XMUT220 Signals and Systems

Sampling and Reconstruction (Derivation of Equations)

Topics

1. Sampling of signals.
2. Derivation of equation for sampling of signals.
3. Reconstruction of signals.
4. Derivation of equation for reconstruction of signals.
5. Down sampling and up sampling.
6. Derivation of equation for down sampling and up sampling.

1. Sampling of Signals

In signals and systems, sampling is the conversion of a continuous-time signal to a discrete-time signal. A common example of sampling is the conversion of a sound wave to a sequence of samples.

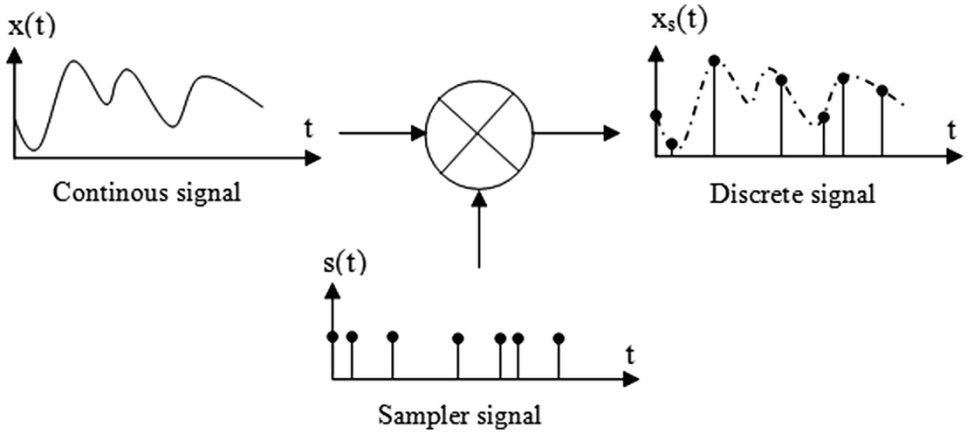


Figure 1: Signal sampling process

As illustrated in the figure above, a continuous signal ($x(t)$) is sampled by a sampler signal ($s(t)$) that results in a discrete signal ($x_p(t)$).

A sample is a value or set of values at a point in time and/or space. A sampler is a subsystem or operation that extracts samples from a continuous signal. A theoretical ideal sampler produces samples equivalent to the instantaneous value of the continuous signal at the desired points.

A continuous signal $x(t)$ can be sampled by an Analogue-to-Digital Converter (A/D) to become a discrete signal composed of a set of time samples.

1.1. Sampling of Signal

The equation that represents sampling of signals:

$$x|m| = x(mt_0)$$

Or

$$x|m| = x(m/F) \qquad (Eq. 1)$$

Where:

- $m = \cdots, -1,\ 0,\ 1, \ldots$
- t_0 is the sampling period, the time interval between two consecutive samples.
- $F = 1/t_0$ is the sampling frequency.

The sampled signal $x_s(t)$ can be represented as the product of the time signal ($x(t)$) and an impulse train ($p(t)$).

$$x_s(t) = x(t) \cdot p(t)$$

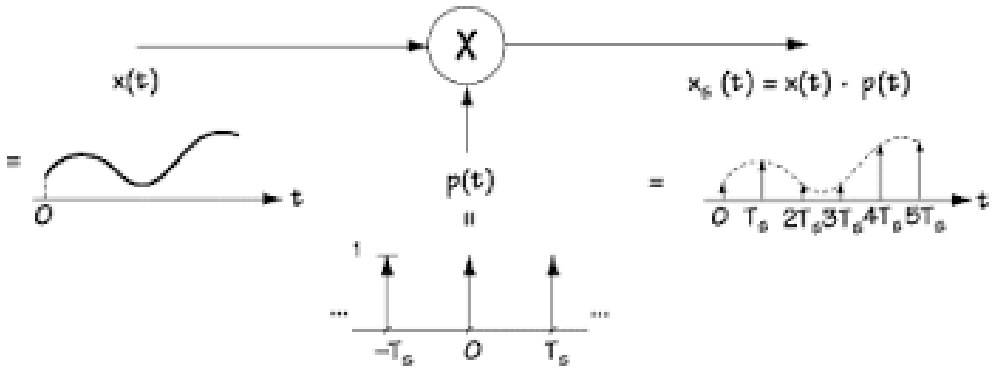


Figure 2: Components of a sampling system

The diagram below shows a sampling by a comb impulse function. In the diagram, the function $f(t)$ is multiplied by the sampling function $s(t;\Delta T)$ to produce the sampled waveform:

$$f^*(t) = f(t).s(t;\Delta T)$$

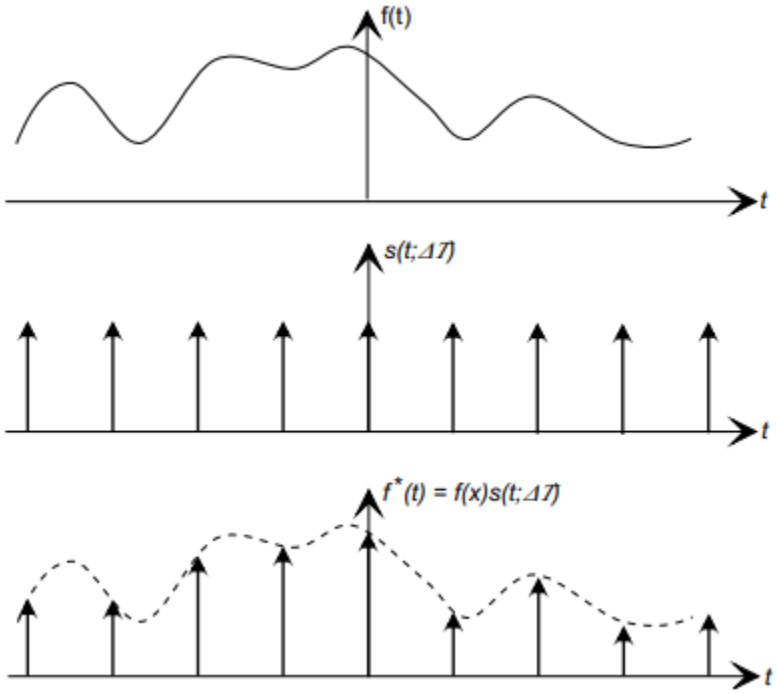


Figure 3: Sampling of signal with comb function

The sampling function (also called a comb function):

$$x_s(t) = x(t) \cdot \mathrm{comb}(t) \qquad (Eq.2)$$

So, substitute the comb function above with equation below to the sampling equation

$$\mathrm{comb} = \sum_{m=-\infty}^{\infty} \delta(t - mt_0)$$

By applying the comb function to our sampling process, this gives the following equation for sampling of signals:

$$x_s(t) = x(t) \cdot \sum_{m=-\infty}^{\infty} \delta(t - mt_0)$$

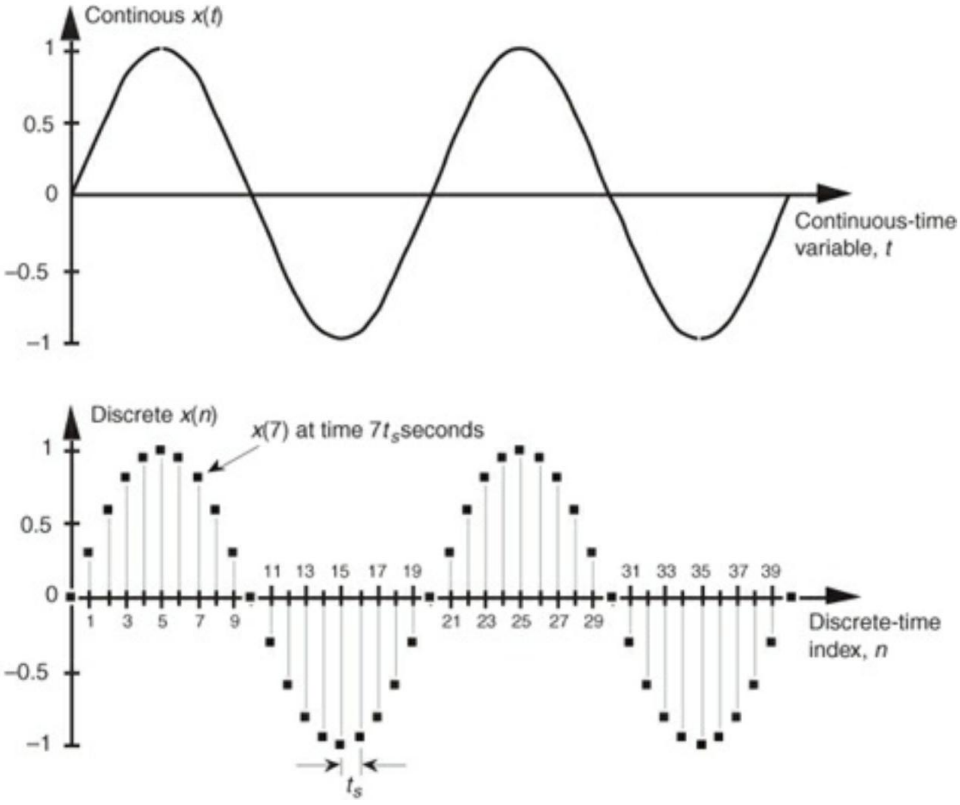


Figure 4: Sampling of signal with infinite discrete-time samples

Since the original signal can be considered to be consisting of infinite discrete-time samples.

$$x(t) = \sum_{m=-\infty}^{\infty} x[m]$$

Thus, the equation for sampled signal becomes:

$$x_s(t) = \sum_{m=-\infty}^{\infty} x[m] \cdot \delta(t - mt_0) \qquad (Eq.\ 3)$$

1.2. Frequency Spectrum of Sampling of Signal

The Fourier transform of the signal $x_s(t)$ to be sampled can be found as:

$$X_s(j\omega) = \int_{-\infty}^{\infty} x_s(t) e^{-j\omega t} dt \qquad (Eq. 4)$$

Substituting equation (3) into equation (4) yields:

$$X_s(j\omega) = \int_{-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} x[m] \cdot \delta(t - mt_0) \right] e^{-j\omega t} dt$$

Thus, rearranging the equation, the Fourier transform of the sampled signal becomes:

$$X_s(j\omega) = \sum_{m=-\infty}^{\infty} x[m] \int_{-\infty}^{\infty} \delta(t - mt_0) \cdot e^{-j\omega t} dt$$

Then, equating the integral part of the equation:

$$X_s(j\omega) = \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-j2\pi f m t_0}$$

Therefore, we can prove from the derivation process before that:

$$X_s(j\omega) = X_s(f)$$

As illustrated in the diagram below, time domain and frequency domain plots of (a) sampled signals, (b) sampling signal and (c) discrete-time sampling samples.

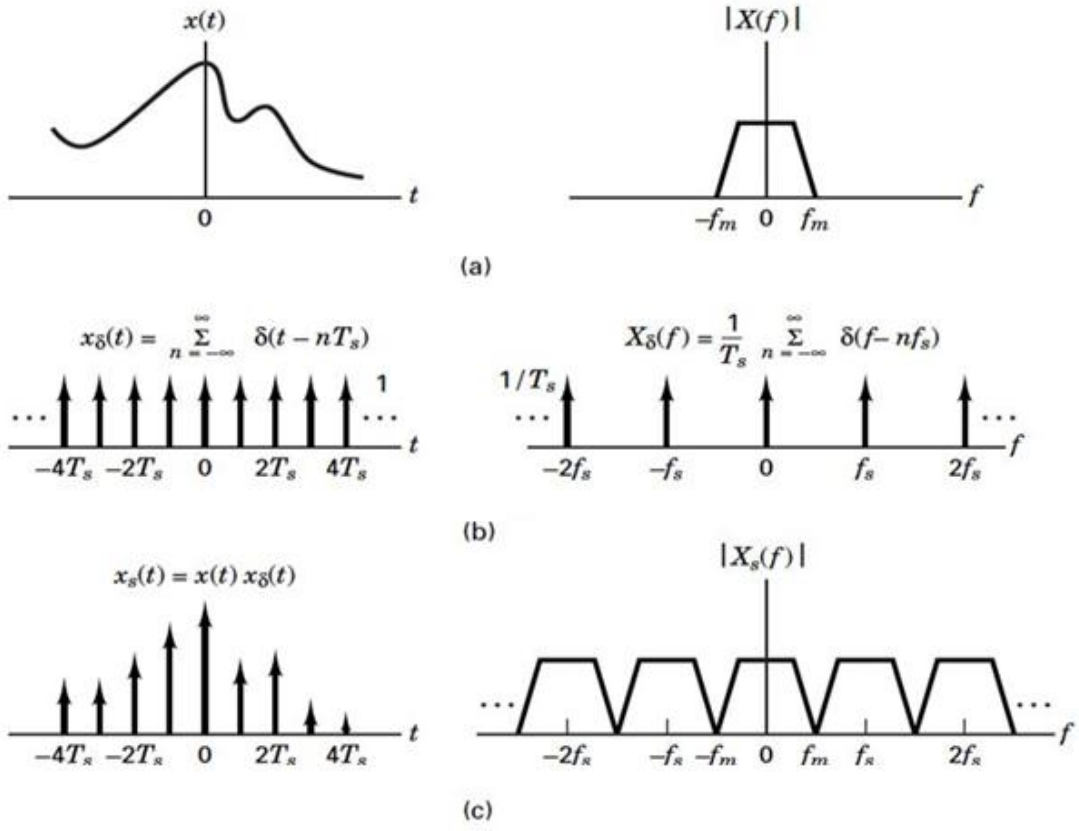


Figure 5: Time domain and frequency domain plots of (a) sampled signals, (b) sampling signal and (c) discrete-time sampling samples.

Note that this spectrum is periodic with period $F = 1/t_0$:

$$X_s(f+F) = \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-j2\pi f m t_0} \cdot e^{-j2\pi F m t_0}$$

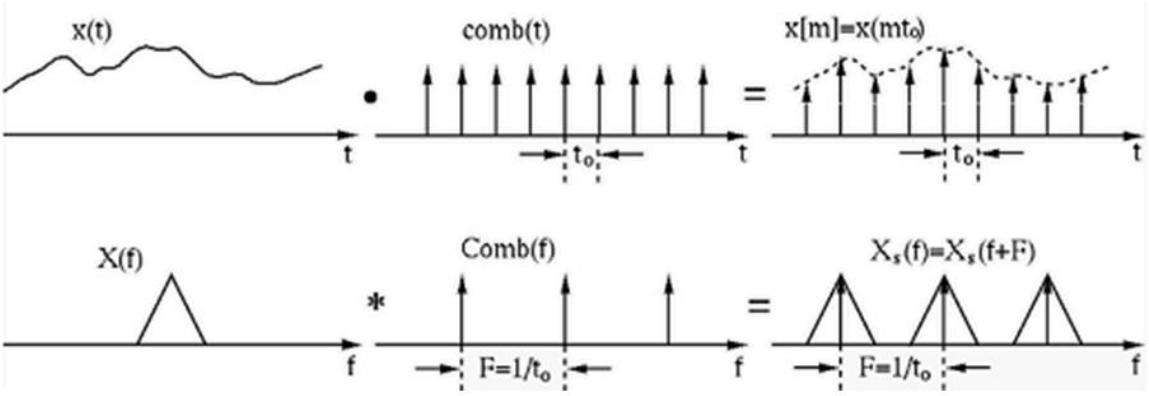
So

$$X_s(f+F) = \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-j2\pi f m t_0}$$

Thus, we can prove that:

$$X_s(f+F) = X_s(f)$$

According to convolution theorem, the sampling process as a multiplication in time domain becomes a convolution in frequency domain.



Signals to be Sampling Sampled

Figure 6: Sampling process with a comb function with both time domain and frequency domain plots of the sampling process.

The sampling process with a comb function is illustrated in the diagram given above. Both time domain and frequency domain plots of the sampling process are shown.

Let $X(f) = \mathcal{F}[x(t)]$, as a result, the comb function is:

$$\text{comb}(f) = \mathcal{F}[\text{comb}(t)]$$

Thus

$$\text{comb}(f) = F \sum_{n=-\infty}^{\infty} \delta(f - nF)$$

Then, the spectrum of the sampled signal is:

$$X_s(f) = X(f) * F \sum_{n=-\infty}^{\infty} \delta(f - nF)$$

If we assume unit sampling rate $F = 1$ and therefore $t_0 = 1/F = 1$, then:

$$X_s(f) = \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-j2\pi f_m} \quad (Eq.5)$$

Or

$$X_s(\omega) = \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-j\omega m} \quad (Eq.6)$$

Time samples of the discrete signal can be obtained from the spectrum $X_s(\omega)$. Applying equation (6), we multiply $e^{j\omega n}$ on both sides of the above equation and integrate with respect to f over 0 to 1:

$$\int_0^1 X_s(\omega) \cdot e^{j\omega n} df = \sum_{m=-\infty}^{\infty} x[m] \int_0^1 e^{j\omega(m-n)f} df$$

So

$$\int_0^1 X_s(\omega) e^{j\omega n} df = \sum_{m=-\infty}^{\infty} x[m] \cdot \delta[m-n] \quad (Eq.7)$$

So, we can prove that:

$$\int_0^1 X_s(\omega) e^{j\omega n} df = x[n]$$

Due to analogy between the transform coefficients:

$$\int_0^1 e^{-j\omega kt}\, df = \int_0^1 e^{-j2knf}\, df$$

And

$$\int_0^1 e^{-j2knf}\, df = \int_0^1 e^{-j2knt}\, dt$$

Then, apply the Euler rule:

$$\int_0^1 e^{-j\omega kt}\, df = \int_0^1 \cos(2\pi kt)\, dt + j \int_0^1 \sin(2\pi kt)\, dt = \delta[k]$$

Integral of these sinusoids of period $1/k$ over 1 is zero except when $k = 0$ when $\cos(2\pi 0t) = 1$.

The previous equation (Eq.7) is the inverse Fourier transform.

$$\int_0^1 e^{-j2knt}\,dt = \delta[k]$$

We rewrite these two equations below:

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

So

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j2\pi f n}$$

Thus, we can prove that:

$$X_s(\omega) = X_s(f)$$

Then,

$$x[n] = \int_0^1 X_s(f) \cdot e^{-j\pi f n} df$$

So

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X_s(\omega) \cdot e^{j\omega n} d\omega \qquad (Eq.8)$$

This equation (8) is the discrete-time Fourier transform (DTFT), which is sometimes also expressed as a function of:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-jn\omega}$$

Or

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega}) \cdot e^{jn\omega} d\omega$$

2. Reconstruction of Signals

The reconstruction of the continuous signal from its samples can be realised in either frequency domain or time domain.

In time domain the reconstruction is implemented by convolution process with some function $h(t)$ to fill the gaps between the discrete samples.

Such process corresponds to certain low-pass filtering with $H(f) = F|h(t)$ in frequency domain, to attenuate all replicas of the signal spectrum $X(f)$ except the middle one (e.g. low-frequency components of the signal).

We look into the methods for reconstruction of signals e.g. zero-order hold, first-order hold and sinc function methods.

2.1. Zero-Order Hold Method

In a zero-order reconstruction method, a square impulse function is used for sampling the signal.

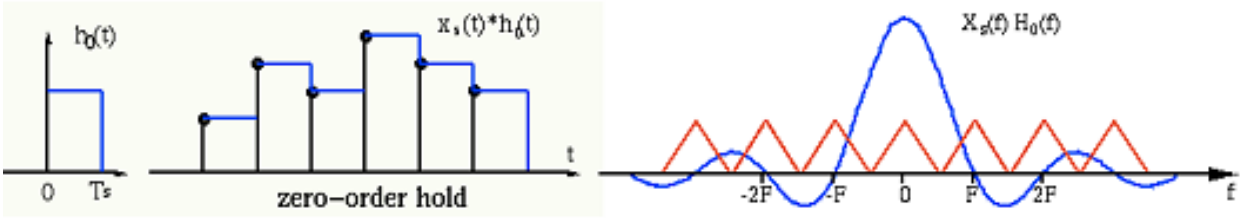


Figure 7: Zero-order hold method

First, consider a square impulse function as shown in the figure above:

$$h(t) = \begin{cases} 1, & [t] < t_0/2 \\ 0, & \text{otherwise} \end{cases}$$

Its Fourier spectrum is a sinc function:

$$H(f) = \mathcal{F}[h(t)] = \frac{\sin(\pi f t_0)}{\pi f}$$

The impulse response of a zero-order hold filter is the same as the square impulse but right-shifted by $t_0/2$:

$$h_0(t) = \begin{cases} 1, & 0 \le t < t_0 \\ 0, & \text{otherwise} \end{cases}$$

Its spectrum is:

$$H_0(f) = \mathcal{F}[h_0(t)] = \left[\frac{\sin(\pi f t_0)}{\pi f}\right] e^{-j2\pi f t_0/2}$$

Note: its values at some particular frequencies $f = 0$ and $f = kF = k/t_0$:

$$H_0(0) = t_0 = \frac{1}{F} \quad \text{and} \quad H_0(kF) = 0 \quad k = \pm 1, \ \pm 2, \cdots$$

Based on $h_0(t)$, the original signal $x(t)$ can be approximately reconstructed:

$$\hat{x}_0(t) = h_0(t) * x_s(t)$$

It is a series of square impulses with their heights modulated by $x[m]$.

The interpolation corresponds a filtering process in frequency domain by the sinc function $H_0(f)$, which behaves as a poor low-pass filter that attenuates all replicas of the original spectrum $X(f)$, and rescales

$X_s(f)$ by a factor of $1/F$ so that its magnitude is back to that of $X(f)$. Recall that a factor of F was introduced during the sampling process.

2.2. First-Order Hold Method

For a first-order hold method, the sampling is performed with a triangular like impulse function.

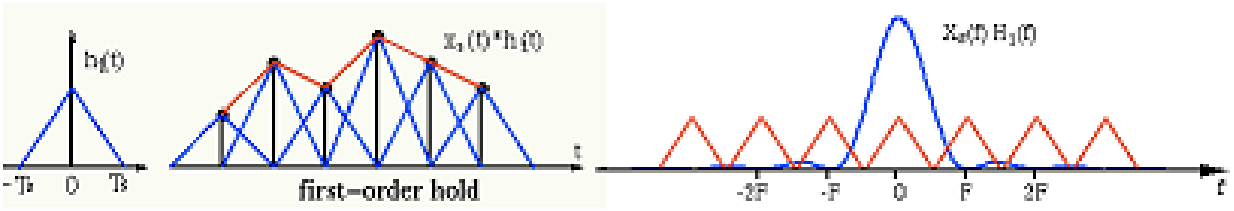


Figure 8: First-order hold method

As shown in the figure above, the impulse response of a first-order hold filter is:

$$h_1(t) = \begin{cases} 1 - |t|/t_0 & |t| < t_0 \\ 0 & \text{otherwise} \end{cases}$$

Note that $h_1(t)$ is related to the square impulse $h(t)$:

$$h_1(t) = \frac{1}{t_0} h(t) * h(t)$$

Or

$$h_1(t) = Fh(t) * h(t)$$

Correspondingly, in frequency domain, we have:

$$\begin{aligned} H_1(f) &= \mathcal{F}|h_1(t)| \\ &= \mathcal{F}[Fh(t) * h(t)] \\ &= FH^2(f) \\ &= F\left[\frac{\sin(\pi f t_0)}{(\pi f)^2}\right]^2 \end{aligned}$$

Note its values at some particular frequencies (f = 0 and $f = kF = k/t_0$):

$$H_1(0) = \frac{F}{F^2} = \frac{1}{F}$$

Where: $H_0(kF) = 0$ and $k = \pm 1, \pm 2, \ldots$

Based on $h_1(t)$, the original signal $x(t)$ can be approximately reconstructed:

$$\hat{x}(t) = h_1(t) * x_s(t)$$

This is the linear interpolation of the sample train (e.g. a straight line connecting every two consecutive samples).

This interpolation corresponds a low-pass filtering in frequency domain by with a scaling factor.

2.3. Sinc Function Method

Reconstruction often uses the sinc function for each sampling impulse. Result of reconstruction is relatively more accurate than the zero order and first order methods.

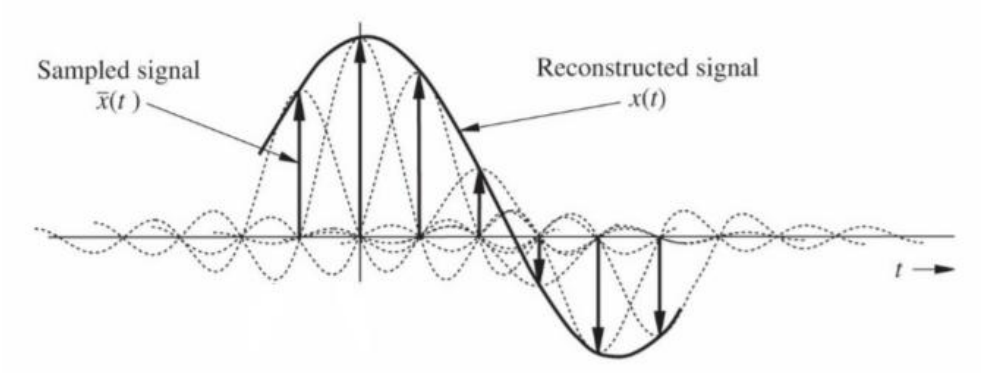


Figure 9: Sinc function interpolation method

The sinc interpolation formula is defined as:

$$x(t) = \sum_{n=-\infty}^{+\infty} x_n \cdot \text{sinc}\left[\frac{\pi}{T}(t - nT)\right]$$

Where: T is the sampling period used to determine x_n from the original signal, and $x(t)$ is the reconstructed signal.

The function $\text{sinc}(x)$ is defined as follow for $\neq 0$, with $\text{sinc}(0) = 1$.

$$\text{sinc}(x) = \frac{\sin(x)}{x}$$

The above formula represents a linear convolution between the sequence x_n and scaled and shifted samples of the sinc function.

3. Down Sampling and Up Sampling

Two topics which are related very closely to discrete-time sampling and reconstruction are the down sampling (decimation) and up sampling (interpolation) methods.

These topics are very commonly applied for data compression application in the digital communication and signal transmissions in the telecommunication.

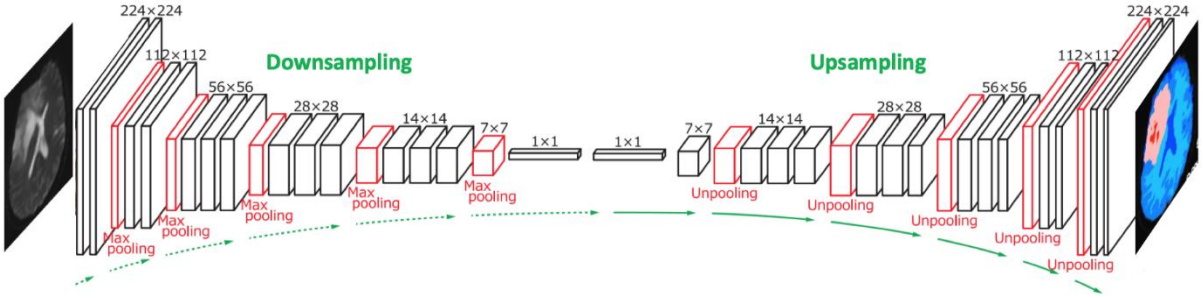


Figure 10: Down sampling and up sampling applications in the data communication system

Down sampling (decimation) is process of reducing the sampling rate. In practice, this usually implies applying low-pass-filter to a signal, then throwing away some of its samples.

Up sampling (interpolation) is the process of inserting zero-valued samples between original samples to increase the sampling rate (this is called “zero-stuffing”).

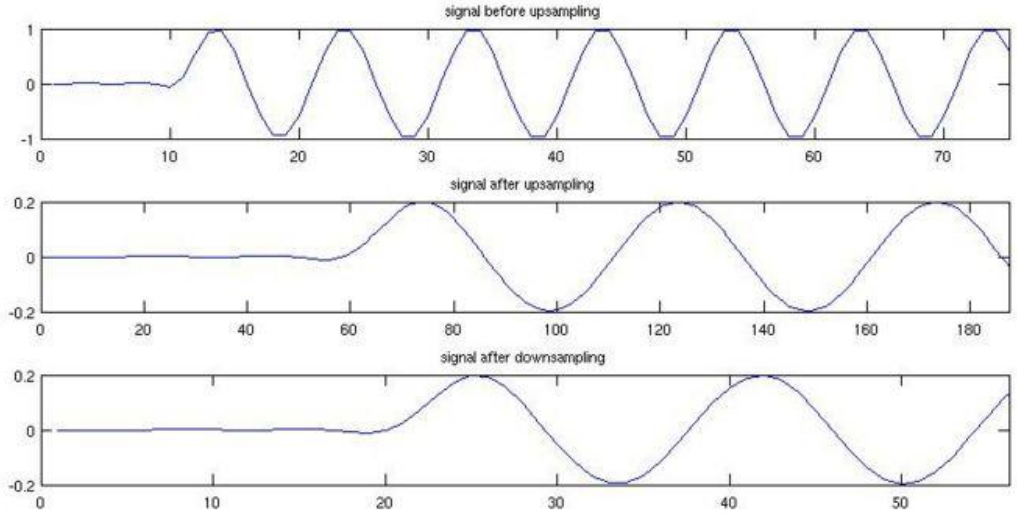


Figure 11: Effect of down sampling and up sampling towards original signals.

Purpose of down sampling:

- Reduce the sampling rate of a discrete-time signal.
- Low sampling rate reduces storage and computation requirements.

The purpose of up sampling:

- Increase the sampling rate of a discrete-time signal.
- Higher sampling rate preserves fidelity.

First, we cover the topic of sampling of the discrete-time signal. Given a discrete-time sequence samples which are resampled at different condition using down sampling or up sampling. When sampled sequence is typically replaced by sequence $x_b[n]$, which is simply every N-th value of $x_p[n]$, that is:

$$x_b[n] = x_p[nN]$$

Or, equivalently

$$x_b[n] = x[nN]$$

In this case, $x_p[n]$ and $x[n]$ are equal at integer multiples of N.

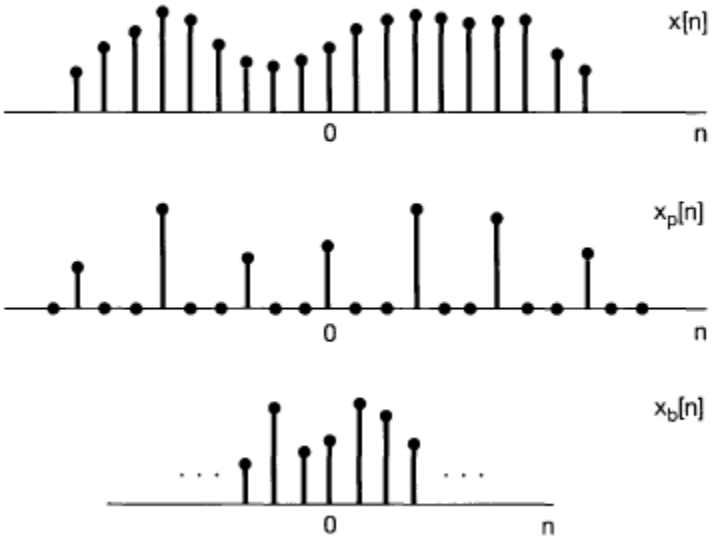


Figure 12: Relationship between $x_p[n]$ corresponding to sampling and $x_b[n]$ corresponding to decimation or interpolation method.

To determine the effect in the frequency domain of down sampling (decimation), we wish to establish the relationship between $X_b(e^{j\omega})$ i.e. the Fourier transform of $X_b[n]$ and $X(e^{j\omega})$.

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_b[k] \cdot e^{-j\omega k}$$

Knowing that $x_b[n] = x_p[nN]$

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_p[kN] \cdot e^{-j\omega k}$$

If we let $n = kN$, or equivalently $k = n/N$, we can write:

$$X_b(e^{j\omega}) = \sum_{n}^{\infty} x_p[n] \cdot e^{-j\omega n/N}$$

Where: n = integer multiple of N

If n is not an integer multiple of N, we can also write:

$$X_b(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_p[n] \cdot e^{-j\omega n/N}$$

Evaluating the right-hand side of the equation above as the Fourier transform of $x_p[n]$

$$\sum_{n=-\infty}^{+\infty} x_p[n] \cdot e^{-j\omega n/N} = X_p(e^{j\omega/N})$$

Thus, we can conclude that the Fourier transform:

$$X_b(e^{j\omega}) = X_p(e^{j\omega/N})$$

3.1. Down Sampling (Decimation)

In down sampling (decimation), the sampling rate is reduced from F_s to F_s/M by discarding $M - 1$ samples for every M samples in the original sequence.

Down sampling (decimation) factor (M) is simply the ratio of the input rate to the output rate.

M = input rate/output rate

For M-fold decimation, or down-sampling, its equation is defined by:

$$y[n] = v[nM] = \sum_{k=-\infty}^{\infty} h[k]\, x[nM - k]$$

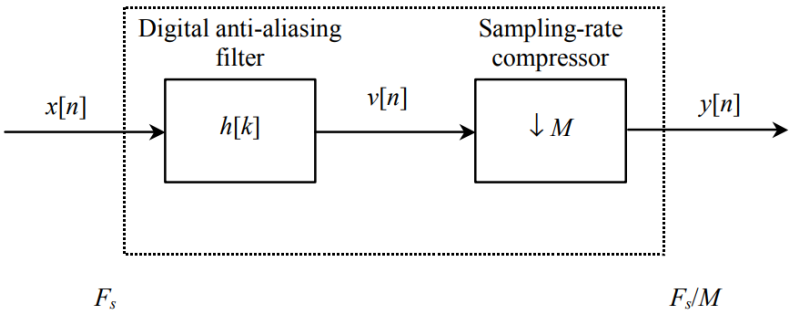


Figure 13: Down sampling (decimation) system

The figure given below shows a down sampling (decimation) by a factor of M = 5.

In the transient response, the sampled decimated signal (x_d) appears at 5 x higher frequency.

Looking at the decimated frequency spectrum, the frequency spectrum of decimated sampling signal appears as being scaled by a factor of 5.

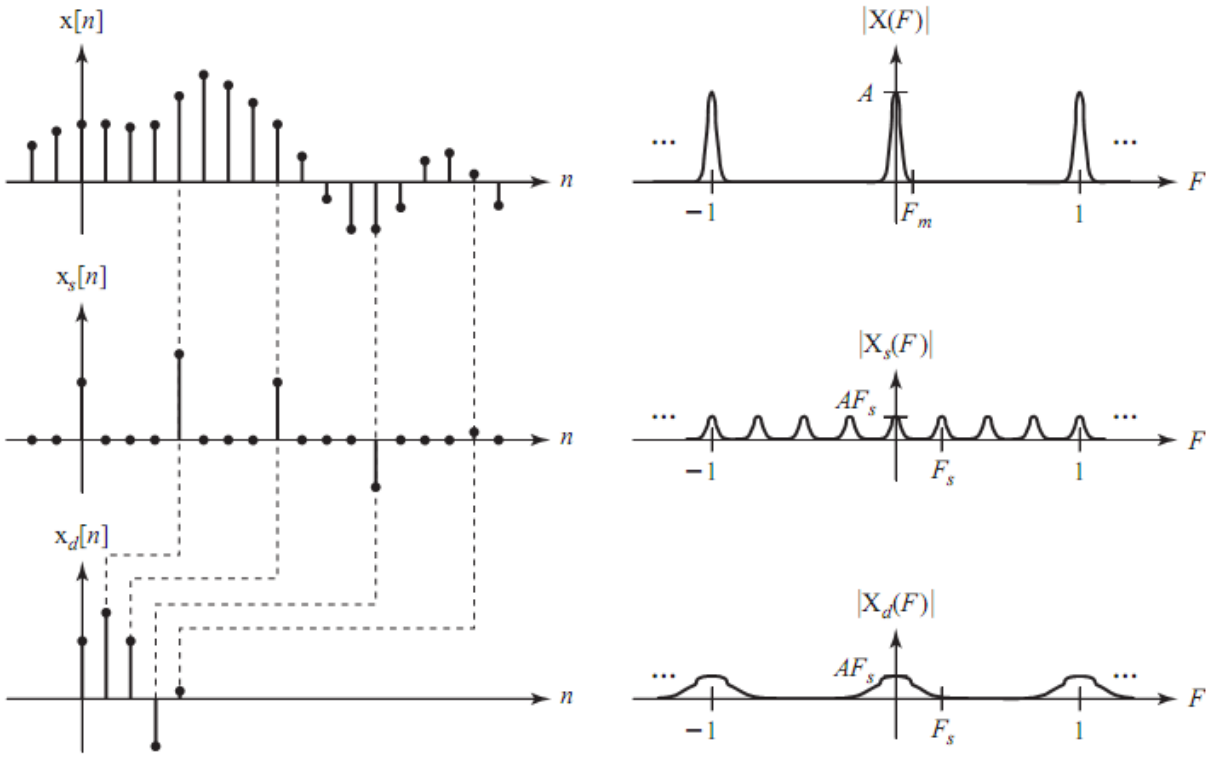


Figure 14: Transient response (left) and frequency spectrum (right) of the down sampling (decimation) in the signal sampling

3.2. Up Sampling (Interpolation)

Up sampling (interpolation) works by inserting $(L-1)$ zero-valued samples for each input sample. The sampling rate therefore increases from F_s to LF_s.

The up sampling (interpolation) factor (L) is simply the ratio of the output rate to the input rate.

$$L = \text{output rate/input rate}$$

The mathematical definition of L-fold interpolation is defined by:

$$y[n] = L \sum_{k=-\infty}^{\infty} h[k]\, w[n-k]$$

Where:

$$w[n] = \begin{cases} x[n/L], & \text{if } n/L \text{ is an integer} \\ 0, & \text{if } n/L \text{ is non} - \text{integer} \end{cases}$$

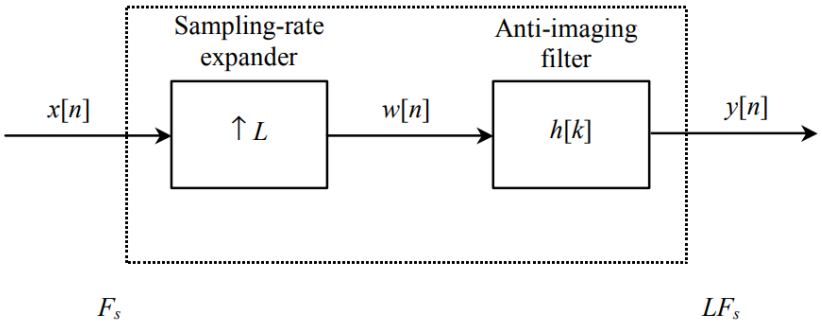


Figure 15: Up sampling (interpolation) system

The following figure illustrates an up sampling (interpolation) by a factor of L = 5.

In the transient response, the sampled interpolated signal (x_i)) appears at 5 x lower frequency.

Looking at the interpolated frequency spectrum, the frequency spectrum of decimated sampling signal appears as being scaled by a factor of 1/5.

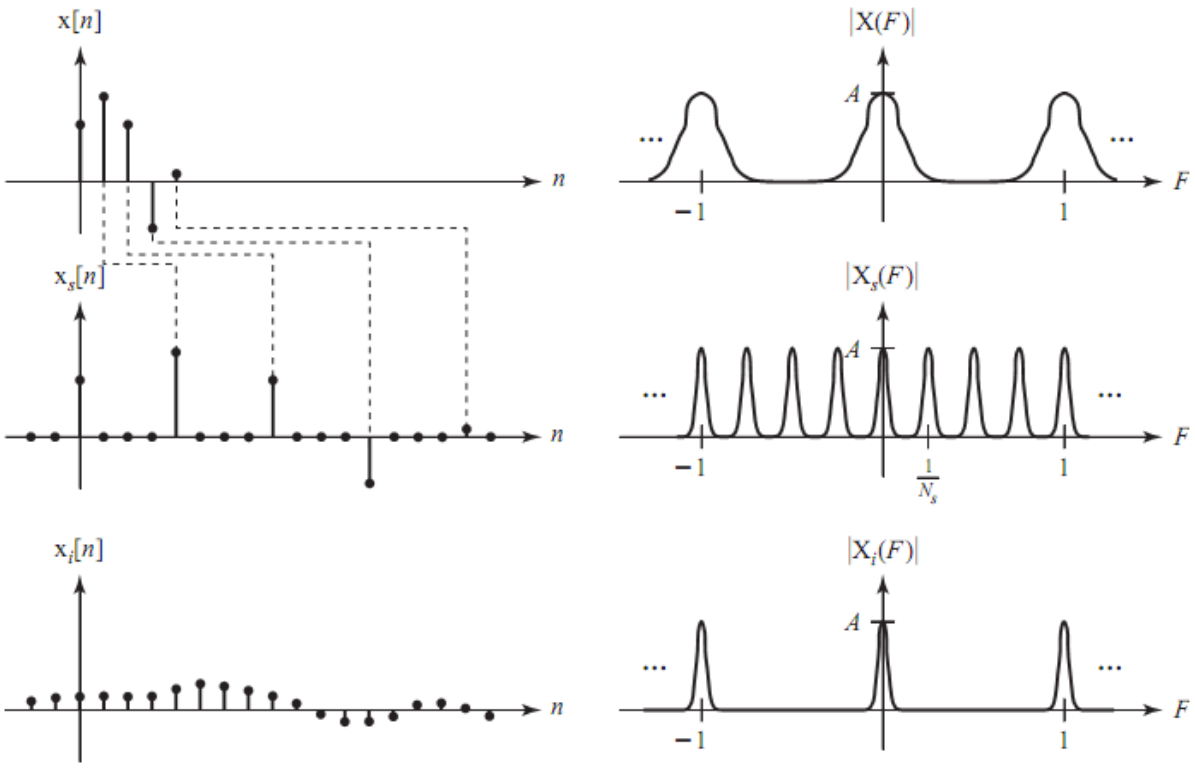


Figure 16: Transient response (left) and frequency spectrum (right) of the up sampling (interpolation) in the signal sampling

3.3. Sampling Rate Conversion

A common use of decimation and interpolation is for sampling-rate conversion. Suppose a digital signal $x[n]$ is sampled at an interval T_1, and we wish to obtain a signal $y[n]$ sampled at an interval T_2.

Then the techniques of decimation and interpolation enable this operation, providing the ratio T_1/T_2 is a rational number i.e. L/M.

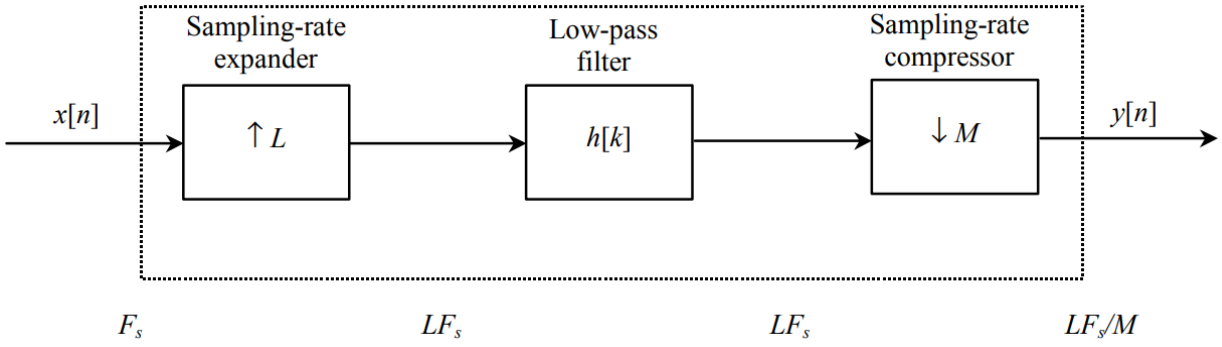


Figure 17: Sampling rate conversion system

Sampling-rate conversion can be accomplished by L-fold expansion, followed by low-pass filtering and then M-fold decimation, as depicted in the figure above.

It is important to emphasize that the interpolation should be performed first and decimation second, to preserve the desired spectral characteristics of $x[n]$.

Furthermore, by cascading the two in this manner, both of the filters can be combined into one single low-pass filter.

An example of sampling-rate conversion would take place when data from a CD format (f_s = 44.1 kHz) is transferred onto a DAT format (f_s = 48 kHz).

Here the sampling-rate is increased from 44.1 kHz to 48 kHz. To enable this process the non-integer factor has to be approximated by a rational number:

$$\frac{L}{M} = \frac{48\ \text{kHz}}{44.1\ \text{kHz}} = \frac{160}{147} = 1.08844$$

Hence, the sampling-rate conversion is achieved by interpolating by L i.e. from 44.1 kHz to [44.1 kHz x 160] = 7056 kHz. Then, decimating by M i.e. from 7056 kHz to [7056 kHz/147] = 48 kHz.