

XMUT220 Signals and Systems

Time & Frequency Characteristics (Frequency Response Analysis)

Topics

1. Introduction to Frequency Response.
2. Frequency Response of Continuous First Order Systems.
3. Frequency Response of Discrete First Order Systems.
4. Frequency Response of Continuous Second Order Systems.
5. Frequency Response of Discrete Second Order Systems.

1. Introduction to Frequency Response

Frequency response is a quantitative measure of the output of a system or device in response to a stimulus, and it is typically used to characterize the dynamics of the system.

It is a measure of magnitude and phase of the output as a function of frequency, in comparison to the input.

- Frequency response plot – both magnitude and phase plots over a range of frequencies.
- Frequency spectral plot – just the magnitude plot over range of frequencies (note: we will look closely its application in sampling and reconstruction).

The following diagram outlines transient response and power spectrum plots of sine wave (left) and square wave (right).

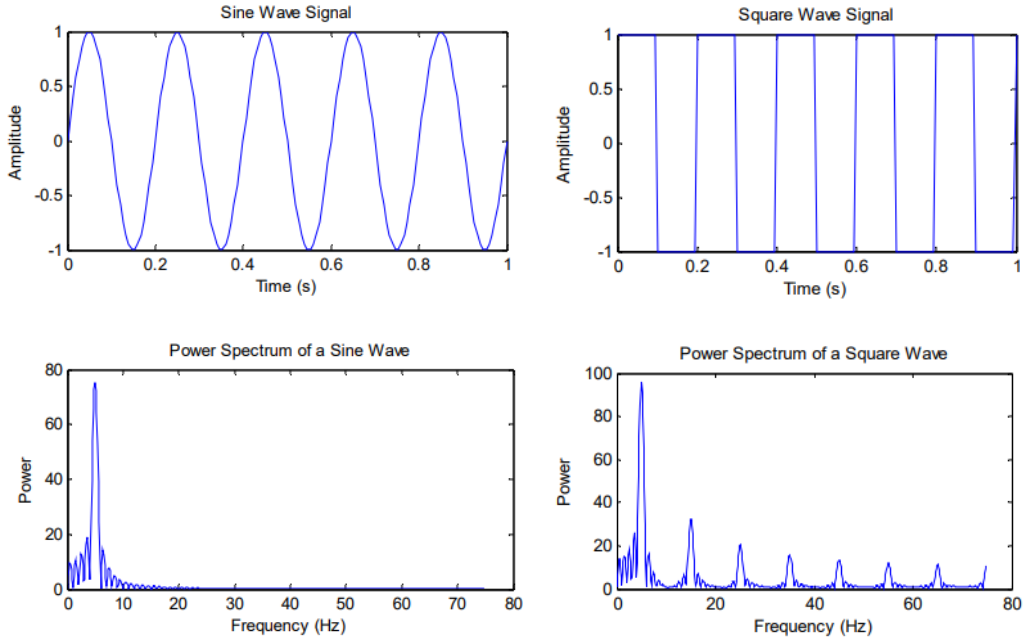


Figure 1: Transient response and power spectrum plots of sine wave (left) and square wave (right).

The frequency response plot is characterized by the magnitude and the phase of the system's response.

Magnitude is typically measured in decibels (dB) or as a decimal, and the phase is measured in radians or degrees, versus frequency in radians/sec or Hertz (Hz).

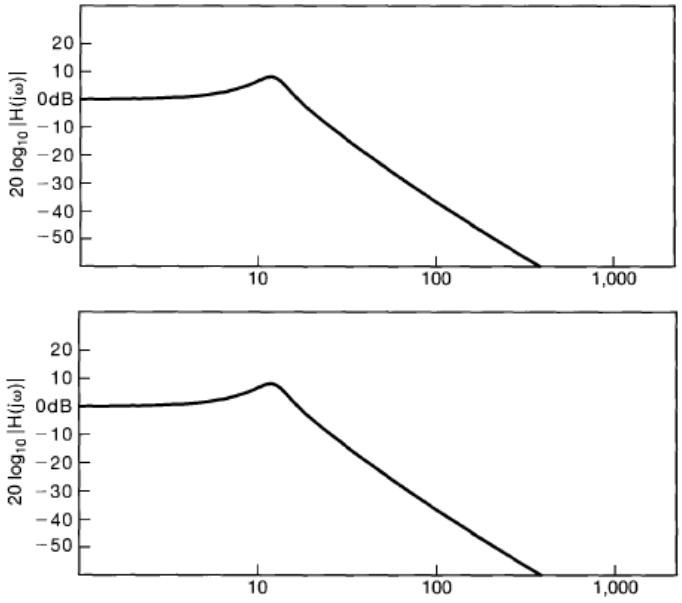


Figure 2: Frequency response of continuous system

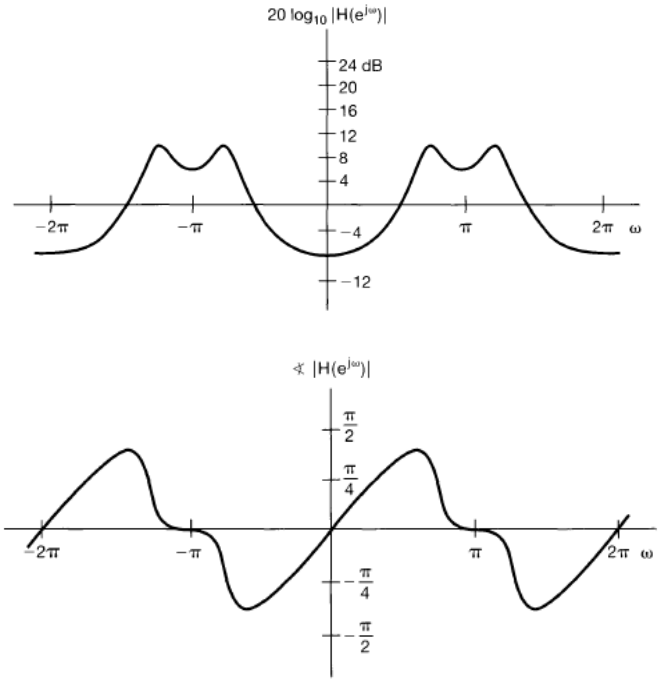


Figure 3: Frequency response of discrete system

The frequency response analysis can be done through three different approaches: Bode plot, Nyquist diagram and Nichols chart.

Bode plot is magnitude and phase measurements on two rectangular plots as functions of frequency.

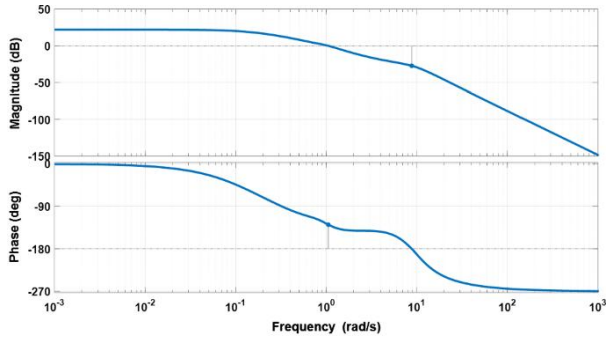


Figure 4: Bode Plot

Nyquist diagram is magnitude and phase angle on a single polar plot with frequency as a parameter.

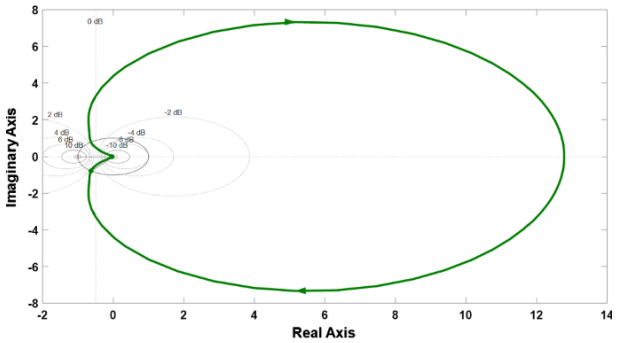


Figure 5: Nyquist Diagram

Nichols chart is magnitude and phase on a single rectangular plot with frequency as a parameter.

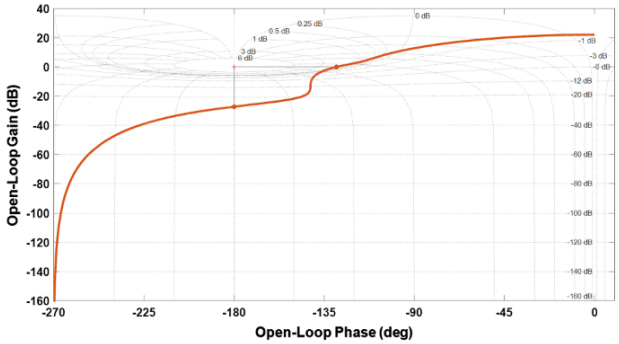


Figure 6: Nichols Chart

2. Frequency Response of Continuous First-Order System

The differential equation for a first-order system is often expressed in the form:

$$\tau\left(\frac{dy(t)}{dt}\right) + y(t) = x(t)$$

Where: t is the time constant of the system and it controls the rate at which the first-order system responds.

The frequency response for the given first-order system is:

$$H(j\omega) = \frac{1}{j\omega\tau + 1}$$

The following diagrams illustrate an impulse response is (top) and a step response (bottom):

$$h(t) = \left(\frac{1}{\tau}\right)e^{\left(-\frac{t}{\tau}\right)}u(t)$$

And

$$s(t) = h(t) * u(t) = \left[1 - e^{-\frac{t}{\tau}}\right]u(t)$$

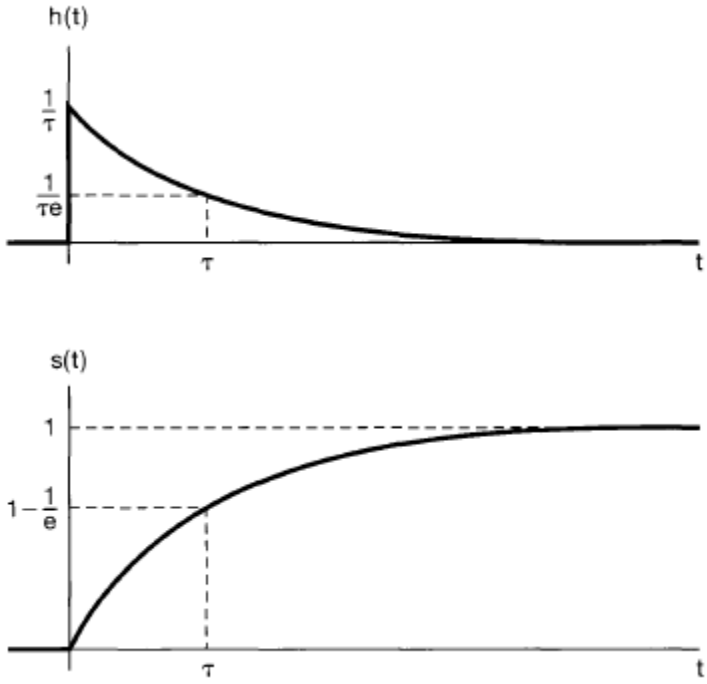


Figure 7: Transient response of continuous first-order system with impulse (top) and step inputs (bottom)

For example, as illustrated in graph, at $t = \tau$ the impulse response has reached $1/e$ times its value at $t = 0$, and the step response is within $1/e$ of its final value.

Therefore, as τ is decreased, the impulse response decays more sharply and the rise time of the step response becomes shorter - i.e. it rises more sharply toward its final value.

Note also that the step response of a first-order system does not exhibit any ringing.

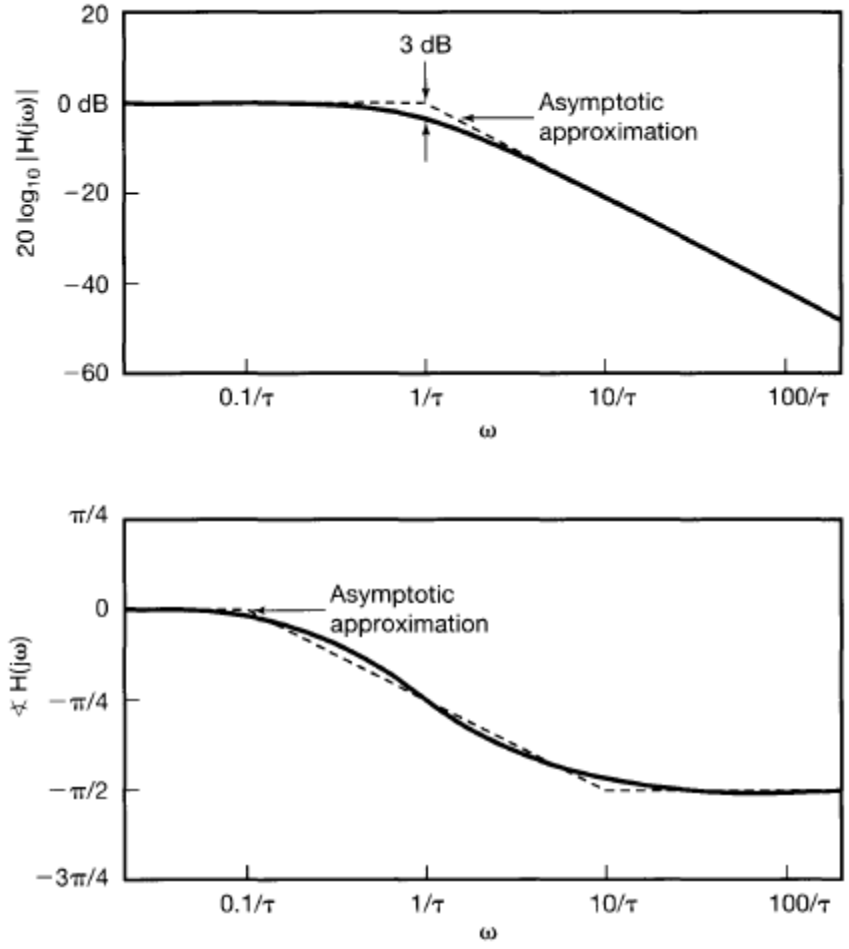


Figure 8: Frequency response of continuous first-order system (magnitude plot (top) and phase plot (bottom))

Figures above depicts the Bode plot of the frequency response of the first order system.

In this figure, the x-axis of the graphs is in logarithmic frequency scale and its unit is radian/second. The y-axis of the magnitude plot is in dB (e.g. $20\log|H(j\omega)|$) and the phase plot is in radian (e.g. $\measuredangle H(j\omega)$).

2.1. Magnitude Response of Continuous First-Order System

To see this, let us first examine the plot of the log magnitude of the frequency response. Specifically, for given first order system, we obtain:

$$20\log|H(j\omega)| = 20\log\left|\frac{1}{\sqrt{(\omega\tau)^2+1}}\right| = -10\log|(\omega\tau)^2+1|$$

From this, we see that for $\omega\tau$ << 1, the log magnitude is approximately zero.

$$20\log|H(j\omega)| \approx 0 \qquad \text{for} \quad \omega \ll 1/\tau$$

While for $\omega\tau$ >> 1, the log magnitude is approximately a linear function of $\log(\omega)$. That is:

$$20\log|H(j\omega)| \approx 20\log(\omega\tau) = -20\log(\omega) - 20\log(\tau) \qquad \text{for} \quad \omega \ll 1/\tau$$

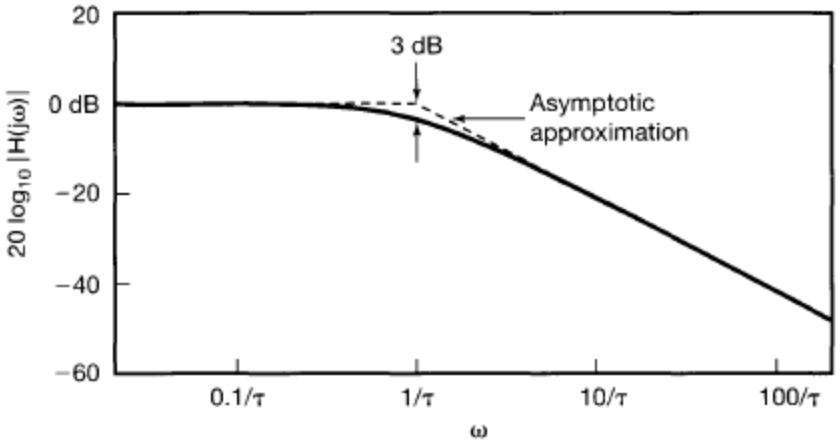
In other words, for the first-order system, the low- and high-frequency asymptotes of the log magnitude are straight lines.

The low-frequency asymptote is just the 0-dB line

While the high-frequency asymptote corresponds to a decrease of 20 dB in $|H(j\omega)|$ for every decade (i.e. factor of 10) in ω.

This is also referred to as a "20-dB-per-decade" asymptote.

Note that the two asymptotic approximations are equal at the point $\log(\omega) = -\log(\tau)$, or equivalently, $\omega = 1/\tau$.



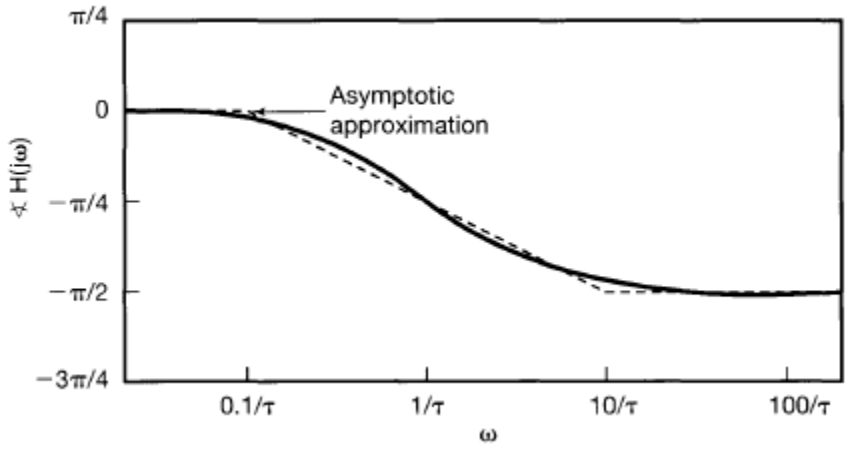


Figure 9: Frequency response of continuous first-order system (e.g. magnitude plot (top) and phase plot (bottom))

Interpreted graphically, this means that the two straight-line asymptotes meet at $\omega = 1/\tau$, which suggests a straight-line approximation to the magnitude plot.

That is, our approximation to $20\log|H(j\omega)|$ equals 0 for $\omega \leq= 1/\tau$ and $20\log(\omega\tau)$ for $\omega \geq= 1/\tau$.

This approximation is also sketched (as a dashed line) in the figure.

2.2. Break Point in Response of Continuous First-Order System

The point at which the slope of the approximation changes is precisely $\omega = 1/\tau$, which, for this reason, is often referred to as the break frequency.

Also, note that at $\omega = 1/\tau$ the two terms $(\omega\tau)^2$ and 1 in the argument of the frequency response equation of the first order system are equal.

Thus, at this point, the actual value of the magnitude is:

$$20\log\left|H\left(j\frac{1}{\tau}\right)\right| = -10\log(2) \cong -3\ \text{dB}$$

Because of this, the point $\omega = 1/\tau$ is sometimes called the 3-dB point.

From the figure, we see that only near the break frequency is there any significant error in the straight-line approximate Bode plot.

Thus, if we wish to obtain a more accurate sketch of the Bode plot, we need only modify the approximation near the break frequency.

2.3. Phase Response of Continuous First-Order System

It is also possible to obtain a useful straight-line approximation to phase of frequency response of the system, $\measuredangle H(j\omega)$:

$$\measuredangle H(j\omega) = -\tan^{-1}(\omega\tau)\begin{cases} 0, & \omega \le 1/\tau \\ -\left(\frac{\pi}{4}\right)[\log(\omega\tau)+1], & 0.1/\tau \le \omega \le 10/\tau \\ -\frac{\pi}{2}, & \omega \gg 10/\tau \end{cases}$$

Note that this approximation is at 0 when $\omega \le 0.1/\tau$ and it is at $-\pi/2$ when $\omega \gg 10/\tau$. Furthermore, it decreases linearly (from 0 to $-\pi/2$) as a function of $\log(\omega)$ in the range:

$$\frac{0.1}{\tau} \le \omega \le \frac{10}{\tau}$$

In this example in the range from one decade below the break frequency to one decade above the break frequency.

Also, the correct asymptotic value of $\measuredangle H(j\omega)$ is 0 for $\omega \le 1/\tau$, and the correct asymptotic value of $\measuredangle H(j\omega)$ is $-\pi/2$ for $\omega \gg 1/\tau$.

Furthermore, the approximation agrees with the actual value of $\measuredangle H(j\omega)$ at the break frequency $= 1/\tau$, at which point:

$$\measuredangle H\left(j\frac{1}{\tau}\right) = -\frac{\pi}{4}$$

This asymptotic approximation is also plotted in the graph, and from it we can see how, if desired, we can modify the straight-line approximation to obtain a more accurate sketch of $\measuredangle H(j\omega)$.

2.4. Frequency Response of Continuous First-Order System (Summary)

From this first-order system, we can again see the inverse relationship between time and frequency.

As we make τ smaller, we speed up the time response of the system [i.e., $h(t)$ becomes more compressed toward the origin, and the rise time of the step response is reduced].

We simultaneously make the break frequency large [i.e., $H(j\omega)$ becomes broader, since $|H(j\omega)| \cong 1$ for a larger range of frequencies].

This can also be seen by multiplying the impulse response by τ and observing the relationship between $\tau h(t)$ and $H(j\omega)$

Thus, $\tau h(t)$ is a function of t/τ and $H(j\omega)$ is a function of $\omega\tau$, and from this we see that changing τ is essentially equivalent to a scaling in time and frequency.

3. Frequency Response of Discrete First-Order System

Consider the first-order causal LTI system described by the difference equation:

$$y[n] - ay[n-1] = x[n]$$

with IaI < 1.

The frequency response of this system is:

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

3.1. Impulse and Step Response of Discrete First-Order System

The impulse response of a first-order system is:

$$h[n] = a^n u[n]$$

This is illustrated in the graph below (top) for several values of *a.*

Also, the step response of the system is:

$$s[n] = h[n] * u[n] = \left[\frac{1 - a^{n+1}}{1 - a}\right] u[n]$$

This is also illustrated in the graph below (bottom).

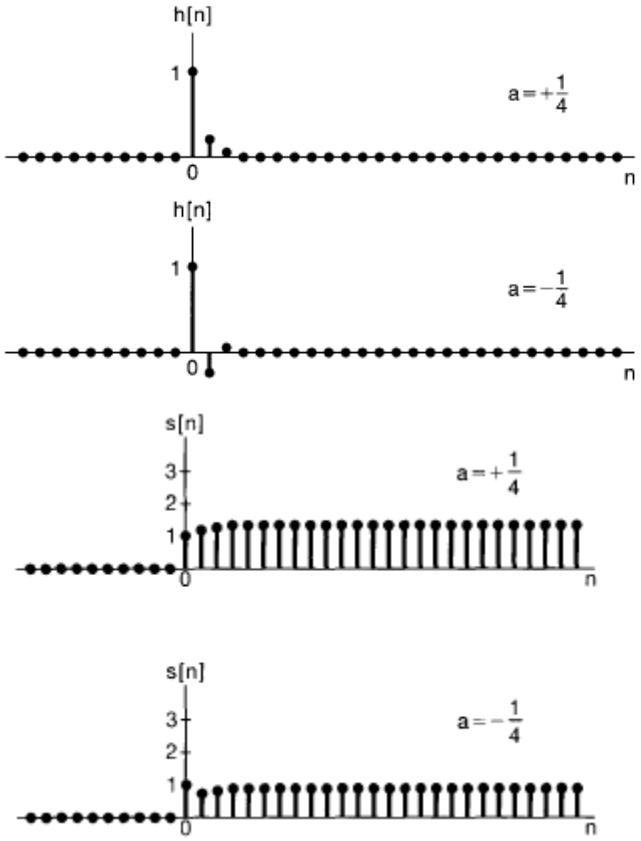


Figure 10: Transient response of discrete first-order system with impulse (top) and step inputs (bottom)

The magnitude of the parameter a plays a role similar to that of the time constant τ in the continuous-time first-order system.

Specifically, $|a|$ determines the rate at which the first-order system responds.

For example, from the equation and graphs of this discrete system, we see that $h[n]$ and $s[n]$ converge to their final value at the rate at which $|a|^n$ converges to zero.

Therefore, the impulse response decays sharply and the step response settles quickly for $|a|$ small.

For $|a|$ nearer to 1, these responses are slower.

Note that unlike its continuous-time counterpart, the discrete first-order system can display oscillatory behavior.

This occurs when a < 0, in which case the step response exhibits both overshoot of its final value and ringing.

3.2. Magnitude and Phase of Frequency Response of Discrete First-Order System

The magnitude and phase of the frequency response of the discrete first-order system are, respectively,

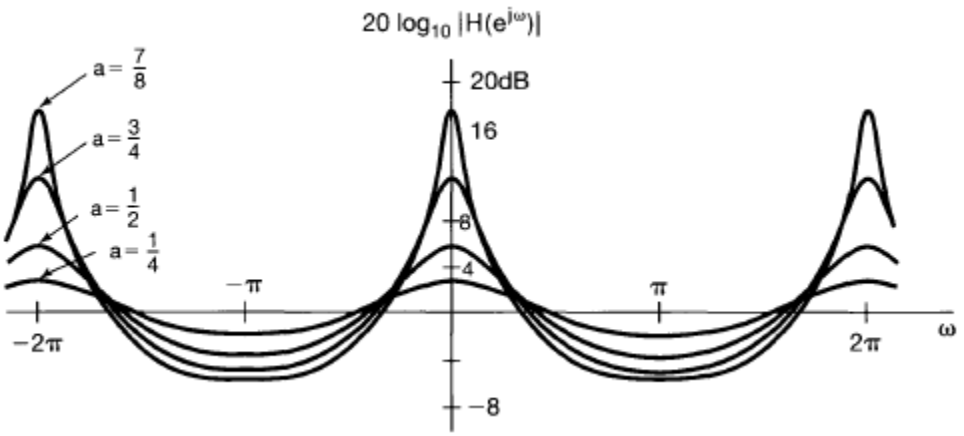
$$\left|H\left(e^{j\omega}\right)\right| = \frac{1}{\sqrt{1 - a^2 - 2\,a\cos\omega}}$$

And

$$\measuredangle H\left(e^{j\omega}\right) = -\tan^{-1}\left[\frac{a\sin\omega}{1 - a\cos\omega}\right]$$

In the given figure (top), we have plotted the log magnitude and the phase of the frequency response of the discrete system for several values of a > 0.

The case of a < 0 is illustrated in the figure (bottom).



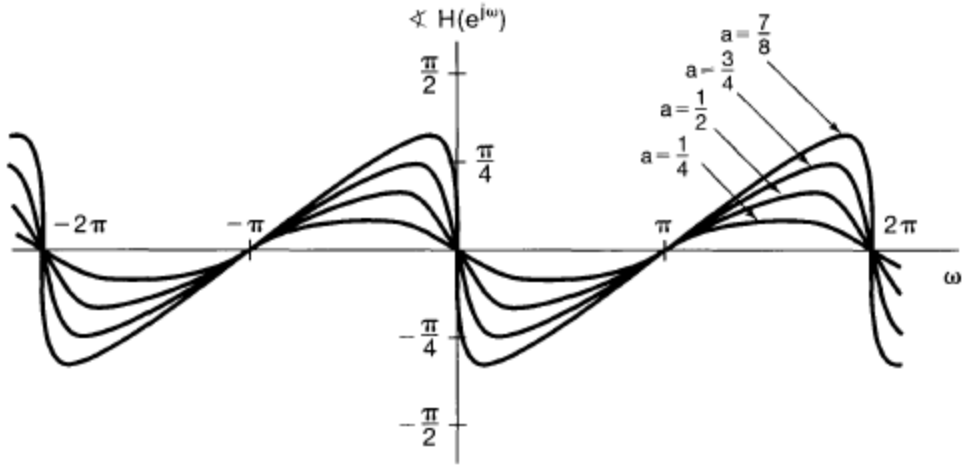
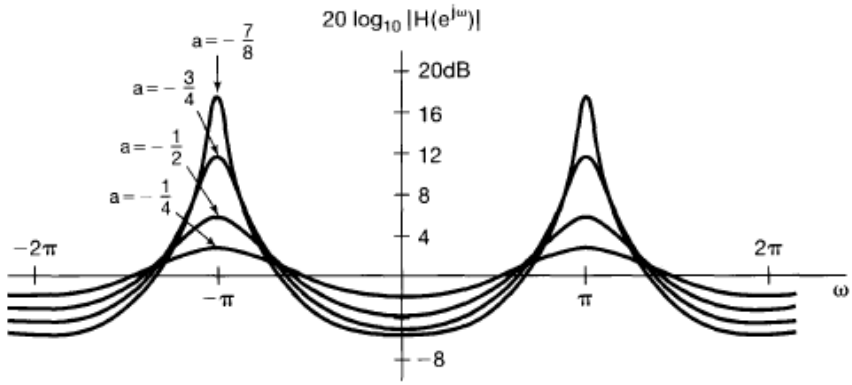


Figure 11: Frequency response of discrete first-order system: magnitude plot (top) and phase plot (bottom) when $a > 0$.

From these figures, we see that for a > 0, the system attenuates high frequencies [i.e., $H(e^{j\omega})$ is smaller for ω near $\pm\pi$ than it is for ω near 0], while when a < 0, the system amplifies high frequencies and attenuates low frequencies.

Note also that for $|a|$ small, the maximum and minimum values, $1/(1+a)$ and $1/(1-a)$, of $H(e^{j\omega})$ are close together in value, and the graph of $H(e^{j\omega})$ is relatively flat.

On the other hand, for $|a|$ near 1, these quantities differ significantly, and consequently $H(e^{j\omega})$ is more sharply peaked, providing filtering and amplification that is more selective over a narrow band of frequencies.



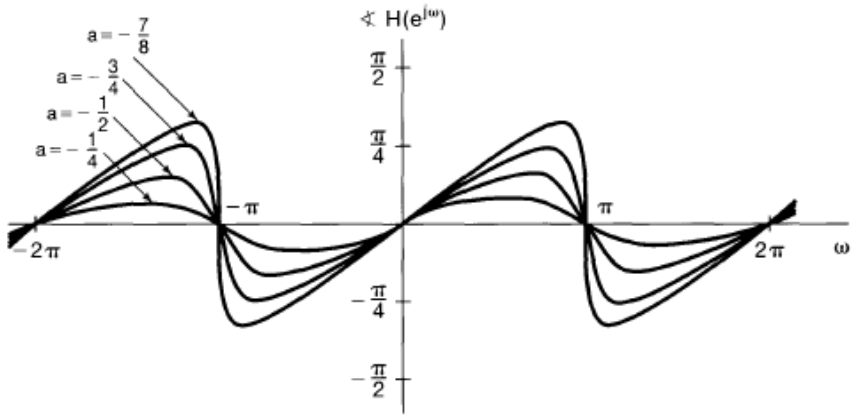


Figure 12: Frequency response of discrete first-order system: magnitude plot (top) and phase plot (bottom) when $a < 0$.

4. Frequency Response of Continuous Second-Order System

The linear constant-coefficient differential equation for a second-order system is:

$$\frac{d^2y(t)}{dt^2} + (2\zeta\omega_n)\frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

Equations of this type arise in many physical systems, including *RLC* circuits and mechanical systems, such as the composed of a spring, a mass, and a viscous damper or dashpot.

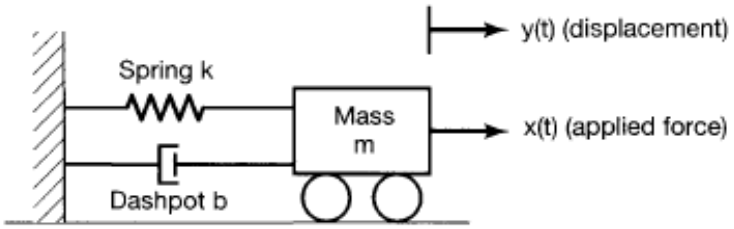


Figure 13: Spring, mass and damper continuous second-order system

In the figure, the input is the applied force $x(t)$ and the output is the displacement of the mass $y(t)$ from some equilibrium position at which the spring exerts no restoring force.

The equation of motion for this system is (e.g. applying the Newton 2nd law, in this case $\sum F = ma$):

$$m\left(\frac{d^2y(t)}{dt^2}\right) = x(t) - (k)y(t) - b\left(\frac{dy(t)}{dt}\right)$$

Or

$$\frac{d^2y(t)}{dt^2} + \left(\frac{b}{m}\right)\frac{dy(t)}{dt} + \left(\frac{k}{m}\right)y(t) = \left(\frac{1}{m}\right)x(t)$$

The frequency response for the second-order system of the standardized second order system equation is:

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

Comparing this to the standardized second-order system equation, we see that if we identify:

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{and} \quad \zeta = \frac{b}{2\sqrt{km}}$$

Then the equation of motion for the system of the given figure reduces to standardized second-order system equation (except for a scale factor of k on $x(t)$).

The denominator of $H(j\omega)$ can be factored to yield:

$$H(j\omega) = \frac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)}$$

Where:

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \quad \text{and} \quad c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

For $\zeta \neq 1$, c_1 and c_2 are unequal, and we can perform a partial-fraction expansion of the form:

$$H(j\omega) = \frac{M}{j\omega - c_1} - \frac{M}{j\omega - c_2}$$

Where: $M = \omega_n/2\sqrt{\zeta^2 - 1}$

4.1. Impulse Response of Continuous Second-Order System

From equation given above, the corresponding impulse response for the system is ($M = \omega_n/2\sqrt{\zeta^2 - 1}$):

$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$$

If $\zeta = 1$, then $c_1 = c_2 = -\omega_n$, and

$$H(j\omega) = \frac{\omega_n^2}{(j\omega + \omega_n)^2}$$

From the table of Fourier transform, we find that in this case the impulse response is:

$$h(t) = \omega_n^2 t e^{-\omega_n t} u(t)$$

Note from impulse response of the system equation and the above equation, that $h(t)/\omega_n$ is a function of $\omega_n t$.

Furthermore, the partial fractioned equation can be rewritten as:

$$H(j\omega) = \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1}$$

From which we see that the frequency response is a function of ω/ω_n.

Thus, changing ω_n is essentially identical to a time and frequency scaling.

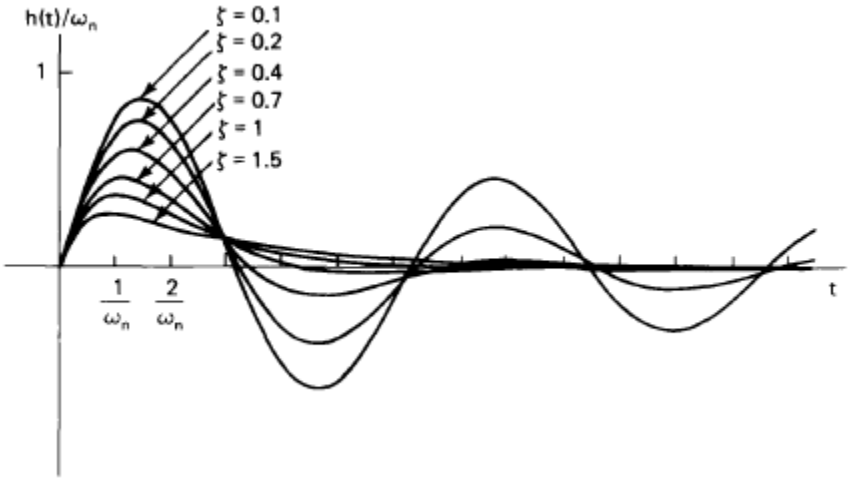


Figure 14: Impulse response of continuous second-order system

4.2. Damping in Frequency Response of Continuous Second-Order System

The parameter ζ is referred to as the damping ratio and the parameter ω_n as the (undamped) natural frequency.

The motivation for this terminology becomes clear when we take a more detailed look at the impulse response and the step response of a second order system.

First, from partial fractioned equation, we see that for $0 < \zeta < 1$, c_1 and c_2 are complex, and we can rewrite the impulse response equation in the form:

$$h(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{2j\sqrt{1-\zeta^2}}\left[e^{j\left(\omega_n\sqrt{1-\zeta^2}\right)t} - e^{-j\left(\omega_n\sqrt{1-\zeta^2}\right)t}\right]u(t)$$

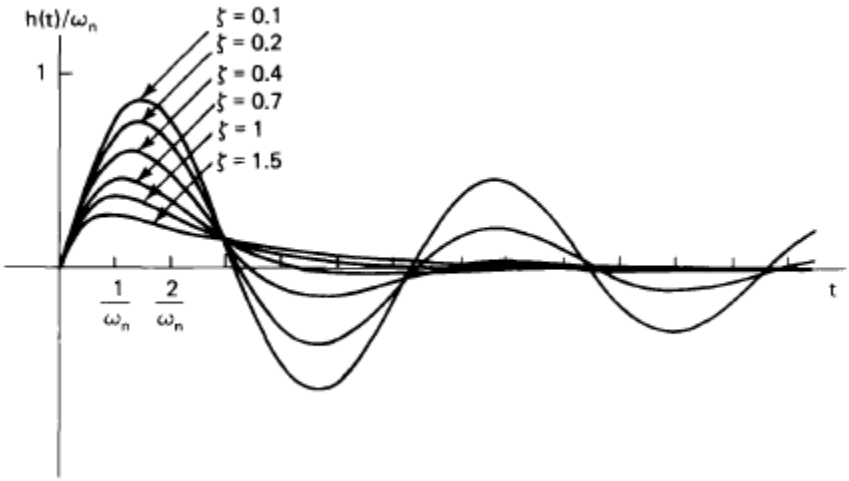
$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\left(\omega_n\sqrt{1-\zeta^2}\right)t\right]u(t)$$

Thus, for $0 < \zeta < 1$, the second-order system has an impulse response that has damped oscillatory behavior, and in this case the system is referred to as being underdamped.

If $\zeta > 1$, both c_1 and c_2 are real and negative, and the impulse response is the difference between two decaying exponentials. In this case, the system is overdamped.

The case of $\zeta = 1$, when $c_1 = c_2$, is called the critically damped case.

The impulse responses (multiplied by $1/\omega_n$) for second-order systems with different values of ζ are plotted in the figure (top).



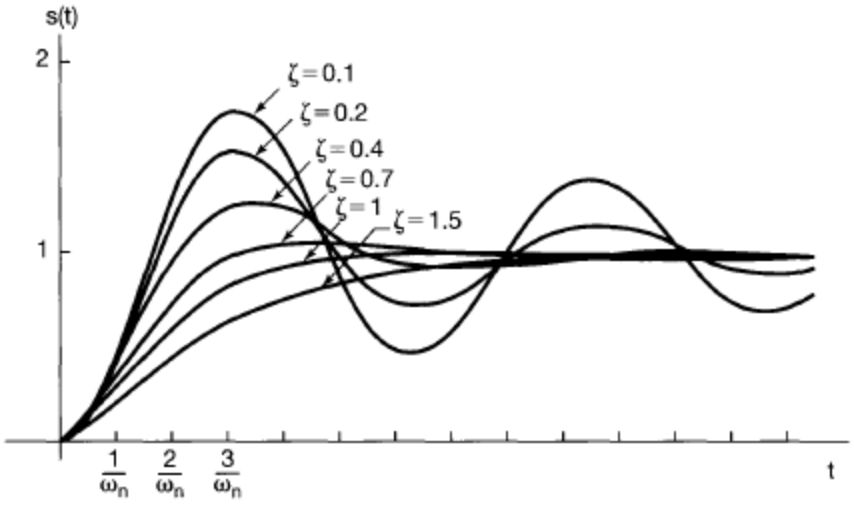
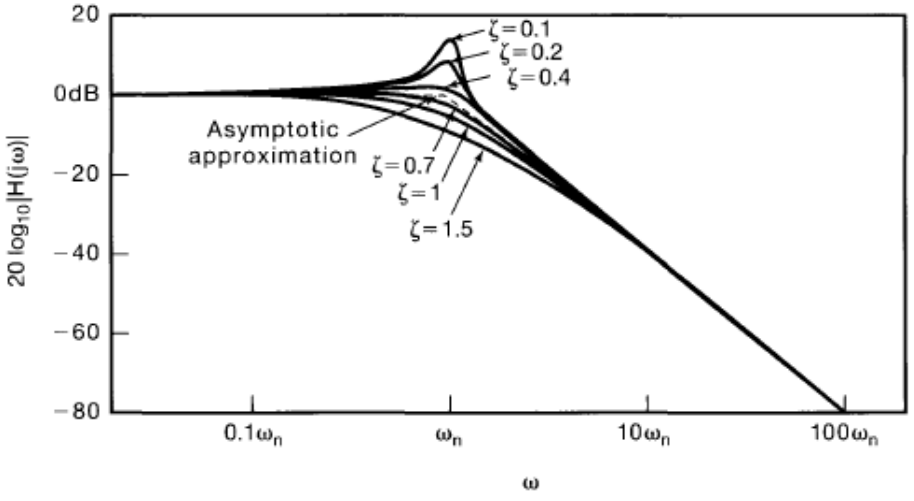


Figure 15: Transient response of continuous second-order system with impulse input (top) and step input (bottom) with various damping ratios (ζ)



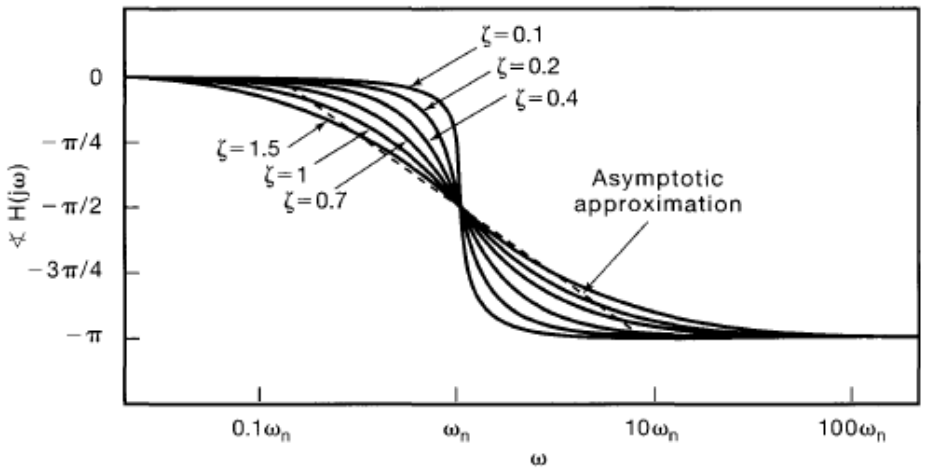


Figure 16: Frequency response of continuous second-order system with various damping ratios (ζ)

4.3. Step Response of Continuous Second-Order System

The step response of a second-order system can be calculated from impulse response of the system equation for $\zeta \neq 1$. This yields the expression ($M = \omega_n/2\sqrt{\zeta^2 - 1}$):

$$s(t) = h(t) * u(t) = \left\{1 + M\left[\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right]\right\}u(t)$$

Where:

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \quad \text{and} \quad c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

For ζ = 1, we can use impulse response equation to obtain:

$$s(t) = [1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}]u(t)$$

The step response of a second-order system is plotted in the given figure for several values of ζ.

From this figure, we see that in the underdamped case, the step response exhibits both overshoot (i.e., the step response exceeds its final value) and ringing (i.e., oscillatory behavior).

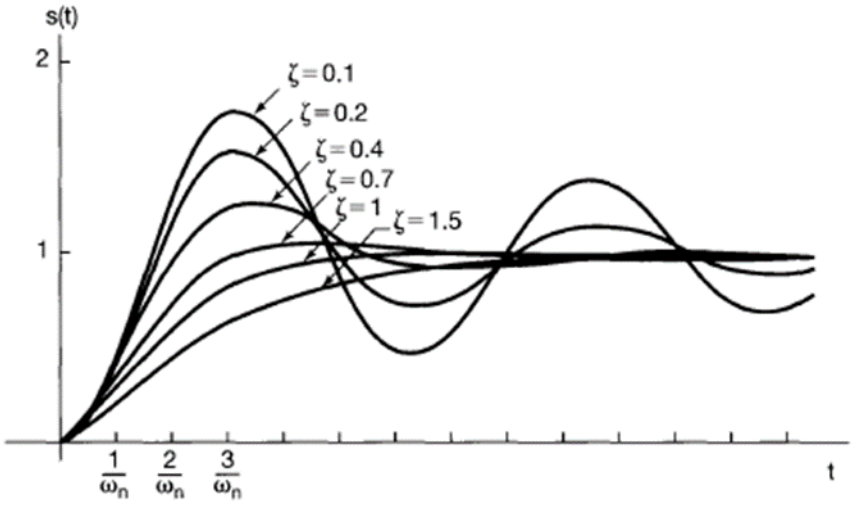


Figure 17: Step response of continuous second-order system

For ζ = 1, the step response has the fastest response (i.e., the shortest rise time) that is possible without overshoot and thus has the shortest settling time.

As ζ increases beyond 1, the response becomes slower. This can be seen from impulse response equations.

As ζ increases, c_1 becomes smaller in magnitude, while c_2 increases in magnitude.

Therefore, although the time constant $(1/|c_2|)$ associated with $e^{c_2 t}$ decreases, the time constant $(1/|c_1|)$ associated with $e^{c_1 t}$ increases.

Consequently, the term involving $e^{c_1 t}$ in step response equation takes a longer time to decay to zero, and thus it is the time constant associated with this term that determines the settling time of the step response.

As a result, the step response takes longer to settle for large values of ζ.

In terms of our spring-dashpot example, as we increase the magnitude of the damping coefficient b beyond the critical value at which ζ in standardized equation of second order system equals 1, the motion of the mass becomes increasingly sluggish.

4.4. Frequency Response of Continuous Second-Order System

Finally, note that, as we have said, the value of ω_n essentially controls the time scale of the responses $h(t)$ and $s(t)$.

For example, in the underdamped case, the larger ω_n is, the more compressed is the impulse response as a function of t, and the higher is the frequency of the oscillations or ringing in both $h(t)$ and $s(t)$.

In fact, from equation for impulse response of second order system, we see that the frequency of the oscillations in $h(t)$ and $s(t)$ is $\omega_n\sqrt{1-\zeta^2}$ which does increase with increasing ω_n.

Note, however, that this frequency depends explicitly on the damping ratio and does not equal (and is in fact smaller than) ω_n, except in the undamped case, ζ = 0.

It is for this reason that the parameter ω_n is traditionally referred to as the undamped natural frequency.

For the spring-dashpot example, we therefore conclude that the rate of oscillation of the mass equals ω_n when no dashpot is present, and the oscillation frequency decreases when we include the dashpot.

4.5. Magnitude Response of Continuous Second-Order System

In the given figure, we have depicted the Bode plot of the frequency response given in the standardized second order system equation for several values of ζ.

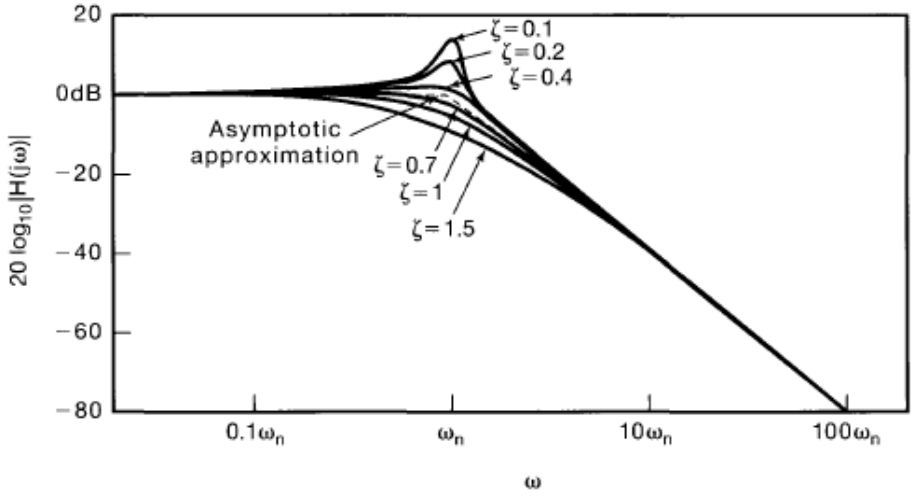


Figure 18: Magnitude plot of the frequency response of continuous second-order system

As in the first-order case, the logarithmic frequency scale leads to linear high- and low-frequency asymptotes for the log magnitude.

Specifically, from the standardized second order system equation, the magnitude of the frequency response of the continuous second order system is:

$$20\log|H(j\omega)| = 20\log\frac{1}{\sqrt{\left[1+\left(\frac{j\omega}{\omega_n}\right)^2\right]^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}}$$

Rearranging the equation given above:

$$20\log|H(j\omega)| = -10\log\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\}$$

From this expression, it follows that:

$$20\log|H(j\omega)| \cong \begin{cases}0, & \text{for } \omega \ll \omega_n \\ -40\log\omega + 40\log\omega_n, & \text{for } \omega \gg \omega_n\end{cases}$$

Therefore, the low-frequency asymptote of the log magnitude is the 0-dB line, while the high-frequency asymptote has a slope of -40 dB per decade i.e. $H(j\omega)$ decreases by 40 dB for every increase in ω of a factor of 10.

Also, note that the two straight-line asymptotes meet at the point $\omega = \omega_n$.

Thus, we obtain a straight-line approximation to the log magnitude by using the approximation given in equation above for $\omega \leq \omega_n$.

For this reason, ω_n is referred to as the break frequency of the second-order system.

This approximation is also plotted (as a dashed line) in given figure.

4.6. Phase Response of Continuous Second-Order System

We can, in addition, obtain a straight-line approximation to $\measuredangle H(j\omega)$, whose exact expression can be obtained from the standardized second order system equation:

$$\measuredangle H(j\omega) = \frac{\frac{2\zeta\omega}{\omega_n}}{1 + \left(\frac{j\omega}{\omega_n}\right)^2}$$

Rearranging the equation given above:

$$\measuredangle H(j\omega) = -\tan^{-1}\left(\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right)$$

The approximation is

$$\measuredangle H(j\omega)\begin{cases} 0, & \omega \le 0.1\omega_n \\ -\dfrac{\pi}{2}[\log(\omega/\omega_n)+1], & 0.1\omega_n \le \omega \le 10\omega_n \\ -\pi, & \omega \ge 10\omega_n \end{cases}$$

which is also plotted in the given figure.

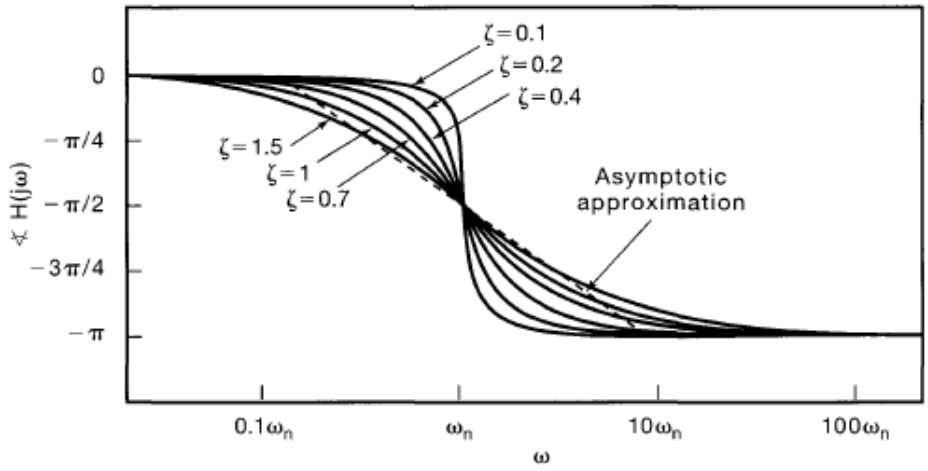


Figure 19: Phase plot of the frequency response of continuous second-order system

Note that the approximation and the actual value again are equal at the break frequency $\omega = 0.1\omega_n$, where $\measuredangle H(j\omega_n) = -\pi/2$

It is important to observe that the asymptotic approximations, approximation of magnitude and phase equations, we have obtained for a second-order system do not depend on ζ.

While the actual plots of $|H(j\omega)|$ and $\sphericalangle H(j\omega)$ certainly do, and thus, to obtain an accurate sketch, especially near the break frequency $\omega = \omega_n$, we must take this into account by modifying the approximations to conform more closely to the actual plots.

4.7. Peak in Frequency Response of Continuous Second-Order System

In particular, note that in this case the actual log magnitude has a peak around $\omega = \omega_n$.

The discrepancy is most pronounced for small values of ζ.

In fact, straightforward calculations using approximation of magnitude equation show that, for $\zeta < \sqrt{2}/2 \cong 0.707$, $|H(j\omega)|$ has a maximum value at:

$$\omega_{max} = \omega_n\sqrt{1-\zeta^2}$$

And the value at this maximum point is:

$$|H(j\omega_{max})| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

For ζ > 0.707, however, $H(j\omega)$ decreases monotonically as ω increases from zero.

The fact that $H(j\omega)$ can have a peak is extremely important in the design of frequency-selective filters and amplifiers.

In some applications, one may want to design such a circuit so that it has a sharp peak in the magnitude of its frequency response at some specified frequency

Thereby providing large frequency-selective amplification for sinusoids at frequencies within a narrow band.

The quality Q of such a circuit is defined to be a measure of the sharpness of the peak.

For a second-order circuit described by an equation of the form of transfer function of the given system equation, the quality is usually taken to be:

$$Q = \frac{1}{2\zeta}$$

From the given figure and the above equation, we see that this definition has the proper behavior e.g. the less damping there is in the system, the sharper is the peak in $|H(j\omega)|$.

5. Frequency Response of Discrete Second-Order System

For a second-order causal LTI system described by:

$$y[n] - 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n]$$

with $0 < r < 1$ and $0 \leq \theta \leq \pi$.

The frequency response for this system is:

$$H\left(e^{j\omega}\right) = \frac{1}{1 - 2r\cos\theta e^{-j\omega} + r^2 e^{-j2\omega}}$$

The denominator of $H\left(e^{j\omega}\right)$ can be factored to obtain:

$$H\left(e^{j\omega}\right) = \frac{1}{[1 - (re^{j\theta})e^{-j\omega}][1 - (re^{-j\theta})e^{-j\omega}]}$$

For $\theta \neq 0$ or π, the two factors in the denominator of $H\left(e^{j\omega}\right)$ are different.

By applying a partial-fraction expansion, this yields:

$$H\left(e^{j\omega}\right) = \frac{A}{1 - (re^{j\theta})e^{-j\omega}} + \frac{B}{1 - (re^{-j\theta})e^{-j\omega}}$$

Where:

$$A = \frac{e^{j\theta}}{2j\sin\theta}, \qquad B = \frac{e^{-j\theta}}{2j\sin\theta}$$

5.1. Impulse Response of Discrete Second-Order System

In this case, the impulse response of the system is

$$h[n] = \left[A\left(re^{j\theta}\right)^n + B\left(re^{-j\theta}\right)^n\right]u[n]$$

Where:

$$A = \frac{e^{j\theta}}{2j\sin\theta}, \qquad B = \frac{e^{-j\theta}}{2j\sin\theta}$$

Knowing that Euler law of $\sin\theta = \left(e^{j\theta} - e^{-j\theta}\right)/2j$:

$$h[n] = r^n\left\{\frac{\sin[(n+1)\theta]}{\sin\theta}\right\}u[n]$$

For $\theta = 0$ or π, the two factors in the denominator of transfer function equation of the system are the same.

When $\theta = 0$,

$$H(e^{j\omega}) = \frac{1}{(1 - re^{-j\omega})^2}$$

And

$$h[n] = (n+1)r^n u[n]$$

When $\theta = \pi$,

$$H(e^{j\omega}) = \frac{1}{(1 + re^{-j\omega})^2}$$

And

$$h[n] = (n+1)(-r)^n u[n]$$

The impulse responses for second-order systems are plotted in the given figure for a range of values of r and θ.

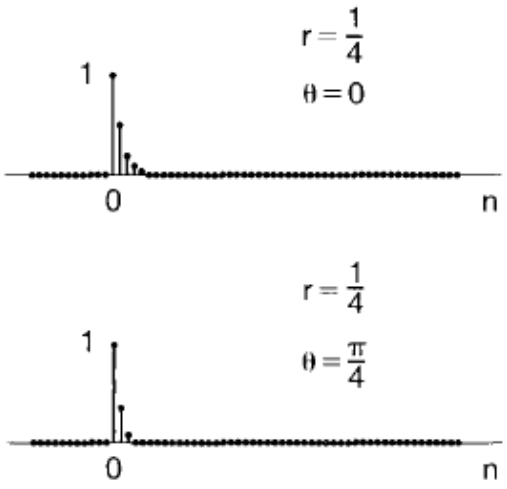


Figure 20: Transient response of discrete second-order system with impulse input. Note: each pair of plots for r = 1/4 and θ = π/4.

From this figure and from impulse response of the system equation, we see that the rate of decay of $h[n]$ is controlled by r i.e., the closer r is to 1, the slower is the decay in $h[n]$.

Similarly, the value of θ determines the frequency of oscillation.

For example, with $\theta = 0$ there is no oscillation in $h[n]$, while for $\theta = \pi$ the oscillations are rapid.

The effect of different values of r and θ can also be seen by examining the step response of transfer function equation of the system.

5.2. Step Response of Discrete Second-Order System

For $\theta \neq 0$ or π,

$$s[n] = h[n] * u[n] = \left[\left(\frac{1-\left(re^{j\theta}\right)^{n+1}}{1-re^{j\theta}}\right)+\left(\frac{1-\left(re^{-j\theta}\right)^{n+1}}{1-re^{-j\theta}}\right)\right]u[n]$$

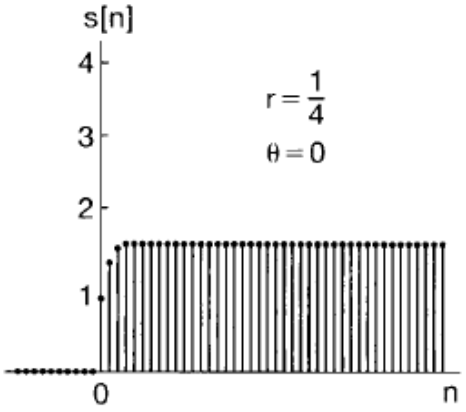
Also, we find that for $\theta = 0$,

$$s[n] = \left[\frac{1}{(r-1)^2}-\frac{r}{(r-1)^2}(r)^n+\frac{r}{r-1}(n+1)(r)^n\right]u[n]$$

While for $\theta = \pi$,

$$s[n] = \left[\frac{1}{(r+1)^2}+\frac{r}{(r+1)^2}(-r)^n+\frac{r}{r+1}(n+1)(-r)^n\right]u[n]$$

The step response is plotted in the figure, again for a range of values of r and θ.



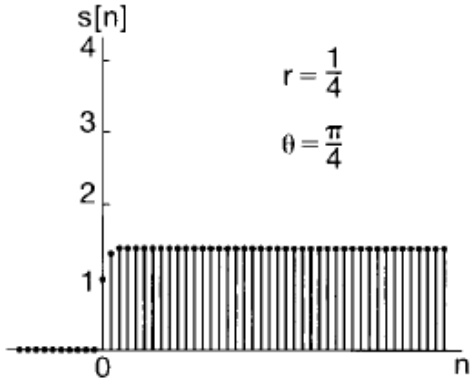


Figure 21: Transient response of discrete second-order system with step input. Note: each pair of plots for r = 1/4 and θ = π/4.

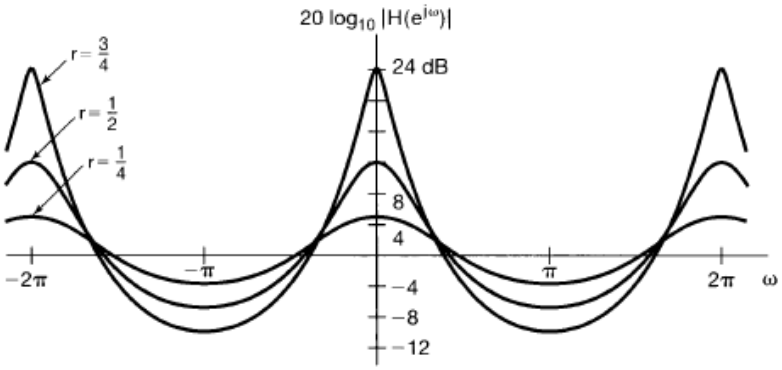
5.3. Frequency Response of Discrete Second-Order System

The transfer function of the discrete second-order system given is the counterpart of the *underdamped* second-order system in continuous time, while the special case of $\theta = 0$ is the critically damped case.

That is, for any value of θ other than zero, the impulse response has a damped oscillatory behavior, and the step response exhibits ringing and overshoot.

The frequency response of this system is depicted in the figure for a number of values of r and θ.

From the figure, we see that a band of frequencies is amplified, and r determines how sharply peaked the frequency response is within this band.



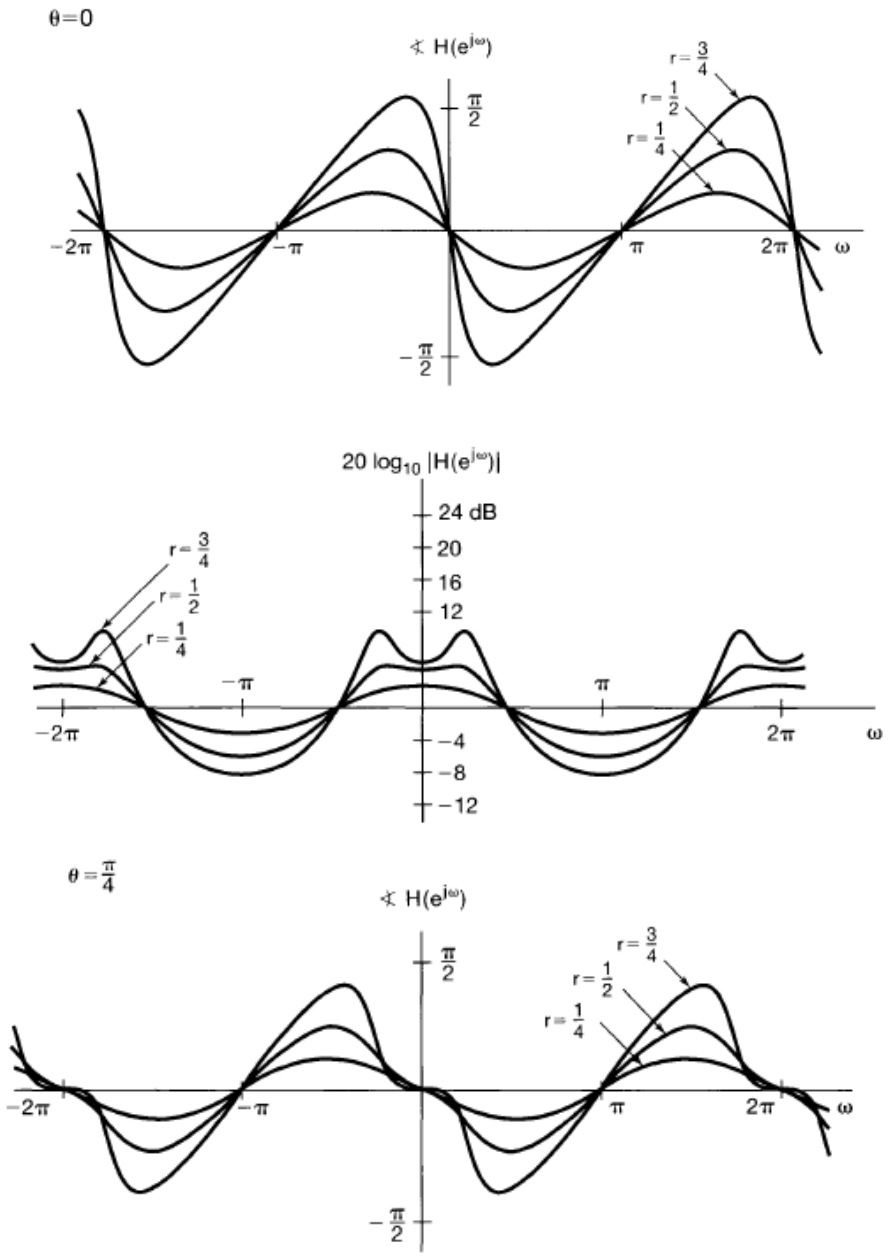


Figure 22: Frequency response of discrete second-order system. Note: θ = 0 (left) and θ = π/4 (right)

As we have just seen, the second-order system described in frequency response equation of the system has factors with complex coefficients (unless $\theta = 0$ or π).

It is also possible to consider second-order systems having factors with real coefficients. Specifically, consider:

$$H(e^{j\omega}) = \frac{1}{(1 - d_1 e^{-j\omega})(1 - d_2 e^{-j\omega})}$$

Where d_1 and d_2 are both real numbers with $|d_1|$, $|d_2| < 1$.

Equation above is the frequency response for the difference equation:

$$y[n] - (d_1 + d_2)y[n-1] + d_1 d_2 y[n-2] = x[n]$$

In this case,

$$H(e^{j\omega}) = \frac{A}{1 - d_1 e^{-j\omega}} + \frac{B}{1 - d_2 e^{-j\omega}}$$

Where:

$$A = \frac{d_1}{d_1 - d_2}, \quad B = \frac{d_2}{d_2 - d_1}$$

Thus,

$$h[n] = [A d_1^n + B d_2^n]u[n]$$

Which is the sum of two decaying real exponentials. Also,

$$s[n] = \left[A\left(\frac{1 - d_1^{n+1}}{1 - d_1}\right) + B\left(\frac{1 - d_2^{n+1}}{1 - d_2}\right)\right]u[n]$$

The system with frequency response given by the frequency response equation of the discrete system corresponds to the cascade of two first-order systems. Therefore, we can deduce most of its properties from our understanding of the first-order case.

For example, the log-magnitude and phase plots for the frequency response equation of the discrete system can be obtained by adding together the plots for each of the two first-order terms.

Also, as we saw for first-order systems, the response of the system is fast if $|d_1|$ and $|d_2|$ are small, but the system has a long settling time if either of these magnitudes is near 1.

Furthermore, if d_1 and d_2 are negative, the response is oscillatory.

The case when both d_1 and d_2 are positive is the counterpart of the overdamped case in continuous time, with the impulse and step responses settling without oscillation.

5.4. Stability of Discrete Systems

In the discrete systems, we have restricted attention to those causal first- and second-order systems that are stable and for which the frequency response can be defined.

In particular, the causal system described by equation of the transfer function of the first order discrete-time system is unstable for $|a| \geq 1$.

$$h[n] = a^n u[n]$$

From the frequency response of the discrete second order system, it is unstable if $r \geq 1$.

$$y[n] - 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n]$$

For the discrete second order system, it is unstable if either $|d_1|$ or $|d_2|$ exceeds 1.

$$y[n] - (d_1 + d_2)y[n-1] + d_1 d_2 y[n-2] = x[n]$$