XMUT220 Signals and Systems

Time & Frequency Characteristics (Time Response Analysis)

Topics

- Introduction to time response analysis.

- First Order Time Domain Responses:
  - Time constant (τ)
  - Rise time (T_r)
  - Settling time (T_s)

- Second Order Time Domain Responses:
  - Natural frequency (ω_n)
  - Damping ratio (ζ)
  - Rise time (T_r)
  - Settling time (T_s)
  - Time-to-peak (T_p)
  - Percentage overshoot ($\%OS$)
  - Steady-state error ($e(\infty)$)

1. Time Domain Analysis

It is the response of a system to an input in the time domain that could determine mostly the characteristics and behaviour of the systems. Especially, if we are interested in the behaviour of the system during the transient response e.g. when the system is transitioning from one condition to another after given an input. For evaluating the

transient response, several quantitative parameters are often used to characterise the response of a system.

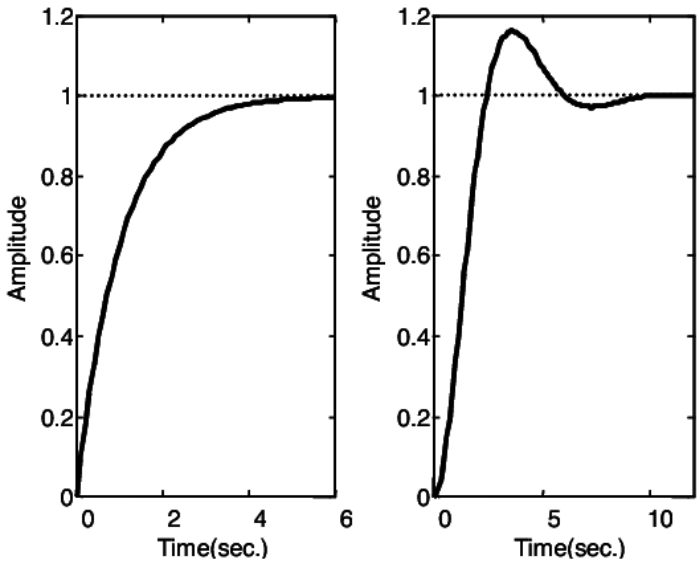


Figure 1: Transient responses of first order system: ramp (left) oscillatory (right)

To simplify the analysis process, we evaluate the time response of the systems in terms of transient response of the first-order and second-order systems.

Higher order systems could be approximated as any combinations of a first-order and a second-order systems e.g. a third-order system could be approximated as a complex system consisting of the a first-order system and a second-order system.

2. First-Order Time Response

A first-order system is defined as a type of system whose input-output relationship (also known as a transfer function) is a first-order differential equation.

The first-order system can be represented with the following equation:

$$\tau\left[\frac{dy(t)}{dt}\right] + y(t) = x(t)$$

Where: the t is the time constant of the system which is used to control the rate at which the first-order system responds.

The transfer function equation for the given first-order system is:

$$H(\omega) = \frac{1}{j\omega\tau + 1}$$

For example, a first-order system may also be written as a transfer function equation given below:

$$\frac{C(\omega)}{R(\omega)} = \frac{K}{j\omega\tau + 1}$$

Or, in term of a normalised form the transfer function equation becomes:

$$\frac{C(\omega)}{R(\omega)} = \frac{a}{j\omega + a}$$

Where: $K = 1$ and $\tau = 1/a$

The time constant (τ) and system gain (K) are used in the time response analysis of first order systems. Other parameters for analysis of first order system are typically rise time (T_r) and settling time (T_s).

For second-order system, these parameters are also considered, together with other parameters of the system e.g. damping ratio, natural frequency, time-to-peak, etc.

For a first-order system given below, derive its time response parameters.

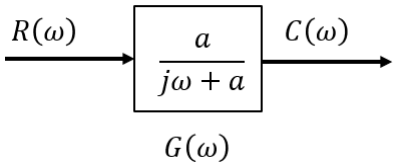


Figure 2: Block diagram of an (open-loop) first-order system

The transfer function of the first-order system shown above is:

$$G(\omega) = \frac{C(\omega)}{R(\omega)} = \frac{a}{j\omega + a}$$

Rearrange the equation given above, the output of the first-order system is:

$$C(\omega) = R(\omega)\left(\frac{a}{j\omega + a}\right)$$

For a unit step, $R(\omega) = u(t) = 1/j\omega$, the equation becomes:

$$C(\omega) = \left(\frac{1}{j\omega}\right)\left(\frac{a}{j\omega + a}\right)$$

Applying a partial fraction method to the equation above:

$$C(\omega) = \left(\frac{1}{j\omega}\right) - \left(\frac{1}{j\omega + a}\right)$$

By using inverse transform from the Fourier transform table, we can obtain response of first-order system when it is subjected to a unit step input in the time domain:

$$c(t) = c_f(t) + c_n(t)$$

Applying the inverse transforms of the parts of the equation given above, the output of a first-order system when subjected to a unit step is equal to:

$$c(t) = 1 - e^{-at}$$

2.1. Time Constant of First-Order System

The following figure shows transient response of first-order system when it is given a step input.

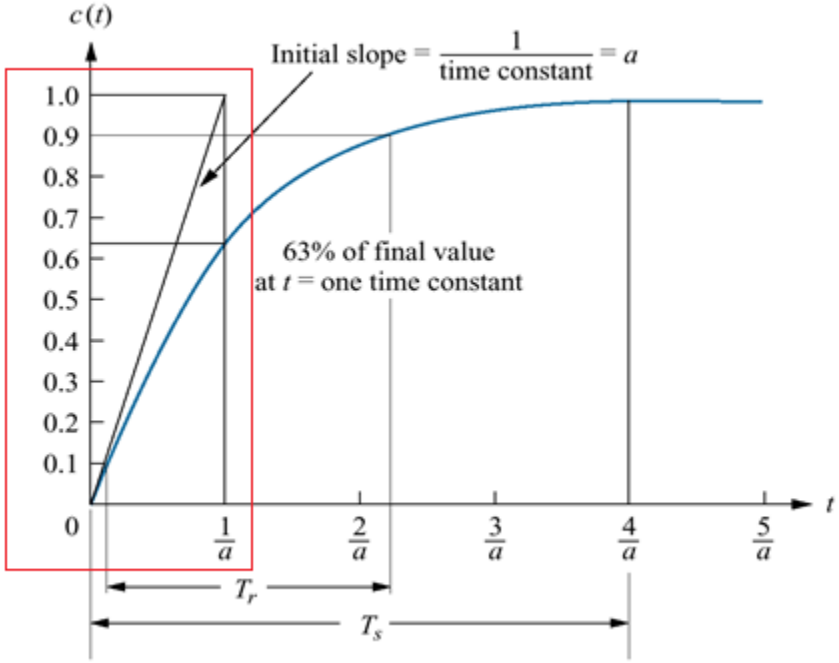


Figure 3: Time constant of first-order system

Time constant (τ) of the first-order system is defined as the time for the step response to rise to 63 % of its final value.

The time constant is derived from the time for e^{-at} to decay to 37% of its final value.

$$\tau_{decay} = e^{-at}\big|_{t=1/a} = e^{-1} = 0.37$$

Thus, for increase response of a first-order system, the time constant is actually equal to rise to about 63% of its final value.

This time constant of the first-order system is achieved when $t = 1/a$:

$$\tau = \tau_{increase} = c(t)|_{t=1/a}$$

Entering the value into the equation, the time constant of the first-order system is:

$$\tau = 1 - e^{-at}\big|_{t=\frac{1}{a}}$$

Solve the equation given above, the time constant of a first-order system is:

$$\tau = 1 - 0.37 = 0.63$$

2.2. Rise Time of First-Order System

Rise time (T_r) of a first-order system is defined as the time for the response to go from 0.1 to 0.9 of its final value:

$$T_r = \ln(0.9) - \ln(0.1)$$

Calculate and normalise the result with the time constant, the rise time of the first-order system is:

$$T_r = \frac{2.3}{a} - \frac{0.1}{a} = \frac{2.2}{a} \quad \text{or } 2.2\tau$$

The rise time of the first-order system is as shown in the following graph.

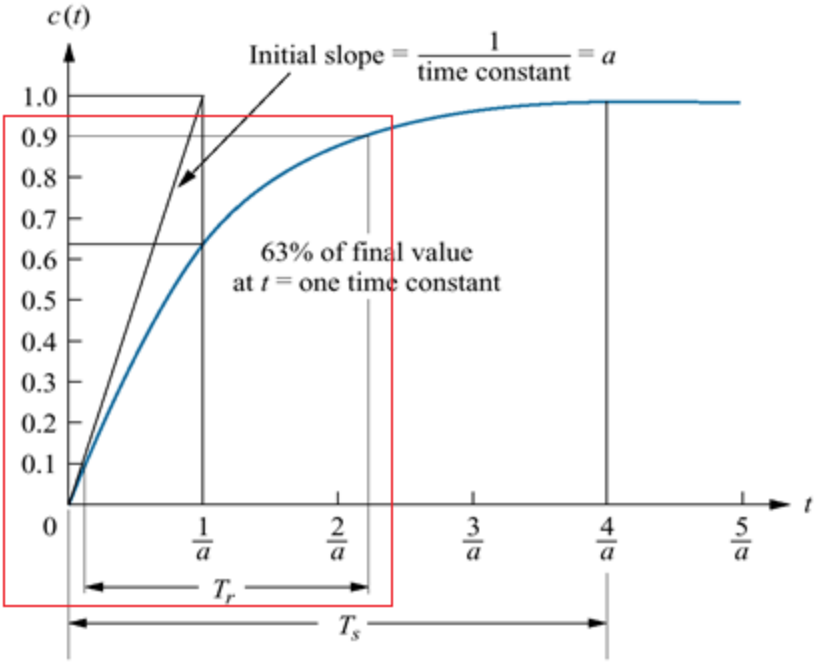


Figure 4: Rise time of first-order system

2.3. Settling Time of First-Order System

Settling time (T_s) of the first-order system is defined as the time for the response to reach, and stay within, 2% of its final value (e.g. 2% is commonly used as a standard and there are also other options for standard i.e. 1%, 0.1%, etc.)

$$T_s = \ln(1) - \ln(0.02)$$

Calculate and normalise the result with the time constant, the settling time of the first-order system is:

$$T_s = \frac{3.9}{a} \quad \text{or} \quad 3.9\tau$$

The settling time of the first-order system is as shown in the following graph.

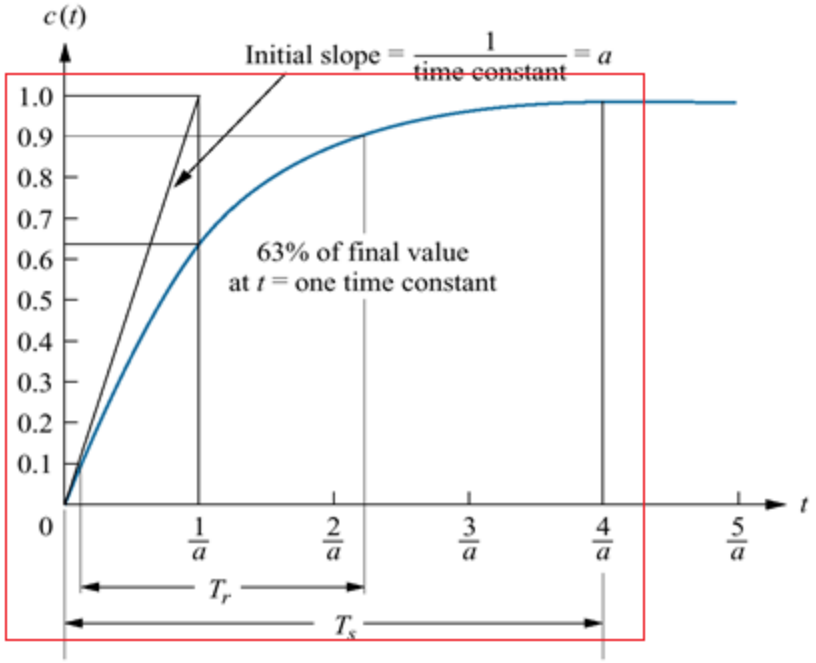


Figure 5: Settling time of first-order system

Example for Tutorial 1 – First-Order System Analysis

You are given the following first-order systems:

$$G(\omega) = \frac{5}{j\omega + 5}$$

a. Calculate its time constant, rise time, and settling time. [6 marks]

b. Simulate the transient response of the system in MATLAB when it is given a step input. [4 marks]

Solution

a. For the given system:

$$G(\omega) = \frac{5}{(j\omega + 5)} = \frac{1}{j\omega/5 + 1}$$

Time constant:

$$\tau = \frac{1}{5} = 0.2 \text{ second}$$

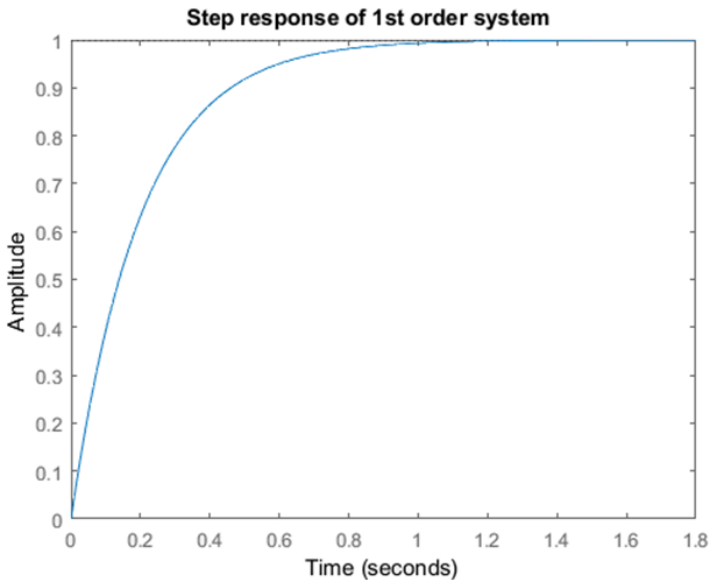
Rise time:

$$T_r = 2.2\tau = \frac{2.2}{a} = \frac{2.2}{5} = 0.44 \text{ second}$$

Settling time:

$$T_s = 4\tau = \frac{4}{a} = \frac{4}{5} = 0.8 \text{ second}$$

b. The transient response of the system is as shown below.



From the plot, it seems the system has τ = 0.22 second, T_r = 0.44 second and T_s = 0.8 second.

3. Second-Order Time Response

A second-order system has a slightly different transient response compared with a first-order system.

3.1. Second-Order Standardised Form

Second-order system is a type of system whose input-output relationship is a second-order differential equation.

A typical standardized linear constant-coefficient differential equation for a second-order system is:

$$\frac{d^2y(t)}{dt^2} + (2\zeta\omega_n)\frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

For the second-order system given above, the standardized second-order system equation is represented as:

$$\frac{Y(\omega)}{X(\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

Given a standardised form of second order equation:

$$G(\omega) = \frac{(\omega_n)^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + (\omega_n)^2}$$

Rearranging the equation given above:

$$G(\omega) = \frac{1}{1 + \left(\frac{2\zeta}{\omega_n}\right)j\omega + \left(\frac{j\omega}{\omega_n}\right)^2}$$

The standard equation for second order equation is:

$$\text{root}_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Referring to the characteristic equation of the transfer function equation above, roots of the equation are:

$$\text{root}_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\frac{2\zeta}{\omega_n} \pm \sqrt{\left(\frac{2\zeta}{\omega_n}\right)^2 - \frac{4}{\omega_n^2}}}{2\left(\frac{1}{\omega_n}\right)^2}$$

Thus, the roots of equation are:

$$\text{root}_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - (\zeta)^2} = \sigma \pm j\omega$$

Roots	Time Response	Type of Response
$-\sigma_1, -\sigma_2$	$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$	Overdamped
$-\sigma_1$	$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$	Critically damped
$-\sigma_d \pm j\omega_d$	$c(t) = A\, e^{-\sigma_d t}\cos(\omega_d t - \phi)$	Underdamped
$\pm j\omega_1$	$c(t) = A\cos(\omega_1 t - \phi)$	Undamped

Note: σ_d is repeated roots and $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Table 1: Roots and time response of second order system response

- For an overdamped response, the system looks like experiencing a heavily damped transient response.
- For a critically damped, it is a critically damped transient response.
- Then, for an underdamped, it is observed as a lightly damped transient response.
- Finally, for an undamped response, the system has no damping response e.g. constant amplitude oscillation transient response.

The graph below shows the various transient responses of several second order systems.

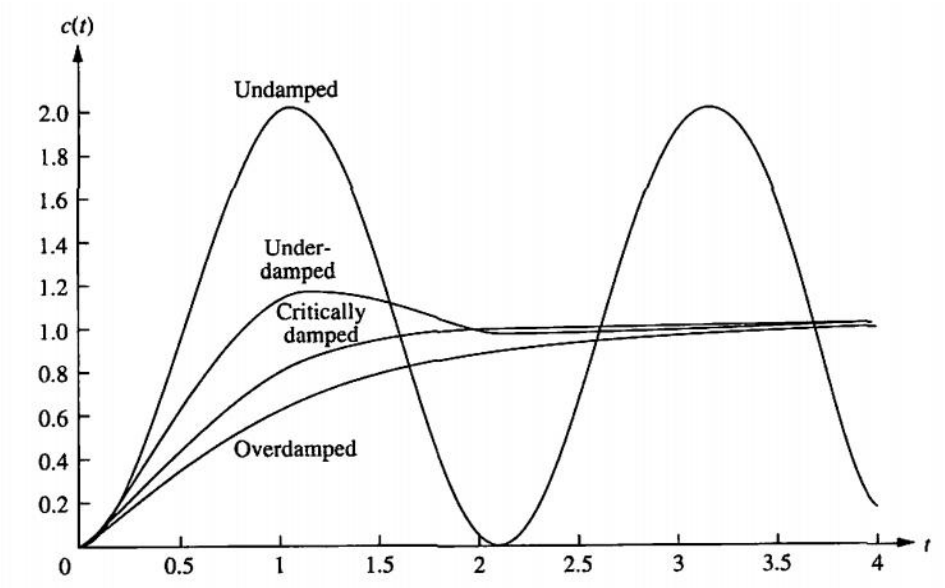


Figure 6: Various transient responses of second order systems

3.1.1. Overdamped response

When the transient response of the system resembles an exponential increase of its amplitude as shown in the figure below, then the response of the system is overdamped. Overdamped response means that a considerable damping is applied on the response of the system.

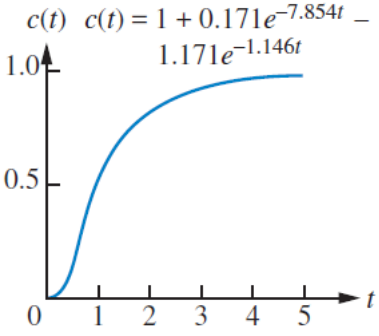


Figure 7: Overdamped response

3.1.2. Critically damped response

Critically damped response is a transition between overdamped and underdamped response. In this response of the system, the value of the damping ratio is 0.707.

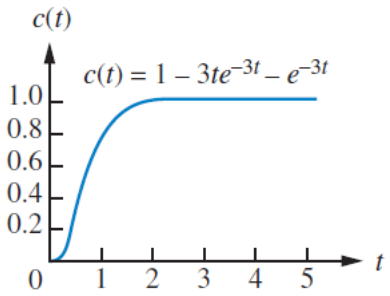


Figure 8: Critically damped response

3.1.3. Underdamped response

When the response of the system experiences slight oscillation at the beginning of the response before it is settling down to the final or intended amplitude, then the response of the system is underdamped. This means that a slight damping influences the response of the system

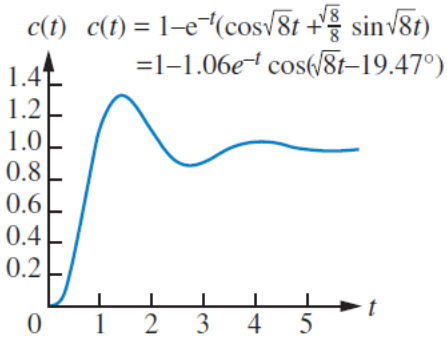


Figure 9: Underdamped response

3.1.4. Undamped response

When the response of the system is a constant amplitude oscillation as shown in the figure below, it means that the system is experiencing no damping.

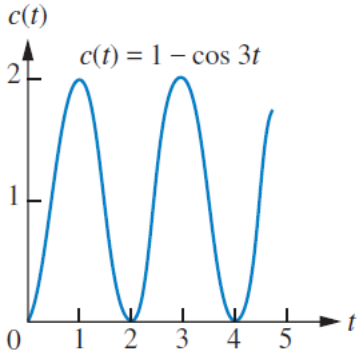


Figure 10: Undamped response

3.2. Natural Frequency

Consider the undamped transient response of the second-order system as shown in the figure below.

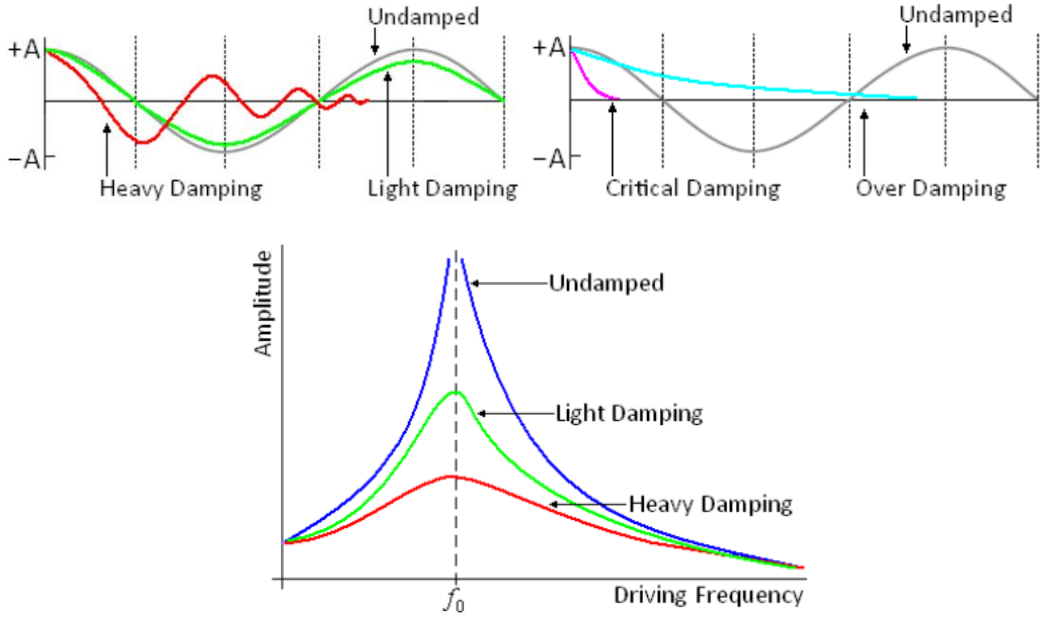


Figure 11: Natural frequency response

By definition, natural frequency is the frequency when there is no damping in the system (a = 0 in this case). For an undamped transient response, we obtain:

$$\frac{C(\omega)}{R(\omega)} = \frac{b}{(j\omega)^2 + b}$$

With roots of the system are:

$$\omega_{1,2} = \pm j\sqrt{b}$$

We know that $\omega = \sqrt{b} = \omega_n$ as a complex number. So, the frequency of oscillation is equal to $\sigma + j\omega$, which is termed the natural frequency.

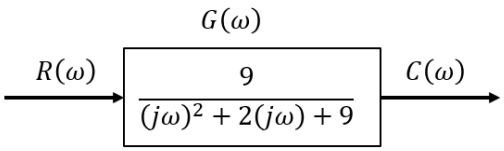
$$y(t) = y_{ss} + (y_0 - y_{ss})e^{-at}\cos(\omega_n t)$$

Furthermore, for a given second-order system, the natural response and forced response are related through the damping ratio as follow:

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency}} = \frac{\sigma}{\omega_n}$$

Example for tutorial 2 - Second Order Time Response

Given the following second order system which is represented with the following transfer function of the system.



Determine:

a. Roots of the characteristic equation of the system. [4 marks]

b. Sketch the transient response of the system. [4 marks]

Solution

a. For the given second-order system, the step response of the system is typically influenced by the complex poles:

$$\frac{C(\omega)}{R(\omega)} = \frac{9}{(j\omega)^2 + 2(j\omega) + 9}$$

Consider the characteristic equation of the system given above:

$$\omega_{1,2} = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(9)}}{2(1)}$$

It is solved to give roots:

$$\omega_{1,2} = -1 \pm j\sqrt{8}$$

The real part generates:

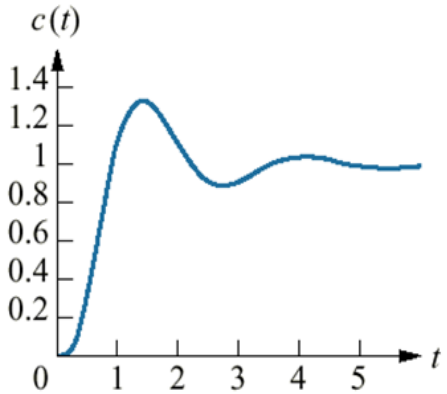
$$\sigma = e^{-t}$$

The complex part generates:

$$\omega_n = k\cos\left(\sqrt{8}t - \phi\right)$$

Where: t = time, k = amplitude, and ϕ = phase shift.

b. The response of the given second order system is shown in the graph below. The transient response of the system is found to be lightly damped (underdamped) response.



3.3. Damping Ratio

Consider a second order system given in the following transfer function:

$$\frac{C(\omega)}{R(\omega)} = \frac{k}{a'j\omega^2 + b'j\omega + c'}$$

For determining damping of the second order system, it is important to normalise the equation above into a unity coefficient form in front of the $(j\omega)^2$ term and k such that:

$$\frac{C(\omega)}{R(\omega)} = \frac{b}{j\omega^2 + aj\omega + b}$$

Note that the constants $a`$, $b`$, $c`$ are not equivalent to a, b.

Example for Tutorial 3 – Normalise Transfer Function

Given a second order system with following transfer function equation:

$$G(\omega) = \frac{10}{3j\omega^2 + 2j\omega + 5}$$

a. Normalise the transfer function equation. [2 marks]
b. Determine the gain at DC, natural frequency ω_n, and damping ratio ζ of the system. [6 marks]

Solution

a. The normalised transfer function equation is:

$$G(\omega) = \left(\frac{10}{5}\right)\left[\frac{\left(\frac{\sqrt{5}}{\sqrt{3}}\right)^2}{(j\omega)^2 + \left[(2)\left(\frac{1}{3\sqrt{3}}\right)(\sqrt{3})\right](j\omega) + \left(\frac{\sqrt{5}}{\sqrt{3}}\right)^2}\right]$$

b. Knowing the normalised transfer function, we can obtain the gain of the system (K) at DC $(\omega = 0)$ that is 10/5, natural frequency (ω_n) is $\sqrt{5/3}$, and damping ratio (ζ) is:

$$2\zeta\omega_n = \frac{2}{3}$$

With $\omega_n = \sqrt{5/3}$, thus $\zeta = 1/3\sqrt{3}$.

Furthermore, the damped transient responses of the second order system can be considered as an exponential decay frequency response as shown in the figure below.

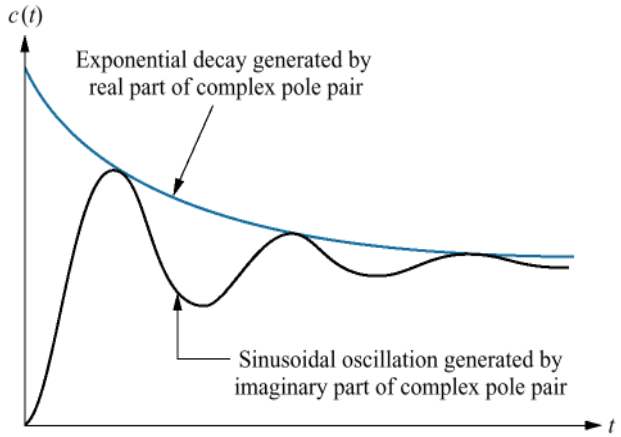


Figure 14: Exponential decay of underdamped second order system response

Considering the transfer function of an underdamped system:

$$\frac{C(\omega)}{R(\omega)} = \frac{b}{(j\omega)^2 + a(j\omega) + b}$$

The real part of root $\sigma + j\omega$ is $\sigma = -a/2$, where $|\sigma|$ is termed the exponential decay frequency.

The damping ratio is defined as measure describing how oscillations in a system decay after a disturbance.

It is equated as the ratio of exponential decay frequency with natural frequency.

$$\zeta = \frac{(a/2)}{b} = \frac{|\sigma|}{\omega_n}$$

Where:

- $|\sigma|$ is the exponential decay frequency
- ω_n is the natural frequency

Given equation for the standardized second-order system with damping ratio and natural frequency:

$$\frac{C(\omega)}{R(\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

Consider the roots of the characteristic equation given above:

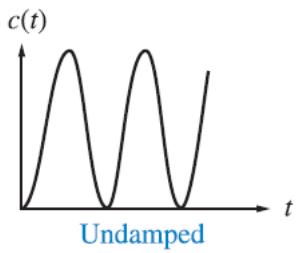
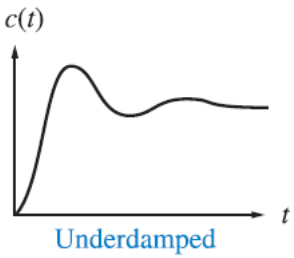
$$\text{roots}_{1,2} = \frac{-2\zeta\omega_n \pm 2\omega_n\sqrt{\zeta^2 - 1}}{2}$$

So, the roots of the equation are:

$$\text{roots}_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = \sigma \pm j\omega$$

Based on their damping ratio, the transient responses of the system are as shown in the figure below. Notice that different damping ratios would contribute to difference response of the system.

- No damping (ζ = 0) would result in a constant amplitude oscillation.
- Lightly damping (0 < ζ < 1) would give a system with initial damped oscillation before settling down to steady-state value.
- Critical damping (ζ = 1) would result in damped oscillation with value of damping ratio is equal to one.
- For heavy damping (ζ > 1), it results in a damped oscillation that settle down to steady-state value.

ζ	Roots	Step Response	Description
0	$\pm j\omega_d$		Constant amplitude oscillation.
$0 < \zeta < 1$	$-\sigma_d \pm j\omega_d$		Initial damped oscillation before settling down to steady state value.

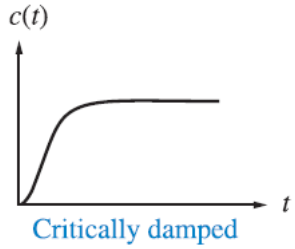
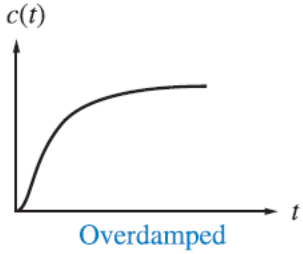
$\zeta = 1$	$-\sigma_d$		Damped oscillation with value of damping ratio is equal to 1.
$\zeta > 1$	$-\sigma_1, -\sigma_2$		Damped oscillation that settles down to steady state value.

Table 2: Damping ratio and step response of second order system

The graph as shown below illustrates several second-order underdamped responses with various damping ratio values.

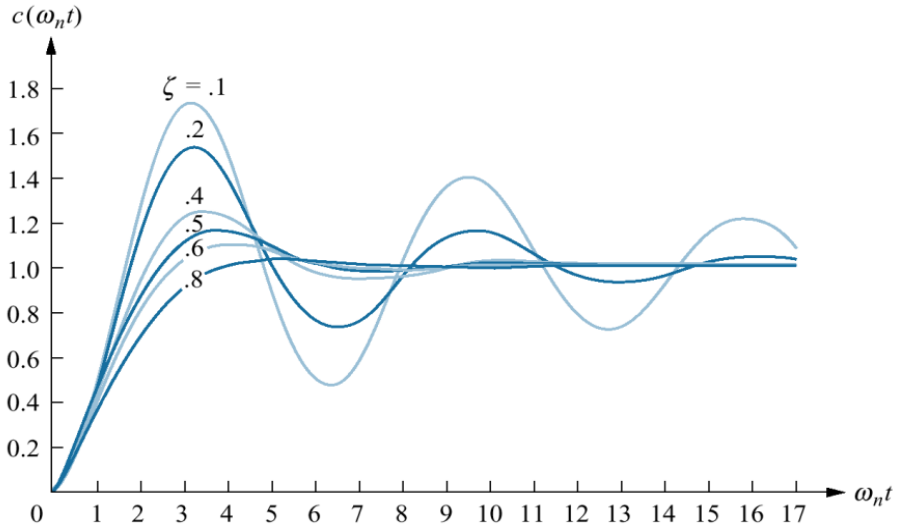


Figure 15: Transient responses of second order systems with various damping ratios

Notice that as the damping ratio is increased, the response of the system becomes less oscillatory in nature.

The system that has a small $\zeta < 0.707$ will typically has an underdamped response.

Whereas the system with ζ = 0.707 is considered to be a system with critically damped response.

Then system with large $\zeta > 0.707$ is categorised as a system with an overdamped response.

4. Second Order Time Response

The graph given below illustrate some of the several second-order underdamped response specifications e.g. rise time (T_r), time-to-peak (T_p), and settling time (T_s).

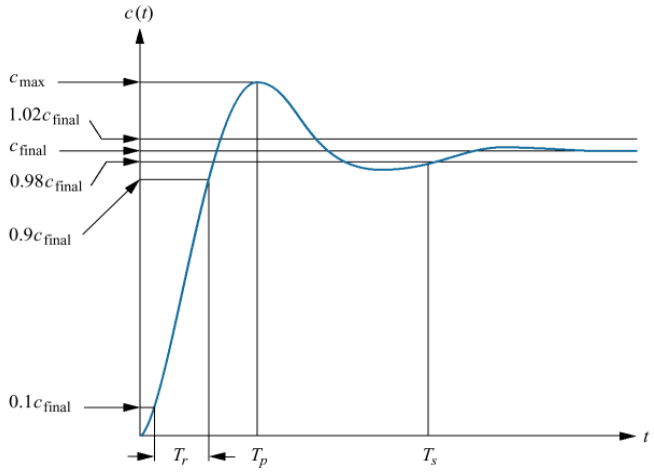


Figure 16: Transient responses of second order system

We cannot use equations for first order system as in most cases these are less accurate for determining the time-domain parameters and characteristics of the second order system.

We still would like to consider the expressions used for determining the rise time and settling time of the first order system for the second order system.

But, we are also interested in parameters which are specific for second order system such as time-to-peak (T_p), percentage overshoot ($\%OS$), and steady-state error ($e(\infty)$).

4.1. Rise Time

Rise time is defined as the time for the response to go from 0.1 to 0.9 of its final value (just like the rise time of the first order response). It is determined from the response of the second order system given in the figure below.

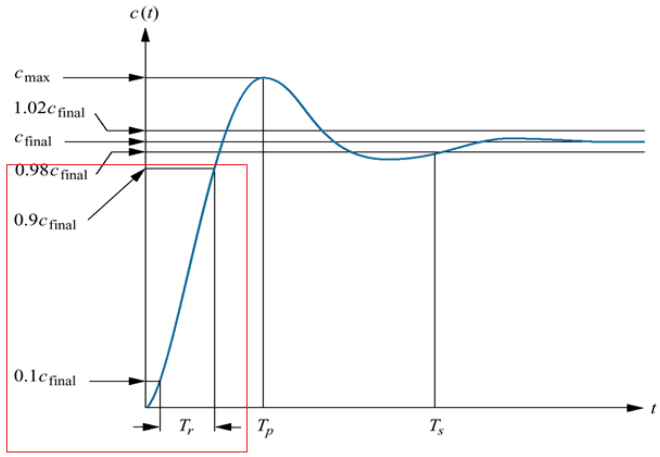


Figure 17: Rise time of second order system

If we use the rise time equation for the first-order system for the second order system, the result is less accurate, especially for underdamped second-order system. So, the rise time (T_r) for a second order (underdamped response) is calculated from.

$$T_r = \frac{(1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)}{\omega_n}$$

The other rise time (T_r) for a second order underdamped response.

$$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}}$$

Where:

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

We can also determine the rise time of the response of the second order system from the graph as illustrated below.

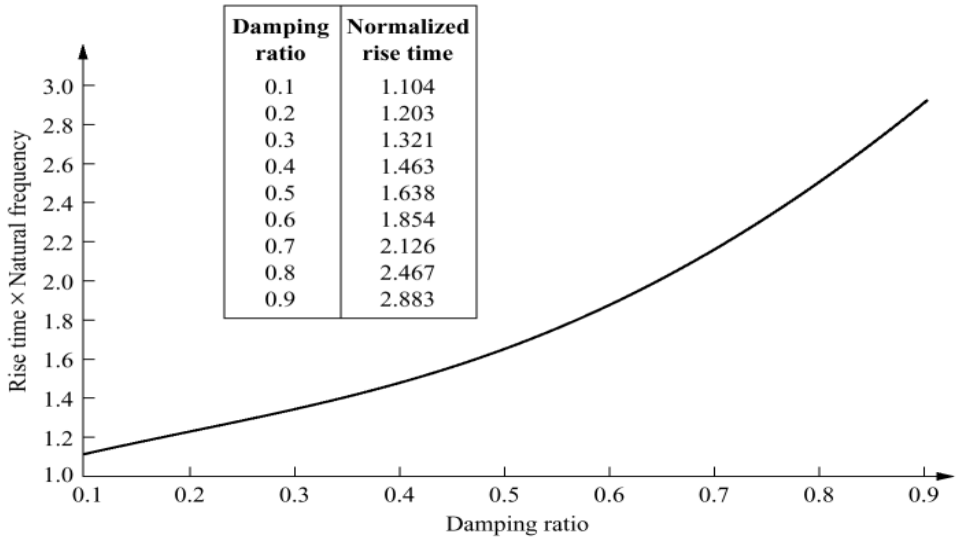


Figure 18: Graph of normalised rise time vs. damping ratio

The graph above is the normalised rise time ($T_r \times \omega_n$) vs. damping ratio (ζ) for a second order underdamped response.

$$t_r\omega_0 = 2.230\zeta^2 - 0.078\zeta + 1.12$$

4.2. Settling Time

Settling time (T_s) is defined as the time for the response to reach, and stay within, 2 % of its final value (e.g. 2% is standard). This parameter and its expression is just like in the first order response.

$$T_s = \ln(1) - \ln(0.02) = \frac{3.9}{a} \text{ or } 3.9\tau$$

The following graph illustrates the settling time in the transient response of the second order system.

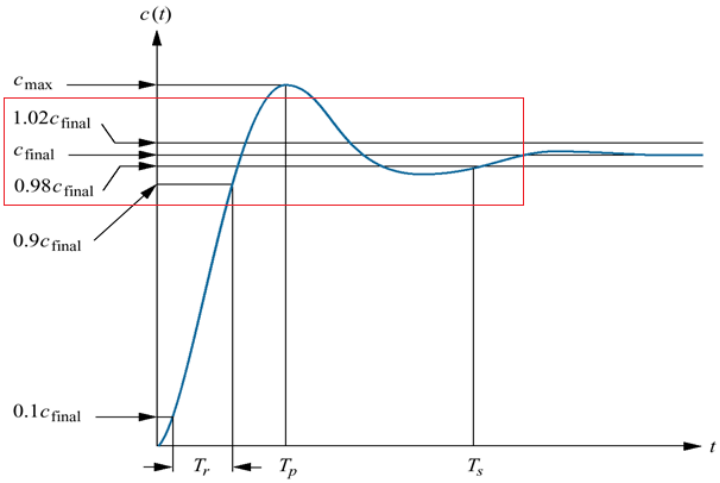


Figure 19: Settling time of second order system

4.3. Time-to-Peak

Time-to-peak (T_p) is the time required to reach the first, or maximum, peak.

$$T_p = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} \qquad \text{where } n = \text{n} - \text{th peak}$$

The time-to-peak of a second order system is as shown in the graph given below.

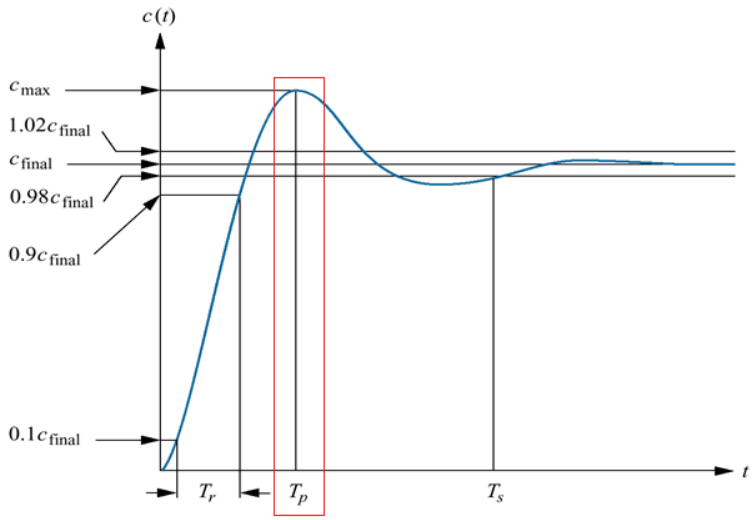


Figure 20: Time-to-peak of second order system

4.4. Percentage Overshoot

Percentage Overshoot ($\%OS$) is the amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100\%$$

The percentage overshoot of the second order system is illustrated in the graph below.

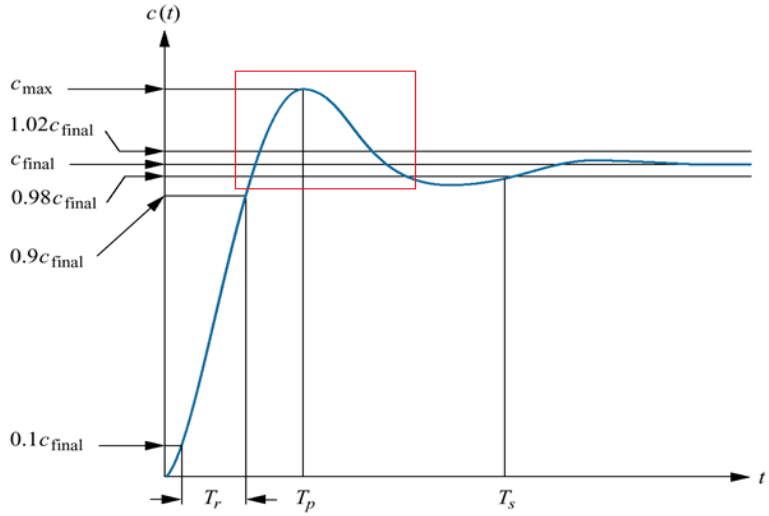


Figure 21: Percentage overshoot of second order system

4.5. Steady State Error

Steady-state error ($e(\infty)$) is the difference between the input and output for a prescribed test input.

$$e(\infty) = \lim_{t\to\infty}[r(t) - c(t)]$$

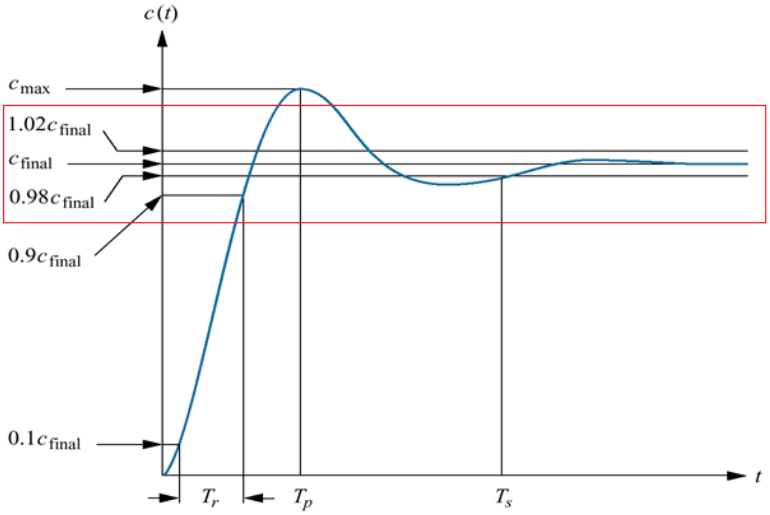


Figure 22: Steady-state error of second order system

Steady-state error of a given a system with feedback is determined from as given below:

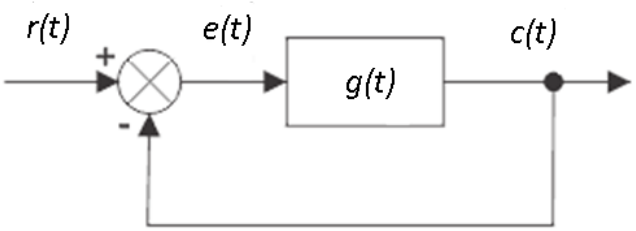


Figure 23: System with feedback

Where:

- $g(t)$ = system
- $r(t)$ = reference signal
- $e(t)$ = error signal
- $c(t)$ = output signal

The equations from the system with feedback determined from:

$$e(t) = r(t) - c(t) \qquad (Eq.1)$$

And

$$c(t) = g(t)e(t) \qquad (Eq.2)$$

Substituting $e(t)$ in equation (2) with equation (1), the steady-state output of the system is:

$$c(t) = g(t)[r(t) - c(t)]$$

Rearrange the equation above for $c(t)$, it is now:

$$c(t) = \frac{g(t)r(t)}{[1 + g(t)]}$$

Applying the final value theorem, when t approach infinity for steady-state condition, i.e. ω is approaching zero:

$$C(\omega) = \lim_{j\omega \to 0} \left[\frac{j\omega R(\omega)G(\omega)}{1 + G(\omega)}\right]$$

Substituting $c(t)$ in equations (1) with (2), the steady-state error of the system is:

$$e(t) = \frac{r(t)}{[1 + g(t)]}$$

Applying the final value theorem, the steady-state error is:

$$E(\omega) = \lim_{j\omega \to 0} \left[\frac{j\omega R(\omega)}{1 + G(\omega)}\right]$$

Example for Tutorial 4 – Second-Order System Response

You are given the following second-order systems:

$$G(\omega) = \frac{16}{(j\omega)^2 + 3j\omega + 16}$$

a. Determine the transient responses of the systems based on the roots of equations. [6 marks]

b. Simulate the system in MATLAB. [4 marks]

c. Find the time response characteristics of the system in terms of damping ratio (ζ) and natural frequency (ω_n). [4 marks]

d. Determine the settling time (T_s), peak time (T_p), rise time (T_r), percentage overshoot ($\%OS$), and steady-state output and error. [12 marks]

Solution

a. For the given system its following transfer function is:

$$G(\omega) = \frac{16}{(j\omega)^2 + 3j\omega + 16}$$

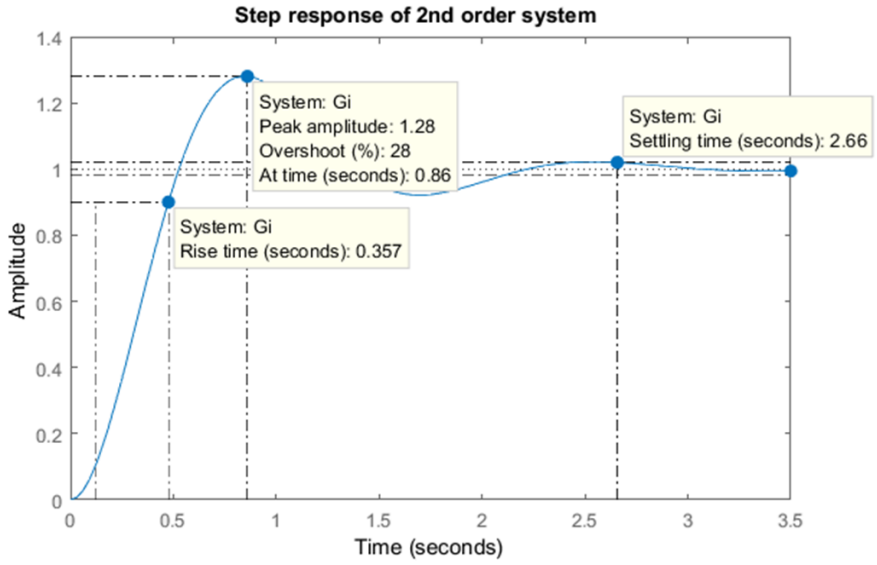
Characteristic equation of the system is $(j\omega)^2 + 3j\omega + 16$.

Roots of the characteristic equation are:

$$\text{root}_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{-3 \pm \sqrt{(3)^2 - 4(1)(16)}}{(2)(1)} = -1.5 \pm j\frac{\sqrt{55}}{2}$$

Since the roots are a pair of complex numbers, the transient response of the system is underdamped response.

b. The result of simulation of the time response of the system in MATLAB is shown in the figure below.



c. The transfer function equation of the system is:

$$G(\omega) = \frac{16}{(j\omega)^2 + 3j\omega + 16}$$

The natural frequency is calculated from:

$$\omega_n^2 = 16 \text{ rad/sec}$$

Thus, $\omega_n = 4$. The damping ratio is calculated from:

$$2\zeta\omega_n = 3$$

Therefore

$$\zeta = \frac{3}{2\omega_n} = \frac{3}{2(4)} = 0.375$$

d. The peak time (first peak, n = 1) is:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{4\sqrt{1-(0.375)^2}} = 0.8472 \text{ second}$$

The percentage overshoot is:

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

$$= e^{-(0.375)\pi/\sqrt{1-(0.375)^2}} = 28.06\ \%$$

The steady-state output of the system is calculated from:

$$C(\omega) = \lim_{\omega\to 0} j\omega R(\omega)G(\omega)$$

For a step input of $R(\omega) = 1/j\omega$, it is:

$$C(\omega) = \lim_{\omega\to 0} R(\omega)G(\omega)$$

$$= \lim_{\omega\to 0} (j\omega)\left(\frac{1}{j\omega}\right)\frac{16}{(j\omega)^2 + 3j\omega + 16} = 1$$

The steady-state error of the system is calculated from:

$$E(\omega) = \lim_{j\omega\to 0}\left[\frac{j\omega R(\omega)}{1 + G(\omega)}\right]$$

For a step input of $1/j\omega$, put the values into the equation, the steady-state error of the system is:

$$E(\omega) = \lim_{j\omega \to 0} \left[\frac{(j\omega)\left(\frac{1}{j\omega}\right)}{1 + \frac{16}{(j\omega)^2 + 3j\omega + 16}} \right] = 0$$