute convolution {4,5}*{1,2,3} using DFTs.

Solution: The convolution has length 2+3-1=4. Zero-pad the two signals to be convolved to {4,5,0,0} and {1,2,3,0}.
DFT{{4,5,0,0}}={9, 4-j5, -1, 4+j5}.
DFT{{1,2,3,0}}={6, -2-j2, 2, -2+j2}.

Multiply these: {54, -18+j2, -2, -18-j2}.
IDFT{{54, -18+j2, -2, -18-j2}}={4,13,22,15}.

So {4,5}*{1,2,3}={4,13,22,15}.

Extra: Fast Fourier Transform (FFT)

FFT: A fast algorithm for computing the
DFT: A transform for computing spectra.

Idea: Divide-and-conquer approach:
Break up a large DFT into smaller ones.
If N_0 is a power of 2, reduce computation from N_0^2 to $(N_0/2)\log_2 N_0$. If $N_0 = 512$, this reduces from 262,144 to 2304 multiplications!

Goal: Compute $\mathbf{X}_k = \sum_{n=0}^{N_0-1} x[n]\, W_{N_0}^{nk}$ where $W_{N_0} = e^{-j2\pi/N_0}$

FFT Formulae

Split x[n] into values at even and odd times:

$$x_e[n] = x[2n],$$

$$x_o[n] = x[2n+1]$$

Then use these formulae to break up N-point DFT into two N/2-point DFTs and some multiplications:

$$\mathbf{X}_k = \underbrace{\sum_{n=0}^{N_0/2-1} x_e[n]\, W_{N_0/2}^{nk}}_{N_0/2\text{-point DFT}} + W_{N_0}^{k} \underbrace{\sum_{n=0}^{N_0/2-1} x_o[n]\, W_{N_0/2}^{nk}}_{N_0/2\text{-point DFT}}$$

$$\mathbf{X}_{k+N_0/2} = \underbrace{\sum_{n=0}^{N_0/2-1} x_e[n]\, W_{N_0/2}^{nk}}_{N_0/2\text{-point DFT}} - W_{N_0}^{k} \underbrace{\sum_{n=0}^{N_0/2-1} x_o[n]\, W_{N_0/2}^{nk}}_{N_0/2\text{-point DFT}}$$

Example: Compute 4-Point DFT using FFT

Goal: Use the FFT to break down the 4-point DFT of {a,b,c,d} into two 2-point DFTs and multiplications by j.

Solution: 2-point DFT: $\mathbf{X}_0 = x[0] + x[1]$ and $\mathbf{X}_1 = x[0] - x[1]$

$x[n]$ ={a,b,c,d}. $x_{\mathrm{e}}[n]$={a,c}. $x_{\mathrm{o}}[n]$={b,d}. W_4^1=-j.

2-point DFT of {a,c} is {a+c, a-c}.

2-point DFT of {b,d} is {b+d, b-d}.

4-point DFT of {a,b,c,d} is then computed from these as:

{(a+c)+(b+d), (a-c)-j(b-d), (a+c)-(b+d), (a-c)+j(b-d)}= result of direct computation of the 4-point DFT, which is {a+b+c+d, a-jb-c+jd, a-b+c-d, a+jb-c-jd}.

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{array}{l} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{array}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4 T	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$		

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ ω_0 $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n + N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}$, $k \ne 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}$, $k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n + 1)a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n + r - 1)!}{n!(r - 1)!} a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—