XMUT220 Signals and Systems

Sampling and Reconstruction (Aliasing)

Topics

1. Fundamental of aliasing.
2. Aliasing in the time domain.
3. Aliasing in the frequency domain.
4. Analysis of aliasing.
5. Case study of aliasing.

1. Introduction to Aliasing

Aliasing in signals and systems:

"*Each of a set of signal frequencies that, when sampled at a given uniform rate, would give the same set of sampled values, and thus might be incorrectly substituted for one another when reconstructing the original signal*".

Aliasing occurs when a sinusoid of one frequency takes on the appearance or identity of a different frequency sinusoid. Similarly, aliasing is always a possibility when sampling a signal, and simple measures should be taken to prevent it from occurring. Signals, given the opportunity, will always alias.

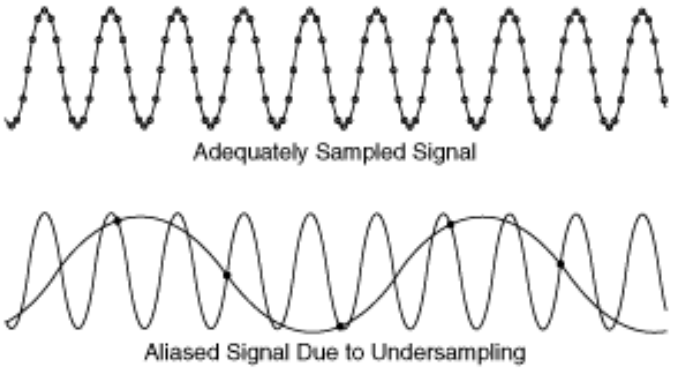


Figure 1: Adequately sampled signal and aliased signal due to undersampling

The fundamental reason for aliasing of signals is the fact that discrete-time sinusoids are not unique functions of frequency. Sinusoids of frequency ω_1 and $\omega_1 + k2\pi$ are identical for all integers k. This is because the sample index n is always integer valued. For example:

$$\cos\left(\frac{\pi}{4}\right) = \cos\left(\frac{9\pi}{4}\right) = \cos\left[\left(\frac{\pi}{4}\right)n + 2\pi n\right] = \cos\left[\left(\frac{\pi}{4}\right)n\right]$$

In the figure below, we illustrate three different continuous-time signals, all of which have identical values at integer multiples of T. That is, clearly, an infinite number of signals can generate a given set of samples.

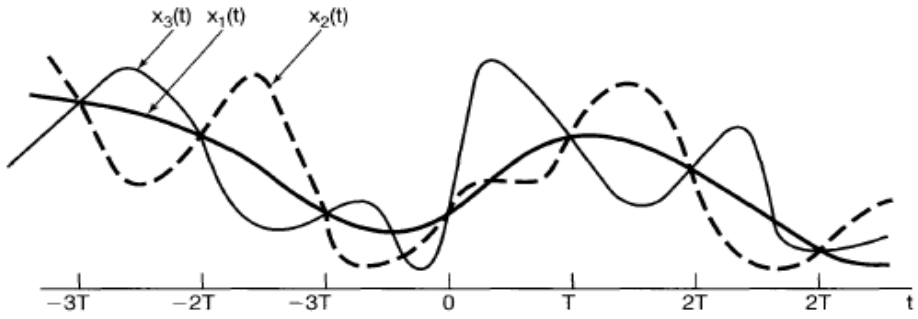


Figure 2: Three different continuous-time signals, all of which have identical values at integer multiples of T.

This property does not apply to continuous-time sinusoids because time is not limited to integers, but takes on a continuum of values. Hence, discrete-time sinusoids are only unique over a 2π interval of frequency.

Typically, we work with the range of discrete-time frequencies $-\pi < \omega \leq \pi$. Aliasing of signals can occur when we sample e.g. convert a continuous-time signal to a discrete-time signal. Continuous-time sinusoids with distinct frequencies are always unique.

When sampling converts a continuous-time sinusoid to a discrete-time sinusoid, the discrete-time sinusoid does not inherit the uniqueness of its continuous-time predecessor. Consider two continuous-time sinusoids:

$$x_1(t) = \cos\left[\left(\frac{\pi}{2}\right)t\right]$$

And

$$x_2(t) = \cos\left[\left(\frac{9\pi}{2}\right)t\right]$$

These sinusoids have distinct frequency and thus are distinct. However, if we sample them at intervals of $T_s = 0.5$ second, then:

$$x_1[n] = x_1(0.5n) = \cos\left[\left(\frac{\pi}{4}\right)n\right]$$

While

$$x_2[n] = x_2(0.5n) = \cos\left[\left(\frac{9\pi}{4}\right)n\right]$$

The discrete-time frequency of $x_1[n]$ and $x_2[n]$ differ by 2π and thus, $x_1[n] = x_2[n]$. We say that the sinusoid with continuous-time frequency $9\pi/2$ rads/sec has "aliased" to the same discrete-time frequency ($\pi/4$) as the sinusoid with continuous-time frequency $\pi/2$ rads/sec.

If two (or more) distinct continuous-time sinusoids contribute to the same discrete-time sinusoid, we have no way of distinguishing the original continuous-time sinusoids.

There is no way to determine the amplitude and phase to assign to the aliased continuous-time sinusoids if we are trying to analyse the signal or convert back to a continuous-time signal.

For example, if we observe that the amplitude of the discrete-time sinusoid at a particular frequency is 2, all we know is that the sum of an unknown number of continuous-time sinusoids produces an amplitude of 2.

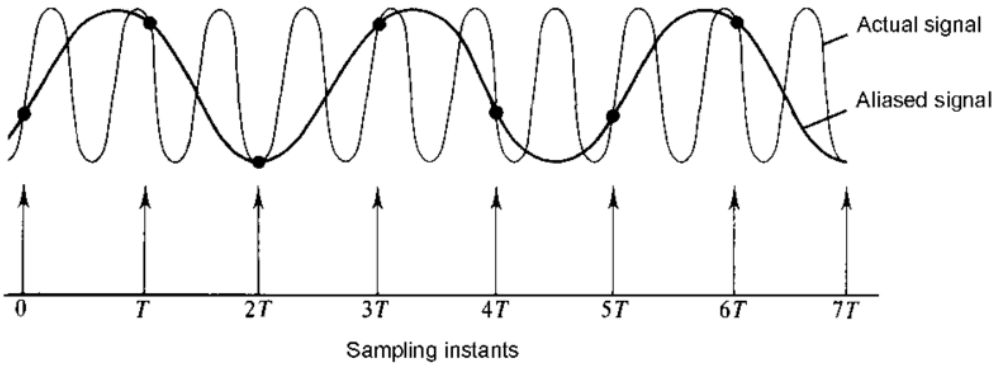


Figure 3: Sampled signal and discrete time samples

There are infinite combinations of continuous-time sinusoids that could sum to produce an amplitude of 2, and there is no way to uniquely determine their amplitudes.

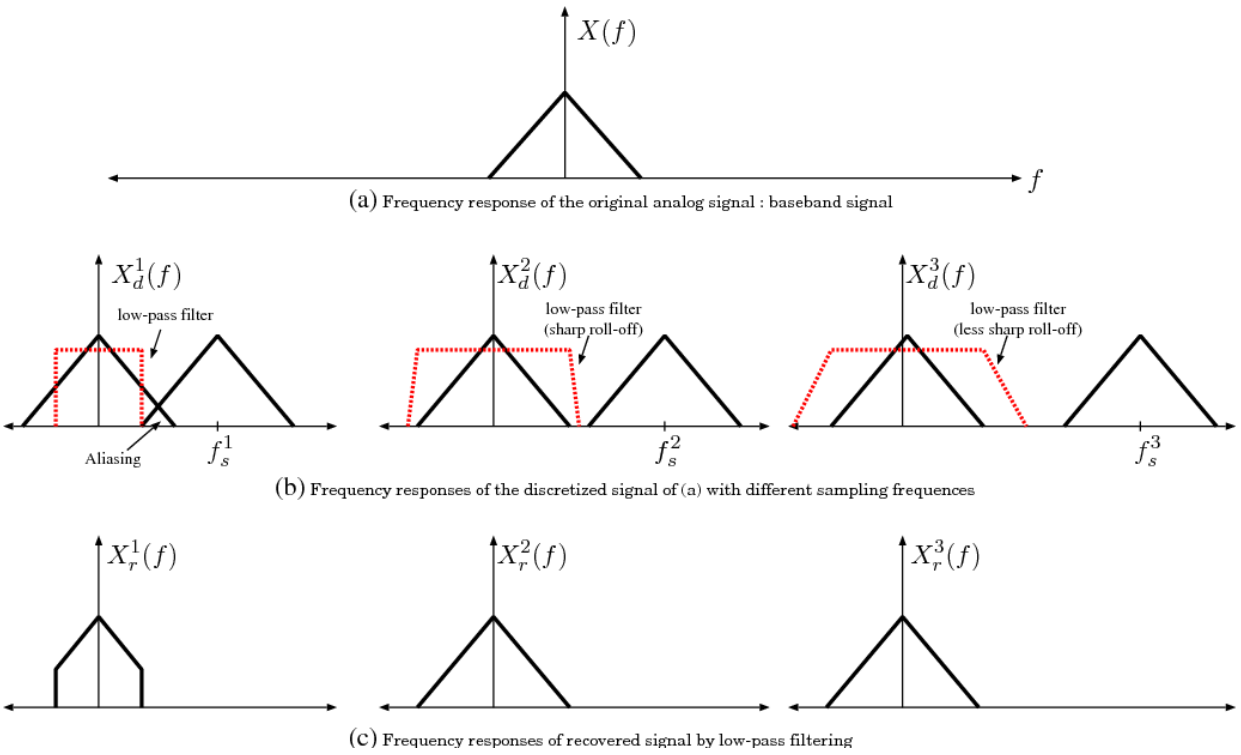


Figure 4: Frequency spectra of aliasing and non-aliasing sampling of a continuous signal

From results of the frequency spectra of continuous-time signal sampling:

- Original signal
- Sampled signal with different sampling frequencies.
- Recovered signal by low-pass filtering.

From frequency spectra of aliased and non-aliased signals:

- Aliased reconstructed signal has different frequency spectrum than the original continuous-time signal.
- Non-aliased reconstructed signal will have the frequency spectrum of original continuous-time signal.

The most common way to prevent aliasing is to limit the range of continuous-time sinusoids so there is a unique or one-to-one mapping between continuous- and discrete-time frequency.

If we use the range $-\pi < \omega \leq \pi$ for discrete-time frequency, then the corresponding range of continuous-time frequency is:

$$-f_s/2 < f \leq f_s/2$$

Where $f_s = 1/T_s$ is the sampling frequency in Hz. This result immediately follows from the fact that:

$$\omega = 2\pi f T_s.$$

This leads us to the Nyquist Sampling Theorem, which states that there is a unique correspondence between discrete- and continuous-time sinusoids.

It experiences no aliasing if the sampling frequency is chosen to exceed twice the highest continuous-time frequency present in the signal. In this case, one can uniquely reconstruct the continuous-time signal from its samples.

In practice, the presence of noise and interference typically makes it impossible to know in advance the highest frequency present in the continuous time signal. A continuous-time low-pass filter is applied to the data before sampling to limit the bandwidth to a desired range. This is called an anti-aliasing filter.

2. Example of Aliasing

An aliasing example is a saxophone playing the sound of note 'A'. The frequency of note ‘A’ is at 440 Hz.

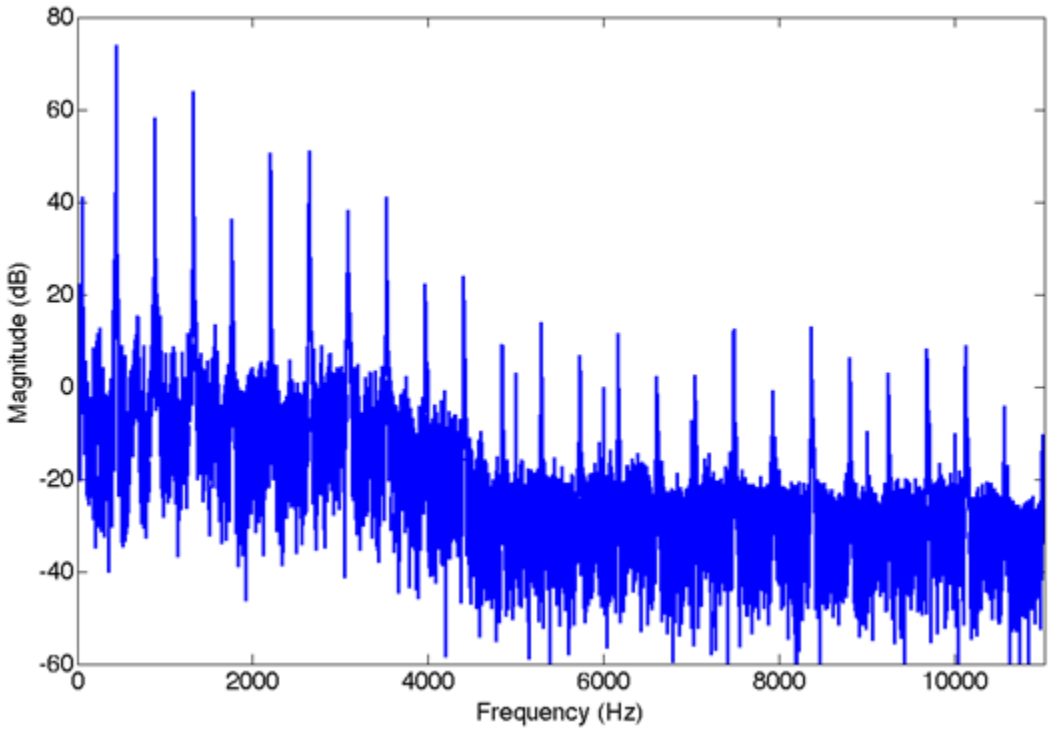


Figure 5: Frequency spectrum of an ‘A’ note in saxophone

As shown above, notice on the peaks at periodic intervals in the frequency spectrum plot of the sampled sound signal. The spectrum of this note, shown in the previous figure, exhibits spectral peaks at harmonics, i.e., integer multiples, of 440 Hz. Notice the peaks at frequencies: 440 Hz, 880 Hz, 1,320 Hz, etc. The amplitude of the harmonics is quite small above 4 kHz.

The spectral peak near 0 Hz occurs at 60 Hz and is due to power line noise contaminating the recording. Then, we sample this signal at 2,756 Hz with and without an antialiasing filter. We sample this signal at 2,756 Hz with an antialiasing filter. Shown in the figure below is the frequency spectrum of the sound of the 'A' note.

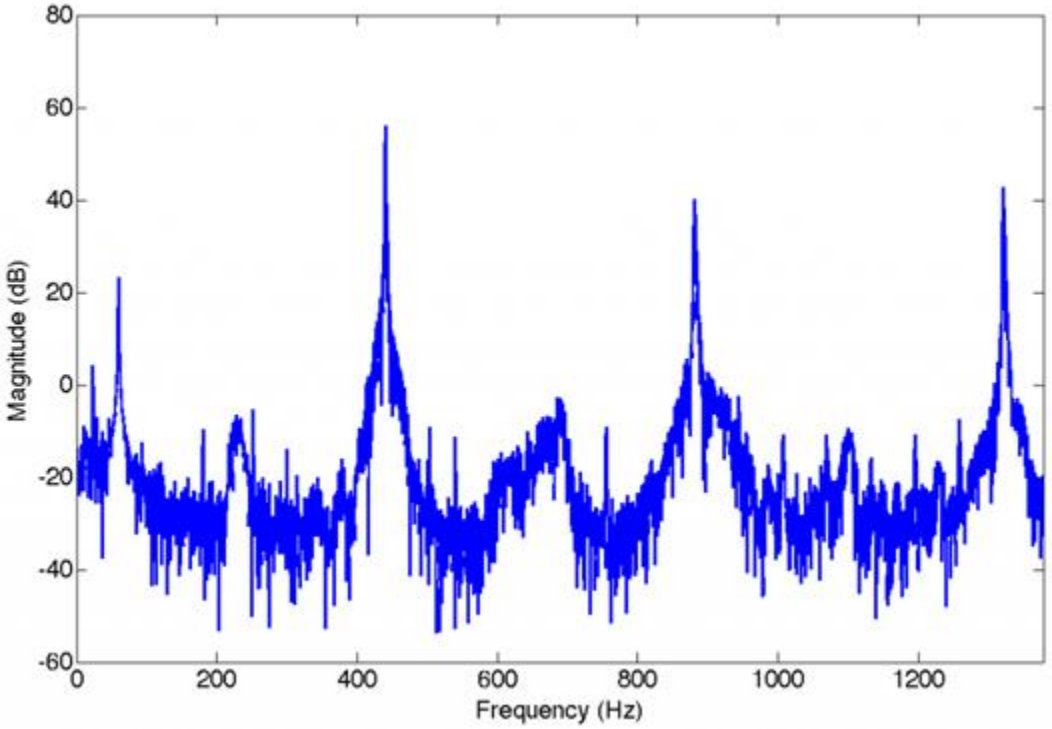


Figure 6: Frequency spectrum of a note 'A' in saxophone with aliasing filter

At this sampling frequency the maximum continuous-time sinusoid frequency that can be represented without aliasing is:

$$F(\text{sampling}) = \frac{F(\max)}{2} = \frac{2{,}756\ \text{Hz}}{2} = 1{,}378\ \text{Hz}$$

Notice the peaks at those harmonics in the spectrum at 440 Hz (1st harmonic), 880 Hz (2nd harmonic), and 1,320 Hz (3rd harmonic). Thus, the antialiasing filter limits the signal to the first three harmonics. The following figures depict the spectrum of the sampled sound without the antialiasing filter applied before sampling.

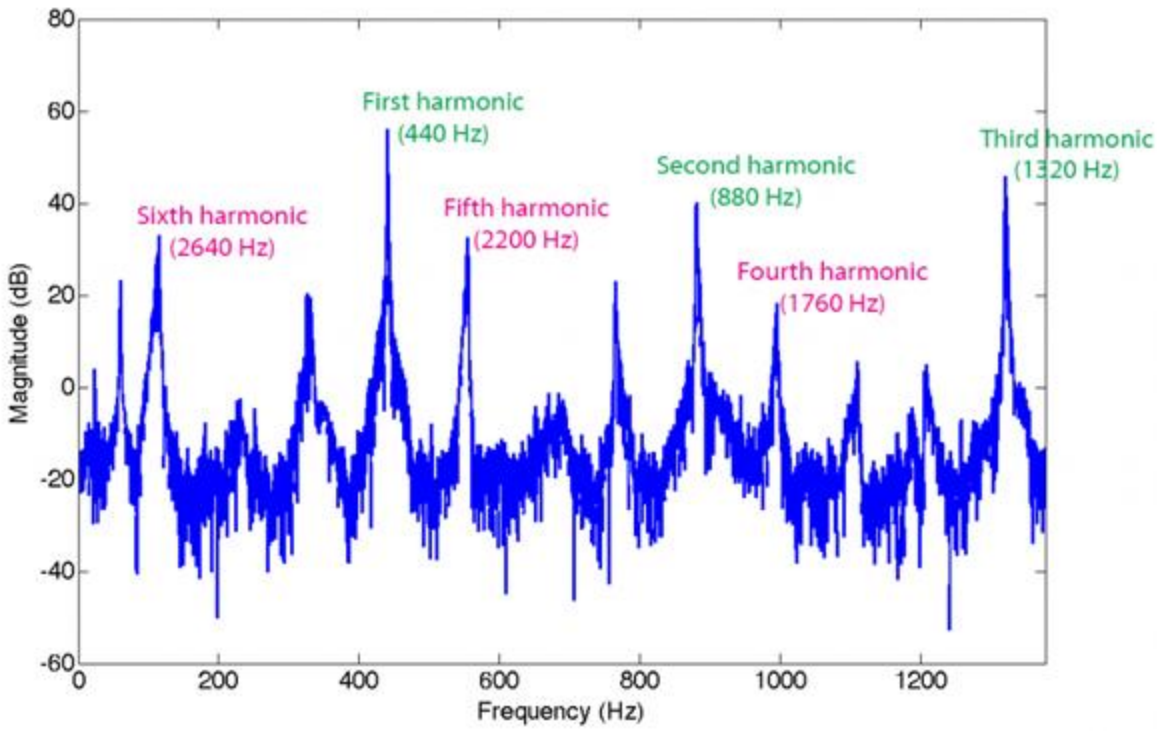


Figure 7: Frequency spectrum of a note 'A' in saxophone without antialiasing filter

The aliased fourth, fifth, and sixth harmonics are labelled in figure above. Furthermore, you can also find the frequency spectrum plot the spectral peaks at 324 Hz and 764 Hz which are aliases of the seventh and eighth harmonics. Note that in general a sinusoid of frequency f_0 aliases to $kf_s - f_0$ and $f_0 - mf_s$, where k, m are integers.

A wind instrument like a saxophone creates a very tonal sound because the only significant contributors are harmonics of the fundamental. Aliasing introduces energy at frequencies which are not harmonically related to the fundamental, and introduces a muddy character to the sound, or a buzzing aspect.

3. Aliasing Example in Time Domain

We are sampling sinusoid signal inputs with the following frequencies: 60 Hz, 340 Hz, or 460 Hz with sampling frequency f_s = 400 Hz. The analogue signals are:

- $x_1(t) = \cos\left(2\pi 60t + \frac{\pi}{3}\right)$
- $x_2(t) = \cos\left(2\pi 340t - \frac{\pi}{3}\right)$
- $x_3(t) = \cos(2\pi 460t + \frac{\pi}{3})$

We sample these signals of $x_i(t)$, i = 1, 2, 3 at sampling rate f_s = 400 Hz

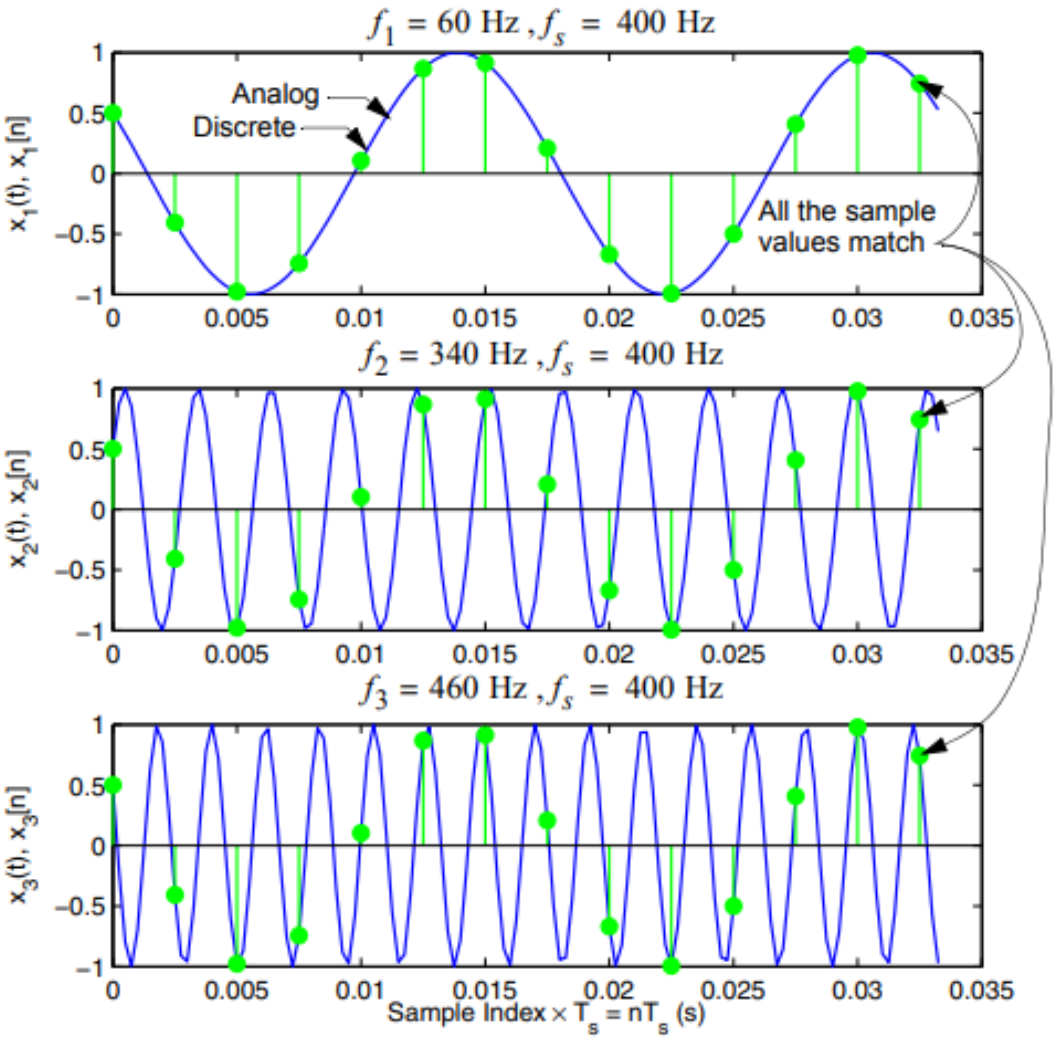


Figure 8: Transient responses of sampling of signals at various frequencies

4. Aliasing Example in Time Domain

We illustrate the time domain plots of the case study shows a continuous time signal with frequency = 1 kHz to be sampled with a sampling signal. Then, the sampling rate of the sampling signal is increased from:

a. The rate of 1 sample per cycle.

b. The rate of 2 samples per cycle.

c. The rate of 7/4 samples per cycle.

d. The rate of 10 samples per cycle.

When sampling rate is 1 sample/cycle, the sampled signal appears in (a) as a flat line signal at a specific amplitude. When sampling rate is 7/4 samples/cycle, the sampled signal in (b) appears as a triangular signal that has been shifted (or modulated) by a specific phase angle.

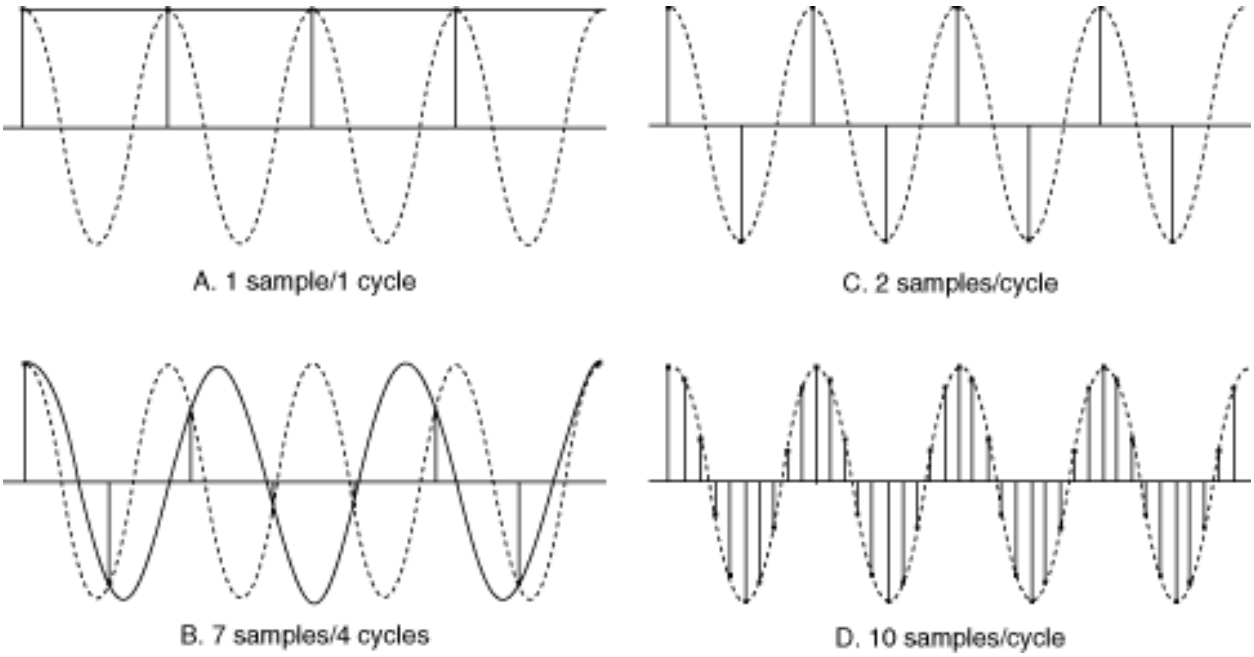


Figure 9: Transient responses due to aliasing in various sampling of a sinusoidal signal

When sampling rate is 2 samples/cycle, the sampled signal in (c) appears as a triangular signal with constant magnitude. When sampling rate is 10 samples/cycle, the sampled signal in (d) appears as a sinusoidal signal with the given specified frequency.

5. Aliasing Example in Time and Frequency Domains

Consider a continuous signal:

$$x(t) = \cos(2\pi ft + \emptyset)$$

Applying Euler theorem formulae to represent the given signal as above

$$x(t) = \frac{1}{2}\left[e^{j(2\pi ft+\emptyset)} + e^{-j(2\pi ft+\emptyset)}\right]$$

Then,

$$x(t) = c_1 e^{j2\pi ft} + c_{-1} e^{-j2\pi ft}$$

Where: $c_1 = \frac{1}{2}e^{j\emptyset}$ and $c_{-1} = \frac{1}{2}e^{-j\emptyset}$

These coefficients are respectively the two non-zero coefficients for the frequency components $e^{j2\pi ft}$ and $e^{-j2\pi ft}$. When this signal is sampled at a rate of $F = 1/t_0$, it becomes a discrete signal:

$$x|m| = \cos(2\pi f m t_0 + \emptyset) = \cos(2\pi f\, m/F + \emptyset)$$

This discrete time signal can be represented as follow:

$$x|m| = \left(\frac{e^{j\emptyset}}{2}\right)e^{j2\pi f_m/F} + \left(\frac{e^{-j\emptyset}}{2}\right)e^{-j2\pi f_m/F}$$

Then,

$$x|m| = c_1 e^{j2\pi f_m/F} + c_{-1}e^{-j2\pi f_m/F}$$

5.1. Analysis of f = 1 Hz

Notice the discrete sampled signal and reconstructed signal in the time domain plot and its frequency spectra in the frequency domain plot. When f = 1 Hz, in this case $f < F/2 = 3$ Hz the two non-zero frequency components. c_1 and c_{-1} are inside the middle period of the spectrum, based on which the signal can be perfectly reconstructed.

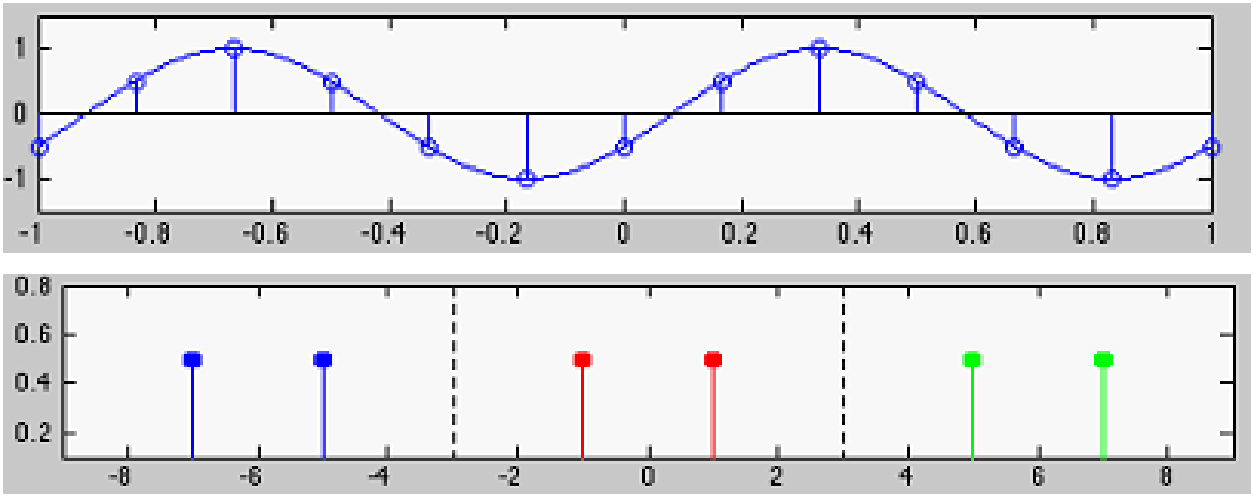


Figure 10: Analysis of aliasing with signal frequency of f = 1 Hz.

5.2. Analysis of f = 2 Hz

When f = 2 Hz, in this case $f < F/2 = 3$ Hz, the two non-zero frequency components moved away from the y-axis. The c_1 and c_{-1} are still inside the middle period, although now they have moved further outward.

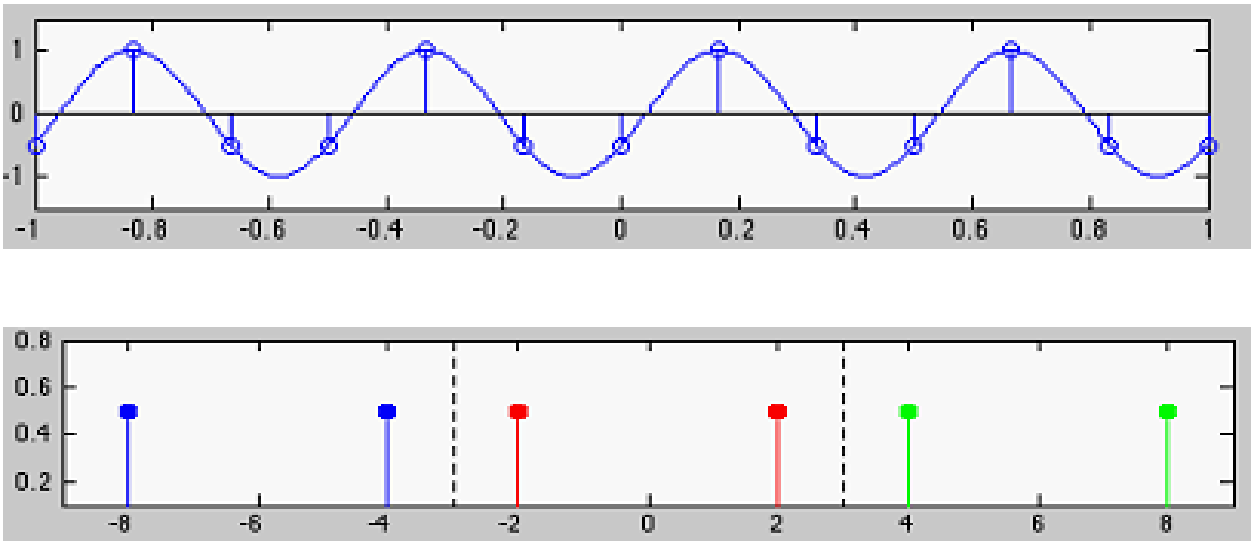


Figure 11: Analysis of aliasing with signal frequency of f = 2 Hz.

5.3. Analysis of f = 3 Hz

When f = 3 Hz, in this case $f = F/3$ Hz, the two non-zero frequency components move to be on top of the sampling boundaries. Depending on the relative phase difference between the signal and the sampling process, the signal is distorted to different amount. In the worst case, when the two samples in a signal period happen to be at the zero-crossings, same as a zero signal $x(t) = 0$.

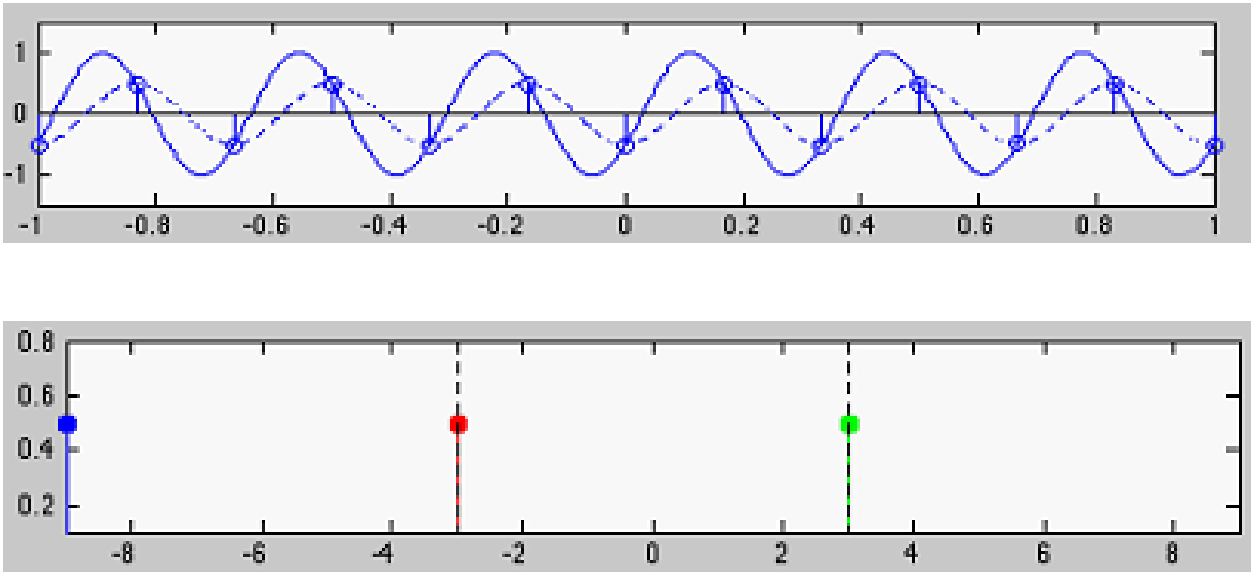


Figure 12: Analysis of aliasing with signal frequency of f = 3 Hz.

Note that for f = 3 Hz, the spectra of the impulses of the signal are superimposed on the sampling boundary lines e.g. the signal is marginally aliased. Notice the dash line (-) in the time domain plot which is the reconstructed signal.

5.4. Analysis of f = 4 Hz

When f = 4 Hz, in this case $f > F/2 = 3$ Hz the signal is:

$$x(t) = \cos(2\pi 4t + \emptyset)$$

In frequency domain, the coefficients $c_1 = e^{j\emptyset}/2$ and $c_{-1} = e^{-j\emptyset}/2$ for f = 4 Hz and f = −4 Hz respectively move out of the middle period, while their replicas in the two neighbouring periods move inside the middle period.

Specifically:

c_1 on the left for the aliased frequency 4 moves in from the left to become coefficient for $-F$ = 4−6 = −2 Hz, and c_{-1} for frequency -4 moves in from the right to become coefficient for −4 +F = 2 Hz.

For analysis of f = 4 Hz, the reconstructed signal based on these coefficients inside the middle period becomes:

$$x'[\mathrm{m}] \quad = c_{-1}e^{j2\pi f_m/F} + c_1 e^{-j2\pi f_m/F}$$

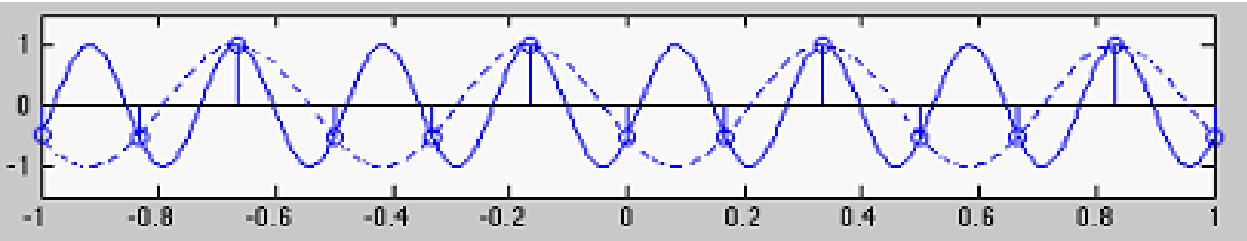
Applying Euler equivalent:

$$x'[\mathrm{m}] = \left(\frac{e^{-j\emptyset}}{2}\right)e^{j2\pi f_m/F} + \left(\frac{e^{j\emptyset}}{2}\right)e^{-j2\pi f_m/F}$$

Then,

$$x'[\mathrm{m}] = \cos(-2\pi f_m t_0 - \emptyset) = \cos(-(2\pi f_m t_0 + \emptyset))$$

This is called aliasing or folding



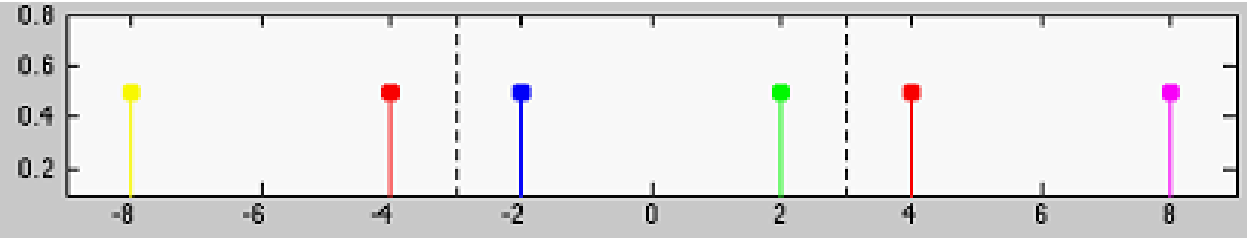


Figure 13: Analysis of aliasing with signal frequency of f = 4 Hz.

5.5. Analysis of f = 5 Hz

When f = 5 Hz, in this case $f > F/2 = 3$ Hz similar folding occurs as the $f = 4$ Hz case occurs. This time the two frequency components moving further away from the centre. But, replicas moving further towards the centre.

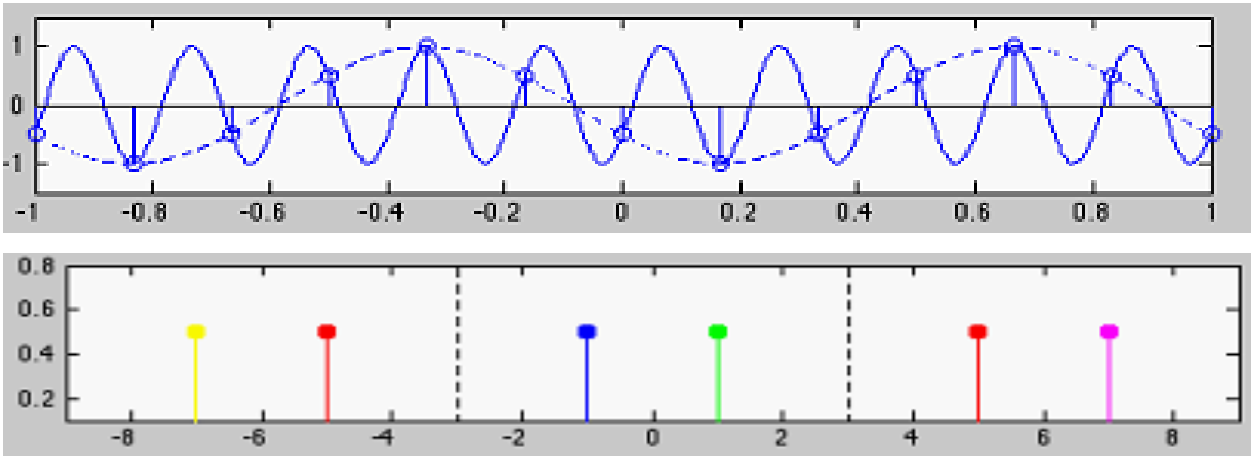


Figure 14: Analysis of aliasing with signal frequency of f = 5 Hz

5.6. Analysis of f = 6 Hz

When f=6 Hz, in this case $f = F$ Hz i.e., one sample is taken per period. The aliased frequency happens to be zero. The reconstructed signal is of course a constant.

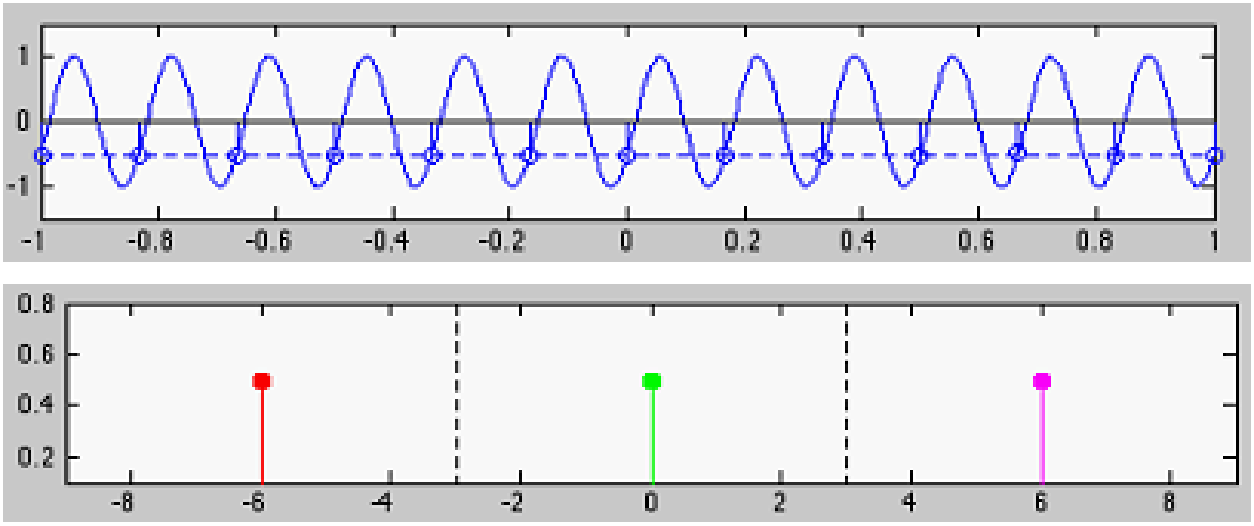


Figure 15: Analysis of aliasing with signal frequency of f = 6 Hz.

5.7. Analysis of f = 7 Hz

When f = 7 Hz, aliasing occurs. Similar to the cases when f = 4 and 5 Hz, except now:

- c_1 is for aliased frequency 7 − F = 7 − 6 = 1 Hz.
- c_{-1} is for aliased frequency −7 + 6 = −1 Hz.

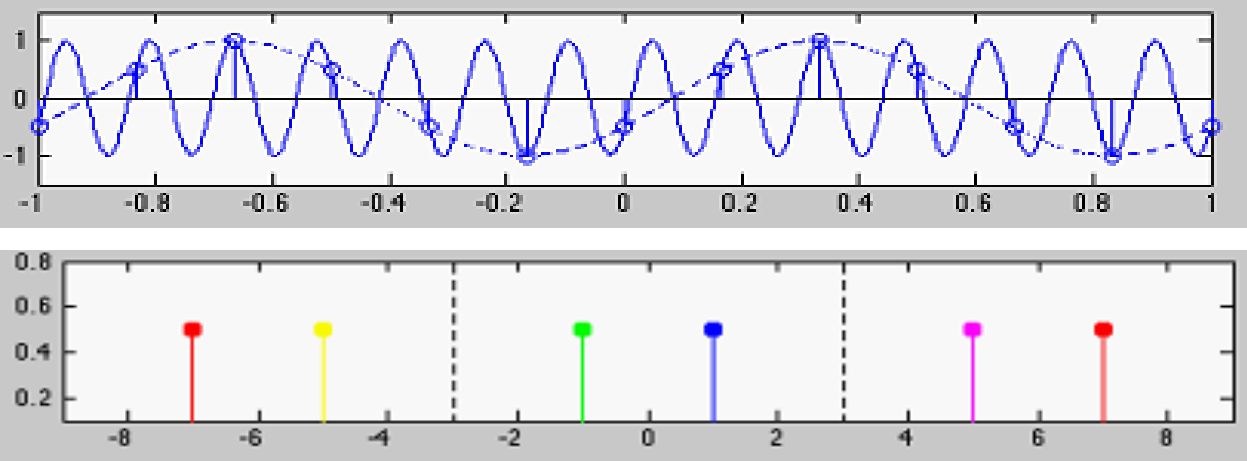


Figure 16: Analysis of aliasing with signal frequency of f = 7 Hz

5.8. Analysis of f = 8 Hz

When f = 8 Hz, aliasing occurs similar to the cases when f = 5 Hz, except now:

- c_1 is for aliased frequency 8 − F = 8 − 6 = 2 Hz.
- c_{-1} is for aliased frequency −8 + F = −8 + 6 = −2 Hz.

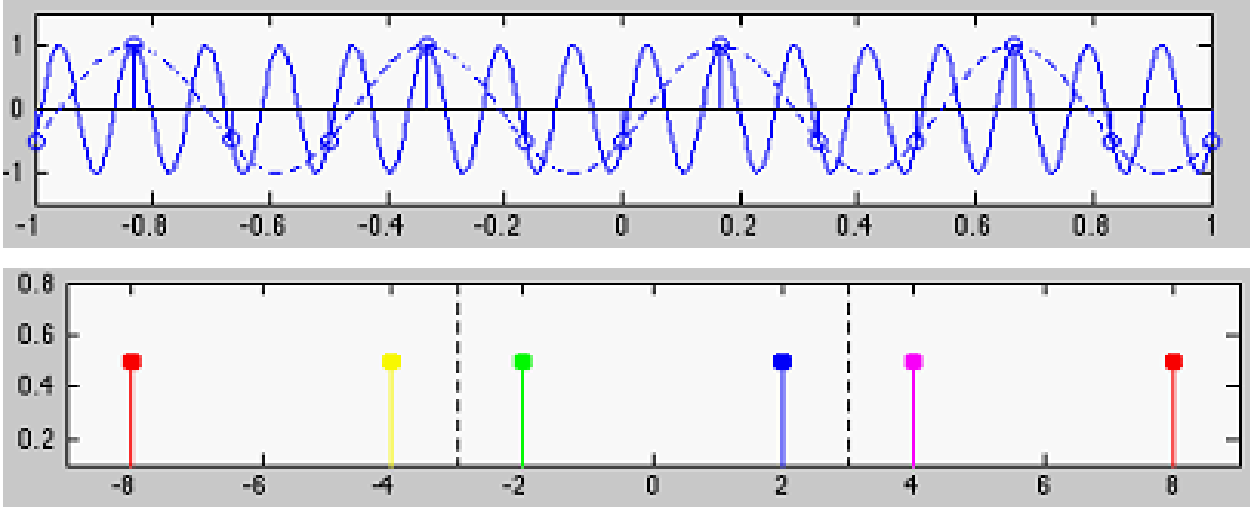


Figure 17: Analysis of aliasing with signal frequency of f = 8 Hz.

5.9. Analysis of f = 9, 10, 11, and 12 Hz

In the remaining four cases for f = 9, 10, 11, and 12 Hz, the two coefficients inside the middle period are identical to those in the previous four cases when f = 3, 4, 5, and 6 Hz, respectively. Except now, they are replicated coefficients moved from not the immediate neighbouring periods but the two periods further out. Consequently, the reconstructed signals based on the middle period are also identical to those in the previous cases.

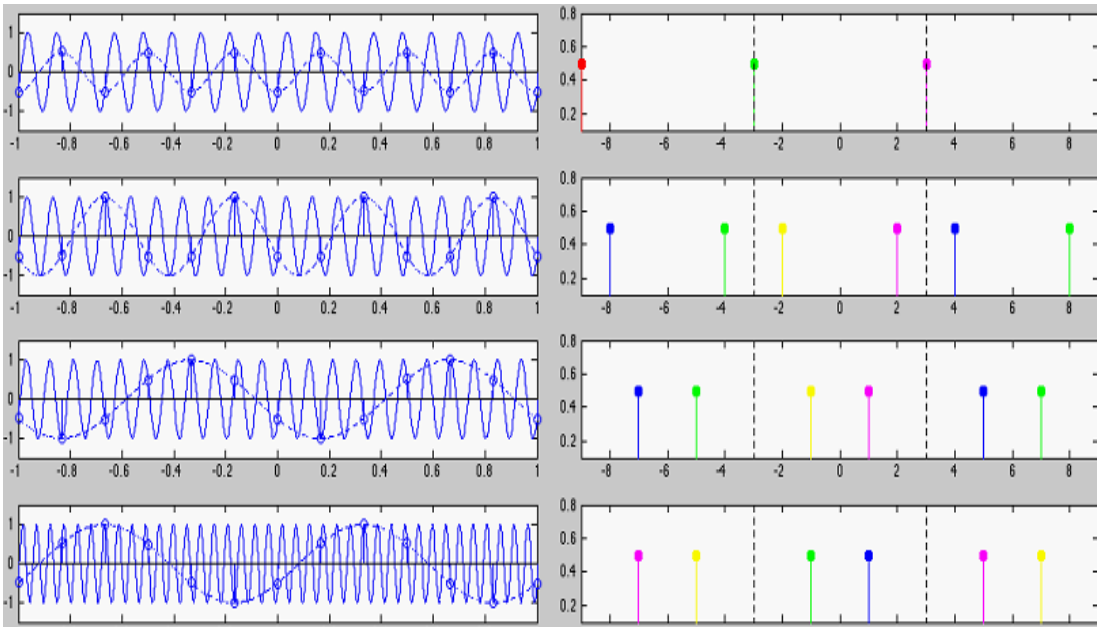


Figure 18: Analysis of aliasing with signal frequency of f = 9, 10, 11, and 12 Hz.