XMUT220 Signals and Systems

Z-Transform (Introduction)

Topics

1. Derivation of Z-transform Equation.
2. Region of Convergence.
3. Inverse z-Transform.

1. Introduction to Fundamentals of z-Transforms

This section introduces the fundamental concepts and characteristics of z-transforms.

1.1. Derivation of z-Transform Equations

The z-transform of a sequence $x[n]$ is:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

An operator $\mathbb{Z}\{\cdot\}$ that transforms a sequence to a function:

$$\mathbb{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z)$$

In both cases z is a continuous complex variable.

Fourier transform from the z-transform by making the substitution $z = e^{j\omega}$.

This corresponds to restricting $|z|$ = 1. Also, with $z = re^{j\omega}$:

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\,(re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{j\omega}$$

z-transform is the Fourier transform of the sequence $x[n]r^{-n}$.

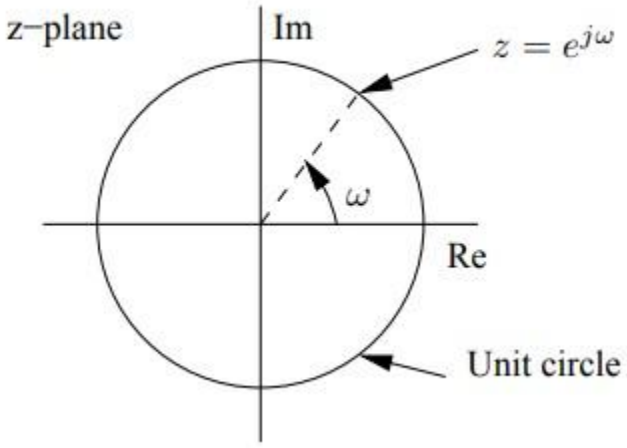

Figure 1: Z-transform of discrete time sequence

For r = 1 this becomes the Fourier transform of $[n]$. The Fourier transform therefore corresponds to the z-transform evaluated on the unit circle. The inherent periodicity in frequency of the Fourier transform is captured naturally under this interpretation. The Fourier transform does not converge for all sequences i.e. the infinite sum may not always be finite.

Similarly, the z-transform does not converge for all sequences or for all values of z. The set of values of z for which the z-transform converges is called the region of convergence (ROC).

The Fourier transform of $[n]$ exists if the sum converges. However, the z-transform of $[n]$ is just the Fourier transform of the sequence $[n]–n$.

The z-transform therefore exists (or converges) if:

$$X(z) = \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

This leads to the condition for the existence of the z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} |x[n]||z^{-n}| < \infty$$

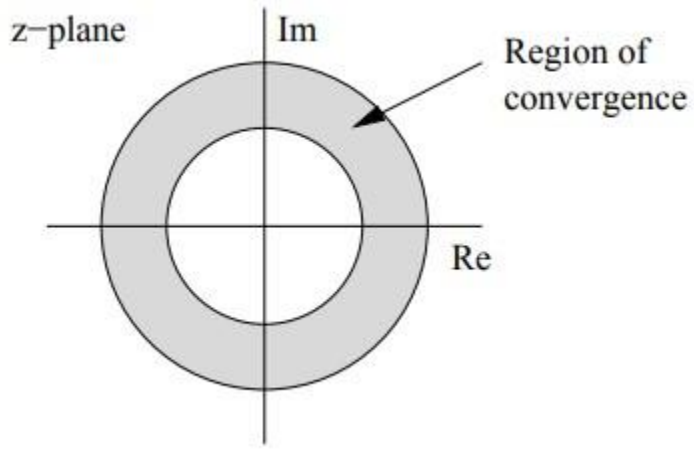

Figure 2: Region of convergence of Z-transform

The ROC therefore consists of a ring in the z-plane:

- In specific cases the inner radius of this ring may include the origin, and the outer radius may extend to infinity.
- If the ROC includes the unit circle |z| = 1, then the Fourier transform will converge.

Most useful z-transforms can be expressed in the form:

$$X(z) = \frac{P(z)}{Q(z)}$$

Where: $P(z)$ and $Q(z)$ are polynomials in z.

The values of z for which $P(z) = 0$ are called the zeros of $X(z)$, and the values with $Q(z) = 0$ are called the poles. The zeros and poles completely specify $X(z)$ to within a multiplicative constant.

Example 1: z-transform of discrete system

Consider the signal:

$$x[n] = a^n u[n]$$

This has the z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=-\infty}^{\infty} (az^{-1})^n$$

Convergence requires that:

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$$

which is only the case if $az-1 < 1$, or equivalently $z > |a|$.

In the ROC, the series converges to since it is just a geometric series:

$$x[n] = a + ar + ar^2 + ar^3 + \cdots + \mathrm{a}r^{n-1}$$

$$= \sum_{k=0}^{n-1} a\, r^k = a\left(\frac{1-r^n}{1-r}\right)$$

As a result

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|$$

The z-transform has a region of convergence for any finite value of a.

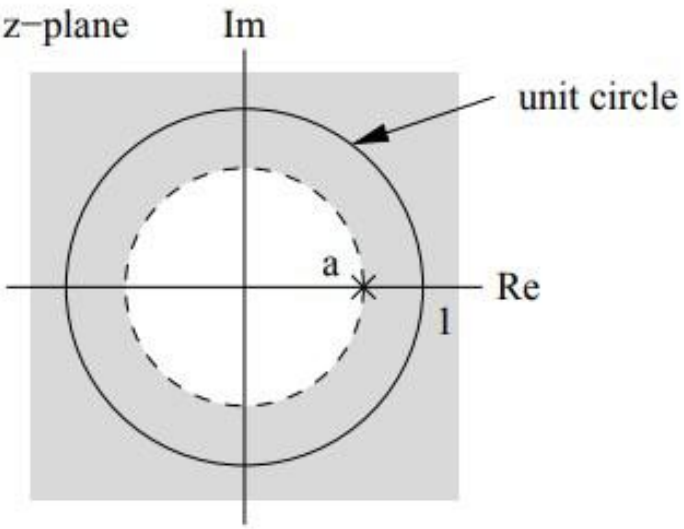

Figure 3: Intersection of region of convergence in Z-transform

The Fourier transform of $x[n]$ only exists if the ROC includes the unit circle, which requires that $|a| < 1$.

On the other hand, if $|a| > 1$ then the ROC does not include the unit circle, and the Fourier transform does not exist.

This is consistent with the fact that for these values of a, the sequence $a^n u[n]$ is exponentially growing, and the sum therefore does not converge.

Example 2: z-transform of discrete system

Now consider the sequence:

$$x[n] = -a^n u[-n-1]$$

This sequence is left-sided because it is nonzero only for $n \leq -1$. The z-transform is:

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n[-n-1]z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

For $|a^{-1}z| < 1$, or $|z| < |a|$, the series converges to:

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|$$

Note that the expression for the z-transform (and the pole zero plot) is exactly the same as for the right-handed exponential sequence — only the region of convergence is different.

Specifying the ROC is therefore critical when dealing with the z-transform.

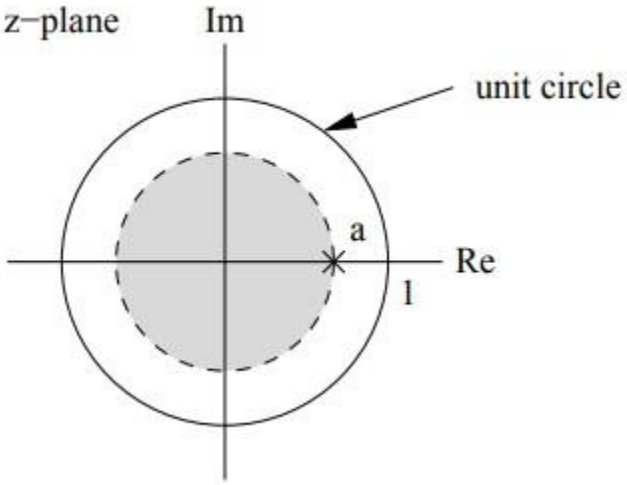

Figure 4: Region of convergence in Z-transform

Example 3: z-transform of discrete system

The signal $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$ is the sum of two real exponentials.

The z-transform is:

$$X(z) = \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \right\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} + \left(-\frac{1}{3}\right)^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n$$

From the example for the right-handed exponential sequence, the first term in this sum converges for $|z| > 1/2$, and the second for $|z| > 1/3$.

The combined transform (z) therefore converges in the intersection of these regions. Namely when $|z| > 1/2$. In this case:

$$X(z) = \frac{1}{1 - 1/2\, z^{-1}} + \frac{1}{1 + 1/3\, z^{-1}} = \frac{2z(z - 1/12)}{(z - 1/2)(z + 1/3)}$$

The pole-zero plot and region of convergence of the signal is:

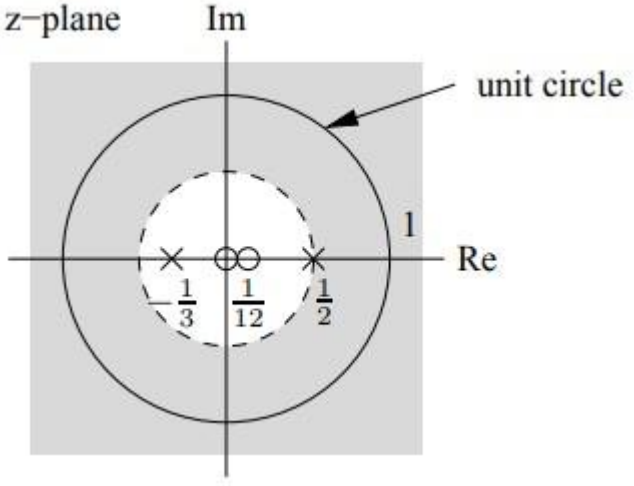

Figure 5: Region of convergence in Z-transform

Example 4: z-transform of discrete system

The signal:

$$f(x) = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

has z-transform:

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

Since there are only a finite number of nonzero terms the sum always converges when az^{-1} is finite. There are no restrictions on a (e.g. $|a| < \infty$), and the ROC is the entire z-plane with the exception of the origin $z = 0$ (where the terms in the sum are infinite).

The N roots of the numerator polynomial are at:

$$z_k = ae^{j(2\pi k/N)}, \quad k = 0,\ 1,\ \cdots,\ N-1$$

since these values satisfy the equation $z^N = a^N$.

The zero at k = 0 cancels the pole at $z = a$, so there are no poles except at the origin, and the zeros are at:

$$z_k = ae^{j(2\pi k/N)}, \quad k = 1,\ \cdots,\ N-1$$

1.2. Property of Region of Convergence

The properties of the ROC depend on the nature of the signal. Assuming that the signal has a finite amplitude and that the z-transform is a rational function.

- The ROC is a ring or disk in the z-plane, centred on the origin ($0 \le r_R < |z| < r_L \le \infty$).
- The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform includes the unit circle. The ROC cannot contain any poles.
- If $x[n]$ is finite duration (i.e. zero except on finite interval $-\infty < N_1 \le n \le N_2 < \infty$), then the ROC is the entire z-plane except perhaps at $z = 0$ or $z = \infty$.
- If $x[n]$ is a right-sided sequence then the ROC extends outward from the outermost finite pole to infinity.

- If $x[n]$ is left-sided then the ROC extends inward from the innermost nonzero pole to z = 0.
- A two-sided sequence (neither left nor right-sided) has a ROC consisting of a ring in the z-plane, bounded on the interior and exterior by a pole (and not containing any poles).
- The ROC is a connected region.

2. Inverse Z-transform

Like Fourier transform, we can also perform inverse transform in z-transform. There are several methods that we can employ to do the inverse transformation e.g. inspection, partial fraction expansion, inverse by partial fraction and power series expansion.

2.1. Inspection method

If one is familiar with (or has a table of) common z-transform pairs, the inverse can be found by inspection. For example, one can invert the z-transform:

$$X(z) = \left(\frac{1}{1 - \frac{1}{2} z^{-1}} \right), \quad |z| > \frac{1}{2}$$

using the z-transform pair:

$$a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}} \qquad \text{for} \quad |z| > |a|$$

By inspection, we recognise that:

$$x[n] = \left(\frac{1}{2} \right)^n u[n]$$

Also, if $X(z)$ is a sum of terms then one may be able to do a term-by-term inversion by inspection, yielding $x[n]$ as a sum of terms.

2.2. Partial Fraction Expansion

For any rational function, we can obtain a partial fraction expansion, and identify the z-transform of each term. Assume that $X(z)$ is expressed as a ratio of polynomials in z^{-1}:

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

It is always possible to factor $X(z)$ as:

$$X(z) = \frac{b_0 \prod_{k=1}^{M}(1 - c_k z^{-1})}{a_0 \prod_{k=1}^{N}(1 - d_k z^{-1})}$$

Where the c_k's and d_k's are the nonzero zeros and poles of $X(z)$.

If the order of equation $M < N$ and the poles are all first order, then $X(z)$ can be expressed as:

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

In this case, the coefficients A_k are given by:

$$A_k = (1 - d_k z^{-1}) X(z)\big|_{z = d_k}$$

If the order of equation $M \geq N$ and the poles are all first order, then an expansion of the form:

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

can be used, and the B_r's be obtained by long division of the numerator by the denominator. The A_k's can be obtained using the same equation as for $M < N$.

The most general form for the partial fraction expansion, which can also deal with multiple-order poles, is:

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k \neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}$$

Ways of finding the C_m's can be found in most standard signal and system textbooks. The terms $B_r z^{-r}$ correspond to shifted and scaled impulse sequences, and invert to terms of the form $B_r \delta[n - r]$. The fractional terms:

$$\frac{A_k}{1 - d_k z^{-1}}$$

correspond to exponential sequences.

For these terms, the ROC properties must be used to decide whether the sequences are left-sided or right-sided.

2.3. Inverse by Partial Fractions

Consider the sequence $x[n]$ with z-transform:

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}$$

Since $M = N = 2$ this can be expressed as:

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

The value B_0 can be found by long division:

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}$$

The coefficients A_1 and A_2 can be found using:

$$A_k = (1 - d_k z^{-1})X(z)\big|_{z = d_k}$$

So:

$$A_1 = \left.\frac{1 + 2z^{-1} + z^{-2}}{1 - z^{-1}}\right|_{z^{-1} = 2} = \frac{1 + 4 + 4}{1 - 2} = -9$$

$$A_2 = \left.\frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{1}{2}z^{-1}}\right|_{z^{-1} = 1} = \frac{1 + 2 + 1}{1/2} = 8$$

Thus

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{(1 - z^{-1})}$$

Using the fact that the ROC is $|z| > 1$, the terms can be inverted one at a time by inspection to give:

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

2.4. Power Series Expansion

If the z-transform is given as a power series in the form:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-1}$$

Thus:

$$= \cdots + x[-2]z^{-1} + x[-1]\, z^{-1} + x[0] + x[1]\, z^{-1} + x[2]\, z^{-2} + \cdots$$

then any value in the sequence can be found by identifying the coefficient of the appropriate power of z^{-1}.

Example 5: Finite-length sequence

The z-transform:

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1})$$

can be multiplied out to give:

$$X(z) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

By inspection, the corresponding sequence is therefore:

$$X[n] = \begin{cases} 1 & n = -2 \\ -\frac{1}{2} & n = -1 \\ -1 & n = 0 \\ \frac{1}{2} & n = 1 \\ 0 & otherwise \end{cases}$$

Or:

$$x[n] = 1\delta[n+2] - \frac{1}{2}\delta[n+1] - 1\delta[n] + \frac{1}{2}\delta[n-1]$$

Example 6: Power series expansion

Consider the z-transform:

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|$$

Using the power series expansion for $\log(1 + x)$, with $|x| < 1$, gives:

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

The corresponding sequence is therefore:

$$x[n] = \begin{cases} (-1)^{n+1}\, a^n/n & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

Example 7: Power series expansion by long division

Consider the transform:

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

Since the ROC is the exterior of a circle, the sequence is right-sided.

We therefore divide to get a power series in powers of z^{-1}:

$$1 + az^{-1} + a^2 z^{-2} + \cdots$$

divided by $1 - z^{-1}$, or :

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \cdots$$

as a result:

$$x[n] = a^n u[n]$$

Example 8: Power series expansion for left-sided sequence

Consider instead the z-transform:

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| < |a|$$

Because of the ROC, the sequence is now a left-sided one. Thus, we divide to obtain a series in powers of z:

$$-a^{-1}z - a^{-2}z^2 - \cdots$$

divided by $-a + z$, thus:

$$z[n] = -a^n u[-n-1]$$

3. Table of z-Transform

The following table shows selective discrete sequences and their transformations in z-transform.

No	Sequence	Transform	ROC
1	$\delta[n]$	1	All z
2	$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$-u[n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4	$\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5	$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6	$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7	$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8	$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9	$\begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
10	$\sin(\omega_0 n)u[n]$	$\frac{\sin\omega_0\, z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
11	$\cos(\omega_0 n)u[n]$	$\frac{1-\cos\omega_0\, z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
12	$[a^n \sin\omega_0 n]u[n]$	$\frac{(a\sin\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\|z\| > \|a\|$
13	$[a^n \cos\omega_0 n]u[n]$	$\frac{1-(a\cos\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\|z\| > \|a\|$

Table 1: Table of transforms of z-transform