XMUT220 Signals and Systems

Z-Transform (Applications)

Topics

1. Introduction to applications of z-transform.
2. Solving discrete-time systems of differential equations.
3. Time-domain response of discrete-time systems.
4. Frequency-domain response of discrete-time systems
5. Analysis of discrete-time systems.

1. Introduction to Applications of z-Transform

There are many applications of z-transform. In this note, we will investigate the applications of z-transform for solving discrete-time systems and differential equations, time-domain response of discrete-time systems, frequency-domain response of discrete-time systems, and analysis of discrete-time systems.

2. Solving Discrete-Time LTI Systems of Differential Equations

First, we will investigate how z-transform is used to solve discrete-time LTI systems as represented by their differential equations and will investigate more applied use of the z-transform in the time and frequency responses of the discrete-time LTI systems and in the analysis of the discrete-time LTI systems.

2.1. Using the z-Transform with Discrete-Time LTI Systems

The output signal of a discrete-time LTI system is related to its input signal and its impulse response through the convolution sum. Specifically, if the impulse response of a discrete-time LTI system is $h[n]$, and if the signal $x[n]$ is applied to the system as input, the output signal $y[n]$ is computed as:

$$y[n] = x[n] * h[n] = \sum_{k}^{\infty} x[k]h[n-k] \qquad (Eq.1)$$

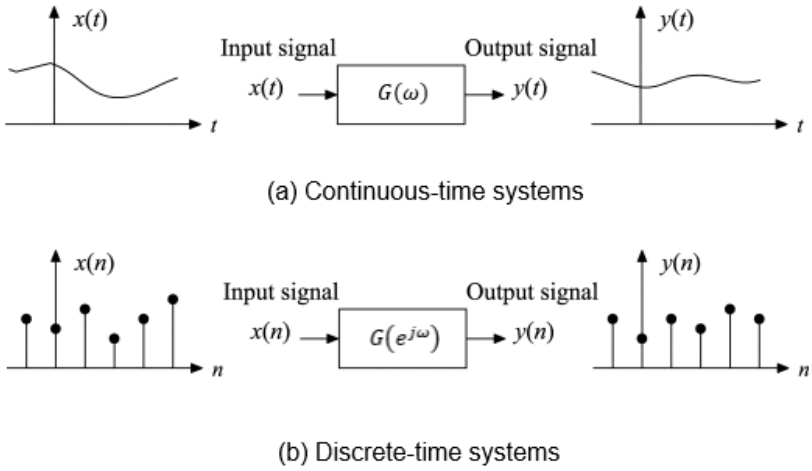


Figure 1: Determining the system function from the difference equation

Based on the convolution property of the *z*-transform, the z-transform of the output signal is equal to the product of the z-transforms of the input signal and the impulse response, that is,

$$Y(z) = X(z)H(z) \qquad (Eq.2)$$

If the input signal $x[n]$ and the impulse response $h[n]$ are specified, we could find the *z*-transforms $X(z)$ and $H(z)$, multiply them to obtain $Y(z)$, and then determine $y[n]$ by means of inverse z-transform. This process provides us an alternative to computing the output by direct application of the convolution sum.

Alternatively, if the input and the output signals $x[n]$ and $y[n]$ are specified, their z-transforms can be computed, and then the z-transform of the impulse response can be determined from them. Solving equation (2) for $H(z)$ we get:

$$H(z) = \frac{Y(z)}{X(z)} \qquad (Eq.3)$$

The function $H(z)$ is the z-domain system function of the system. We already know that a discrete-time LTI system can be described fully and uniquely by means of its impulse response $h[n]$. Since the system function $H(z)$ is just the z-transform of the impulse response $h[n]$, it also represents a complete description of the discrete-time LTI system.

2.2. Relating the system function to the difference equation

We have established the fact that a discrete-time LTI system can be described by means of a constant-coefficient linear difference equation in the standard form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad (Eq.4)$$

Therefore, it must be possible to obtain the other two forms of description of the discrete-time LTI system, namely the system function and the impulse response, from the knowledge of its difference equation.

If we take the z-transform of both sides of equation (4), the equality would still be valid:

$$Z\left\{\sum_{k=0}^{N} a_k y[n-k]\right\} = Z\left\{\sum_{k=0}^{M} b_k x[n-k]\right\} \qquad (Eq.5)$$

Using the linearity property of the z-transform, the equation (5) can be written as:

$$\sum_{k=0}^{N} a_k (Z) y[n-k] = \sum_{k=0}^{M} b_k (Z) x[n-k] \qquad (Eq.6)$$

and using the time-shifting property of the z-transform leads to:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \qquad (Eq.7)$$

The transforms $X(z)$ and $Y(z)$ do not depend on the summation index k, and can therefore be factored out of the summations in the equation (7) resulting in:

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k} \qquad (Eq.8)$$

Finally, the system function can be obtained from the equation (8) as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad (Eq. 9)$$

Finding the system function from the difference equation:

1. Separate the terms of the difference equation so that $y[n]$ and its time-shifted versions are on the left of the equal sign, and $x[n]$ and its time-shifted versions are on the right of the equal sign, as in the equation (4).
2. Take the z-transforms of both sides of the difference equation and use the time-shifting property of the z-transform as in the equation (7).
3. Determine the system function as the ratio of $Y(z)$ to $X(z)$ as in the equation (9).
4. If the impulse response is needed, it can now be determined as the inverse *z*-transform of $H(z)$.

At this point we will make two important observations:

1. In the above development leading up to the z-domain system function we have relied on the convolution operation in the equation (2). In the equation (9), we have used the convolution property of the z-transform. We know that the convolution operation is only applicable to problems involving linear and time-invariant systems.

   Therefore, the system function concept is meaningful only for systems that are both linear and time-invariant. This notion was introduced in earlier discussions involving system functions as well.

2. Furthermore, it was justified that a constant-coefficient difference equation corresponds to a linear and time-invariant system only if all initial conditions are set to zero.

Consequently, we conclude that, in determining the system function from the difference equation, all initial conditions must be assumed to be zero.

If we need to use *z*-transform based techniques to solve a difference equation subject to non-zero initial conditions, we can do that with the unilateral *z*-transform, but not by the system function.

Example 1: Finding the system function from the difference equation

Let a discrete-time LTI system be defined by the difference equation

$$y[n] - 0.4y[n-1] + 0.89y[n-2] = x[n] - x[n-1]$$

Find the system function $H(z)$ for this system. [Answer: $H(z) = z(z-1)/(z^2 - 0.4z + 0.89)$] [6 marks]

Solution

We will assume zero initial conditions and take the z-transforms of both sides of the difference equation to get:

$$Y(z) - 0.4z^{-1}Y(z) + 0.89z^{-2}Y(z) = X(z) - z^{-1}X(z)$$

Rearranging the equation above in terms of $H(z) = Y(z)/X(z)$, thus:

$$Y(z)[1 - 0.4z^{-1} + 0.89z^{-2}] = Z(z)[1 - z^{-1}\,]$$

From which the system function is obtained as:

$$H(z) = \frac{1 - z^{-1}}{1 - 0.4z^{-1} + 0.89z^{-2}}$$

or, using non-negative powers of z:

$$H(z) = \frac{z(z-1)}{z^2 - 0.4z + 0.89}$$

We will make another important observation based on the result obtained in Example 8.33: The characteristic equation for the system considered in the Example 1 is:

$$z^2 - 0.4z + 0.89 = 0$$

and the solutions of the characteristic equation are the modes of the system. When we find the system function $H(z)$ from the difference equation we see that its denominator polynomial is identical to the characteristic polynomial.

The roots of the denominator polynomial are the poles of the system function in the *z*-domain, and consequently, they are identical to the modes of the difference equation of the system.

Recall that we have reached some conclusions about the relationship between the modes of the difference equation and the natural response of the corresponding system.

The same conclusions would apply to the poles of the system function. Specifically, if all poles p_k of the system are real-valued and distinct, then the natural response is in the form:

$$y[n] = \sum_{k=1}^{N} c_k p_k^n$$

Complex poles appear in conjugate pairs provided that all coefficients of the system function are real-valued. A pair of complex conjugate poles:

$$p_{1a} = r_1 e^{j\Omega_1} \qquad \text{and} \qquad p_{1b} = r_1 e^{-j\Omega_1}$$

yields a response of the type:

$$y_1[n] = d_1 r_1^n \cos(\Omega_1 n + \theta_1)$$

Finally, a pole of multiplicity m at $z = p_1$ leads to a response in the form:

$$y_1[n] = c_{11} p_1^n + c_{12} n p_1^n + \ldots + c_1 m n^{m-1} p_1^n + \text{other terms}$$

Justifications for these relationships were through the use of the modes of the difference equation.

Sometimes we need to reverse the problem represented in Example 1 and find the difference equation from the knowledge of the system function. The next example will demonstrate this.

Example 2: Finding the difference equation from the system function

Find the difference equation of the discrete-time LTI system defined by the system function:

$$H(z) = \frac{z^2 - 5z + 6}{z^3 + 2z^2 - z - 2}$$

[Answer: $y[n] + 2y[n-1] - y[n-2] - 2y[n-3] = x[n-1] - 5x[n-2] + 6x[n-3]$] [8 marks]

Solution

Let us first write $H(z)$ as a rational function of z^{-1} by multiplying both its numerator and denominator with z^{-3}:

$$H(z) = \frac{z^{-1} - 5z^{-2} + 6z^{-3}}{1 + 2z^{-1} - z^{-2} - 2z^{-3}}$$

We know that the system function is the ratio of the output transform to the input transform, that is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} - 5z^{-2} + 6z^{-3}}{1 + 2z^{-1} - z^{-2} - 2z^{-3}}$$

So that we can write:

$$(1 + 2z^{-1} - z^{-2} - 2z^{-3})Y(z) = (z^{-1} - 5z^{-2} + 6z^{-3})X(z)$$

Thus

$$Y(z) + 2z^{-1}Y(z) - z^{-2}Y(z) - 2z^{-3}Y(z) = z^{-1}X(z) - 5z^{-2}X(z) + 6z^{-3}X(z)$$

or equivalently

$$y[n] + 2y[n-1] - y[n-2] - 2y[n-3] = x[n-1] - 5x[n-2] + 6x[n-3]$$

3. Time-Domain Response of Discrete-Time Systems

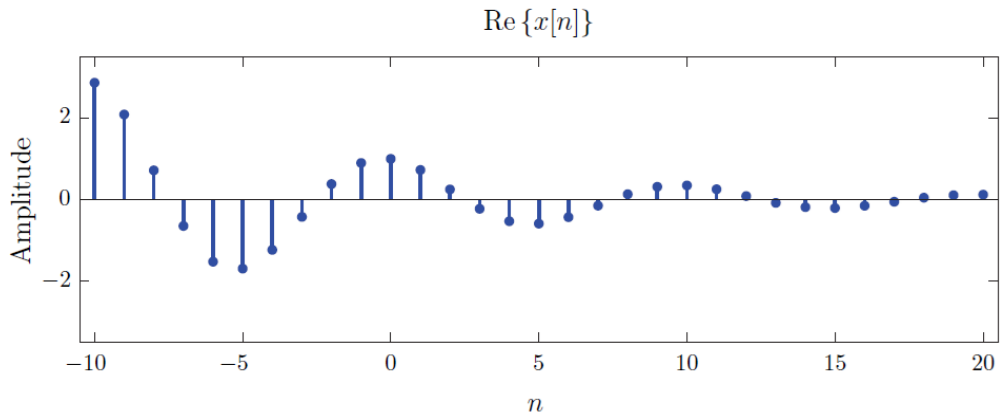
We will investigate into two case studies of time-domain response of the discrete-time systems. First, response of the discrete-time LTI system to exponential signal and second, exponentially damped sinusoid.

3.1. Response of a Discrete-Time LTI system to complex exponential signal

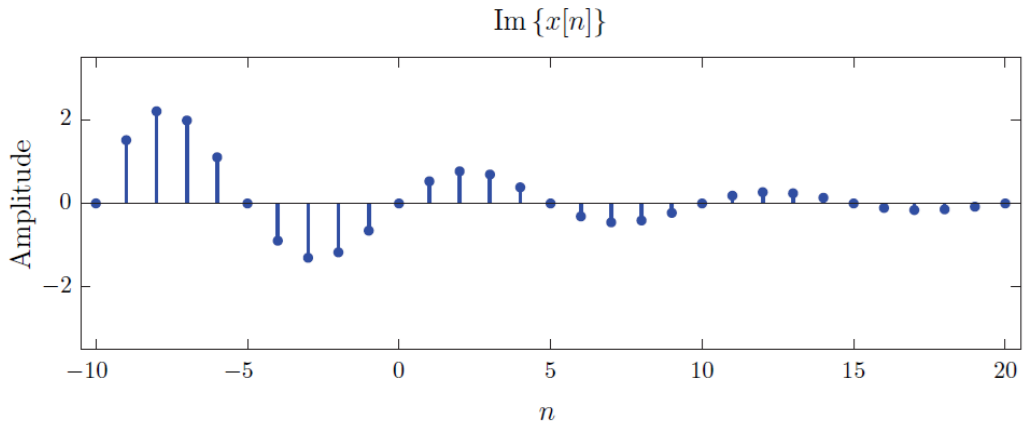
An interesting interpretation of the system function concept is obtained when one considers the response of a discrete-time LTI system to a complex exponential signal in the form:

$$x[n] = z_0^n$$

where z_0 represents a point in the z-plane within the ROC of the system function.



(a): Real part of input signal $x[n]$



(b): Imaginary part of input signal $x[n]$

Figure 2: A complex exponential input signal

Let $h[n]$ be the impulse response of the discrete-time LTI system under consideration. The output signal is determined using the convolution sum:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \qquad (Eq.10)$$

Substituting $x[n-k] = z_0^{n-k}$ into the equation (1) and simplifying the resulting summation we obtain:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] z_0^{n-k} = z_0^n \sum_{k=-\infty}^{\infty} h[k] z_0^{-k} \qquad (Eq. 11)$$

In the equation (11) above, the summation corresponds to the system function $H(z)$ evaluated at the point $z = z_0$, so the response of the discrete-time LTI system to the input signal $x[n] = z_0^n$ is:

$$y[n] = z_0^n H(z_0) \qquad (Eq. 12)$$

We conclude that the response of a discrete-time LTI system to the exponential signal $x[n] = z_0^n$ is a scaled version of the input signal. The scale factor is the value of the system function at the point $z = z_0$.

Example 3: Response of a discrete-time LTI system to a complex exponential signal

Consider a discrete-time LTI system with the system function:

$$H(z) = \frac{z+1}{z^2 - \frac{5}{6}z + \frac{1}{6}}$$

a. Write $x[n]$ in Cartesian form. [Answer: $x[n] = (0.9)^n \cos(0.2\pi n) + j(0.9)^n \sin(0.2\pi n)$] [10 marks]

b. Simulate the real and imaginary parts of $x[n]$ in MATLAB. [4 marks]

c. Find the response of the system to an exponential signal: [6 marks]

$$x[n] = \left(0.9e^{j0.2\pi}\right)^n$$

[Answer: $y[n] = 4.7526(0.9)^n \cos(0.2\pi n - 1.7963) + j4.7526(0.9)^n \sin(0.2\pi n - 1.7963)$]

d. Simulate the real and imaginary parts of $y[n]$ in MATLAB. [4 marks]

Solution

a. The input signal is complex-valued.

$$x[n] = \left(0.9e^{j0.2\pi}\right)^n$$

Using Euler's formulae,

$$r(\cos\theta + j\sin\theta) = r^{j\theta}$$

we can write $x[n]$ as:

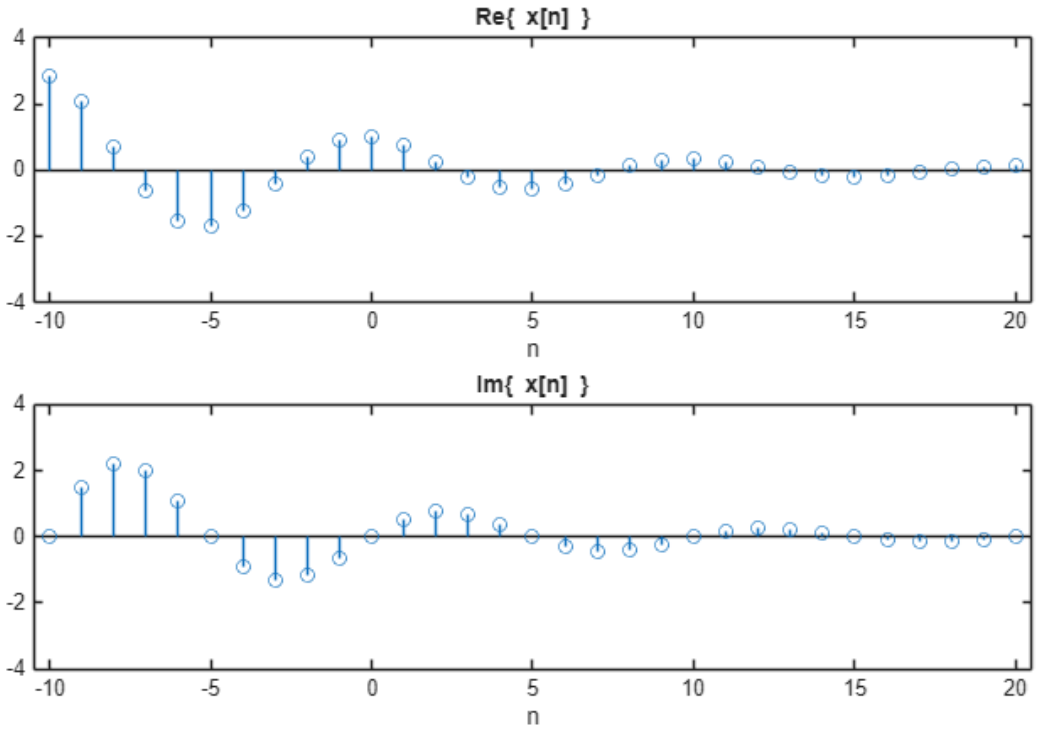
$$x[n] = (0.9)^n \cos(0.2\pi n) + j(0.9)^n \sin(0.2\pi n)$$

b. Real and imaginary parts of the input signal, $x[n]$ are simulated using the MATLAB script listed below.

MATLAB Script:

```
% Create a vector of sample indices.
n = [-10:20];
% Compute the input signal.
x = (0.9*exp(j*0.2*pi)).^n;
% Graph real and imaginary parts of the input signal.
clf;
subplot(2,1,1);
stem(n,real(x));
axis([-10.5,20.5,-4,4]);
xlabel('n');
title('Re\{ x[n] \}');
subplot(2,1,2);
stem(n,imag(x));
axis([-10.5,20.5,-4,4]);
xlabel('n');
title('Im\{ x[n] \}');
```

The following figures show the real and imaginary parts of input signal $x[n]$ of the MATLAB simulation.



c. The value of the system function at $z_0 = 0.9e^{j0.2\pi}$ is:

$$H(0.9e^{j0.2\pi}) = \frac{0.9e^{j0.2\pi} + 1}{(0.9e^{j0.2\pi})^2 - \frac{5}{6}(0.9e^{j0.2\pi}) + \frac{1}{6}}$$

$$= -1.0627 - j4.6323$$

$$= 4.7526e^{-j1.7963}$$

The output signal $y[n]$ is found using equation below as:

$$y[n] = z_0^n H(z_0)$$

$$= \left(0.9e^{j0.2\pi}\right)^n (4.7526e^{-j1.7963})$$

$$= 4.7526(0.9)^n e^{j(0.2\pi n - 1.7963)}$$

or, in Cartesian form:

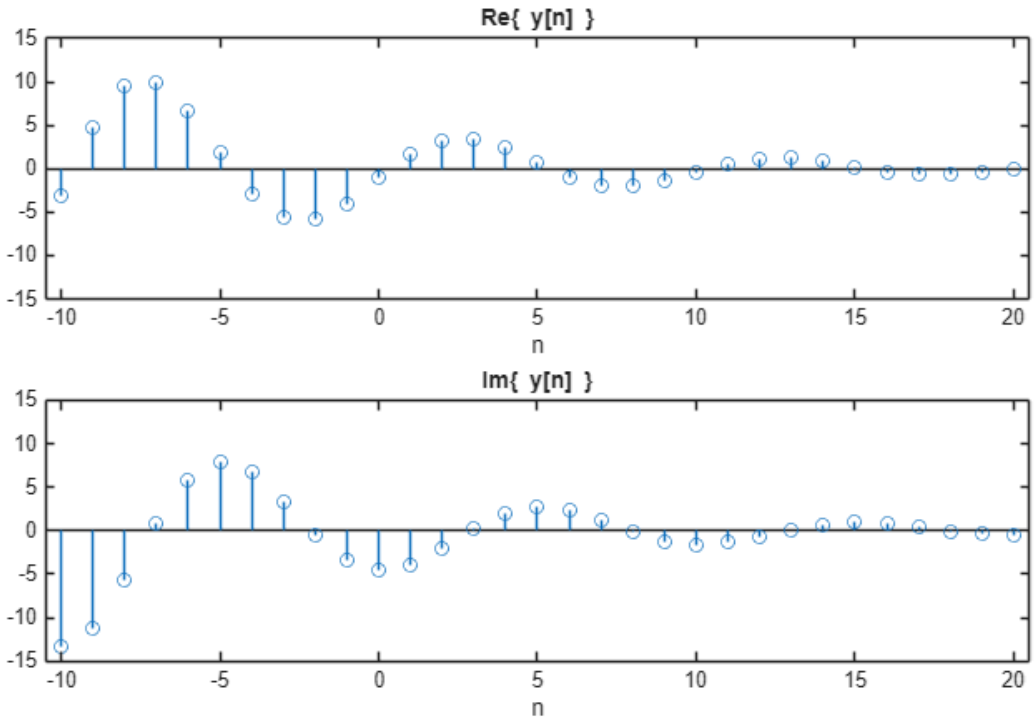
$$y[n] = 4.7526(0.9)^n \cos(0.2\pi n - 1.7963) + j4.7526(0.9)^n \sin(0.2\pi n - 1.7963)$$

d. Real and imaginary parts of the output signal $y[n]$ is simulated using the MATLAB script listed below.

MATLAB Script:

```
% Define an anonymous function for H(z).
H = @(z) (z+1)./(z.*z-5/6*z+1/6);
% Create a vector of sample indices.
n = [-10:20];
% Compute the input signal.
x = (0.9*exp(j*0.2*pi)).^n;
% Evaluate the system function for z=0.9\,e^{j0.2\pi}
Htmp = H(0.9*exp(j*0.2*pi))
% Compute the output signal.
y = Htmp*x;
% Graph real and imaginary parts of the output signal.
clf;
subplot(2,1,1);
stem(n,real(y));
axis([-10.5,20.5,-15,15]);
xlabel('n');
title('Re\{ y[n] \}');
subplot(2,1,2);
stem(n,imag(y));
axis([-10.5,20.5,-15,15]);
xlabel('n');
title('Im\{ y[n] \}');
```

The following figures show the real and imaginary parts of output signal $y[n]$ of the MATLAB simulation.



3.2. Response of a DTLTI system to exponentially damped sinusoid

Next, we will consider an input signal in the form of an exponentially damped sinusoid such as:

$$x[n] = r_0^n \cos(\Omega_0 n) \qquad (Eq.\,13)$$

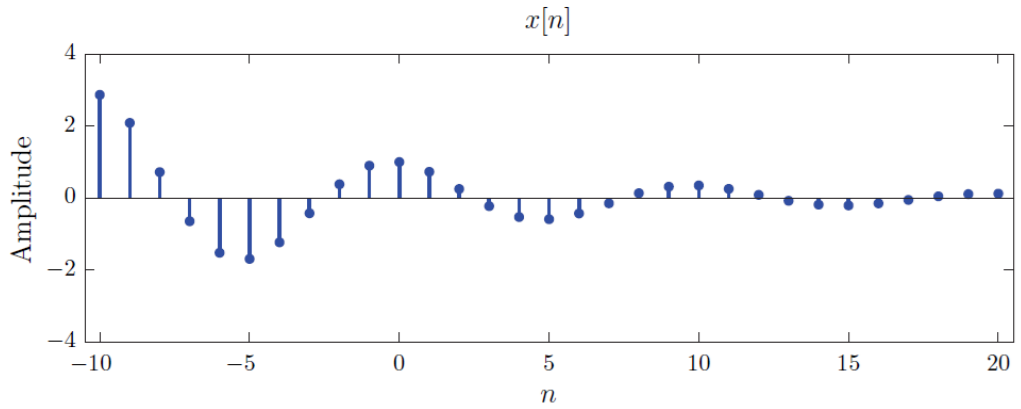


Figure 3: A complex exponentially damped sinusoid input signal

To find the response of a discrete-time LTI system to this signal, let us write $x[n]$ using Euler's formula:

$$x[n] = 0.5r_0^n e^{j\Omega_0 n} + 0.5r_0^n e^{-j\Omega_0 n} \qquad (Eq.14)$$

Let z_0 be defined as:

$$z_0 = r_0 e^{j\Omega_0}$$

It follows that:

$$z_0^* = r_0 e^{-j\Omega_0}$$

and the equation (14) can be written as:

$$x[n] = 0.5Z_0^n + 0.5(Z_0^*)^n \qquad (Eq.15)$$

The response of the system is determined using the linearity of the system:

$$\begin{aligned} y[n] &= \text{Sys}\{0.5Z_0^n + 0.5(Z_0^*)^n\} \\ &= 0.5\text{Sys}\{Z_0^n\} + 0.5\text{Sys}\{(Z_0^*)^n\} \qquad (Eq.16) \end{aligned}$$

We already know that the response of the system to the term z_0^n is:

$$\text{Sys}\{z_0^n\} = z_0^n H(z_0) \qquad (Eq.17)$$

and its response to the term $(z_0^*)^n$ is:

$$\text{Sys}\{(z_0^*)^n\} = (z_0^*)^n H(z_0^*) \qquad (Eq.18)$$

so that the output signal $y[n]$ can be written as:

$$y[n] = 0.5z_0^n H(z_0) + 0.5(z_0^*)^n H(z_0^*) \qquad (Eq.19)$$

The result obtained in the equation (19) can be further simplified. Let the value of the system function evaluated at the point $z = z_0$ be written in polar complex form as:

$$H(z_0) = H_0 e^{j\theta_0} \qquad (Eq.20)$$

Where: H_0 and θ_0 represent the magnitude and the phase of the system function at the point $z = z_0$ respectively:

$$H_0 = |H(z_0)| \qquad (Eq.21)$$

and

$$\theta_0 = \angle H(z_0) \qquad (Eq.22)$$

It can be shown that for a system with a real-valued impulse response, the value of the system function at the point $z = z_0^*$ is the complex conjugate of its value at the point $z = z_0$, that is,

$$H(z_0^*) = [H(z_0)]^* = H_0 e^{-j\theta_0} \qquad (Eq.\,23)$$

Using the equation (20) and (23) in the equation (19), the output signal $y[n]$ becomes:

$$\begin{aligned} y[n] &= 0.5 z_0^n H_0 e^{j\theta_0} + 0.5 (z_0^*)^n H_0 e^{-j\theta_0} \\ &= 0.5 \left(r_0^n e^{j\Omega_0}\right)^n H_0 e^{j\theta_0} + 0.5 \left(r_0^n e^{-j\Omega_0}\right)^n H_0 e^{-j\theta_0} \\ &= 0.5 r_0^n H_0 \left[e^{j(\Omega_0 n + \theta_0)} - e^{-j(\Omega_0 n + \theta_0)}\right] \\ &= r_0^n H_0 \cos(\Omega_0 n + \theta_0) \qquad (Eq.\,24) \end{aligned}$$

Comparison of the input signal in the equation (13) and the output signal equation (24) reveals the following:

1. The amplitude of the signal is multiplied by the magnitude of the system function evaluated at the point $z = r_0 e^{j\Omega}$.
2. The phase of the cosine function is incremented by an amount equal to the phase of the system function evaluated at the point $z = r_0 e^{j\Omega}$.

Example 4: Response of a discrete-time LTI system to an exponentially damped sinusoid

Considering again the system function used in the Example 3,

$$H(z) = \frac{z + 1}{z^2 - \frac{5}{6}z + \frac{1}{6}}$$

determine the response of the system to an exponentially damped sinusoid input signal:

$$x[n] = (0.9)^n \cos(0.2\pi n)$$

a. Obtain the output signal, $y[n]$ in Cartesian form. [Answer: $y[n] = 4.7526\,(0.9)^n \cos(0.2\pi n - 1.7963)$] [6 marks]

b. Simulate the input signal $x[n]$ and output signal $y[n]$ in MATLAB. [4 marks]

Solution

a. In the Example 3, we have evaluated the system function at the point $z = 0.9e^{j0.2\pi}$ and have obtained:

$$H(0.9e^{j0.2\pi}) = H_0 e^{j\theta_0} = 4.7526e^{-j1.7963}$$

Using the equation given below,

$$y[n] = H_0 r_0^n \cos(\Omega_0 n + \theta_0)$$

With $H_0 = 4.7526$ and $\theta_0 = -1.7963$ radians, we obtain the output signal, $y[n]$ as:

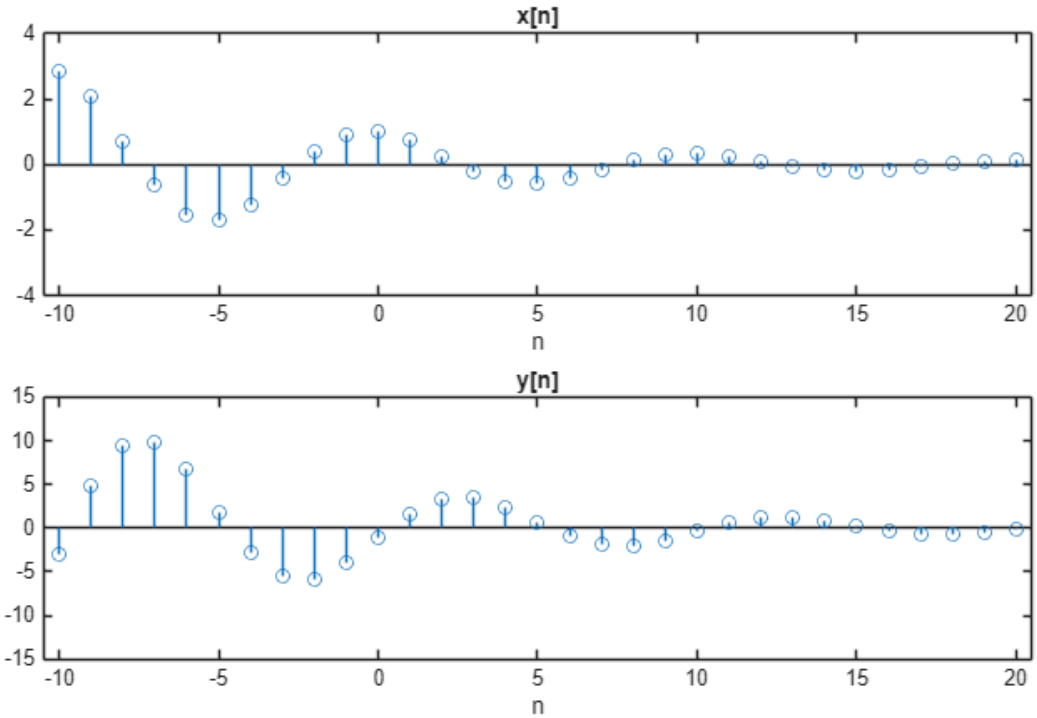
$$y[n] = 4.7526(0.9)^n \cos(0.2\pi n - 1.7963)$$

b. The MATLAB simulation results for the input and the output signals are simulated using the script listed below.

MATLAB script:

```
% Define an anonymous function for H(z).
H = @(z) (z+1)./(z.*z-5/6*z+1/6);
% Create a vector of sample indices.
n = [-10:20];
% Compute the input signal.
x = (0.9).^n.*cos(0.2*pi*n);
% Evaluate the system function for z=0.9*exp(j*0.2*pi).
Htmp = H(0.9*exp(j*0.2*pi))
H0 = abs(Htmp)
Theta0 = angle(Htmp)
% Compute the output signal.
y = H0*(0.9).^n.*cos(0.2*pi*n+Theta0);
% Graph input and output signala.
clf;
subplot(2,1,1);
stem(n,x);
axis([-10.5,20.5,-4,4]);
xlabel('n');
title('x[n]');
subplot(2,1,2);
stem(n,y);
axis([-10.5,20.5,-15,15]);
xlabel('n');
title('y[n]');
```

The following figures show the input signal $x[n]$ and the output signal $y[n]$ of the MATLAB simulation.



3. Frequency Response of Discrete-Time Systems

To examine the frequency response of discrete-time systems, we can specialise the z transform to the DTFT through the transformation $z \to e^{j\Omega}$ with Ω being a real variable representing discrete-time radian frequency.

The fact that Ω is real means that in determining frequency response the only values of z that we are now considering are those on the unit circle in the z-plane because $|e^{j\Omega}| = 1$ for any real Ω.

This is directly analogous to determining the frequency response of a continuous-time system by examining the behaviour of its s-domain transfer function as s moves along the ω-axis in the s-plane, and a similar graphical technique can be used.

Suppose the transfer function of a system is:

$$H(z) = \frac{z}{(z-p_1)(z-p_2)} = \frac{z}{z^2 - \frac{z}{2} + \frac{5}{16}} \qquad (Eq.25)$$

Where:

$$p_1 = \frac{1+j2}{4} \quad \text{and} \quad p_2 = \frac{1-j2}{4}$$

The transfer function has a zero at the origin and two complex-conjugate finite poles as shown in the figure below.

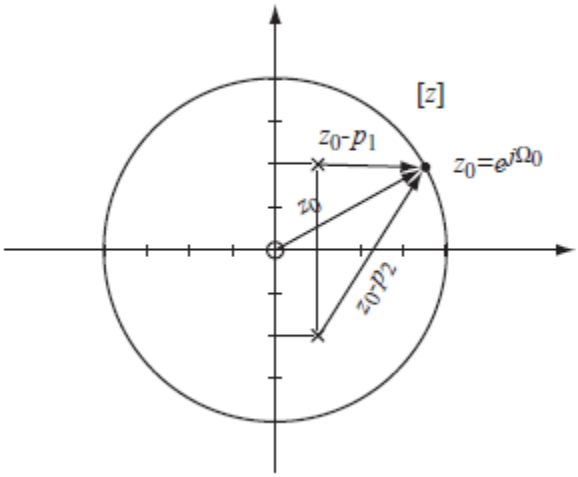


Figure 4: z-domain pole-zero diagram of a system transfer function

The frequency response of the system at any particular radian frequency Ω_0 is determined (to within a multiplicative constant) by the vectors from the finite poles and finite zeros of the transfer function to the *z*-plane point $z_0 = e^{j\Omega_0}$.

The magnitude of the frequency response is the product of the magnitudes of the zero vectors divided by the product of the magnitudes of the pole vectors. In this case,

$$\left|H\left(e^{j\Omega}\right)\right| = \frac{\left|e^{j\Omega}\right|}{|e^{j\Omega} - p_1||e^{j\Omega} - p_2|} \qquad (Eq.26)$$

It is apparent that as $e^{j\Omega}$ approaches a pole, p_1 for example, the magnitude of the difference $e^{j\Omega} - p_1$ becomes small, making the magnitude of the denominator small and tending to make the magnitude of the transfer function larger. The opposite effect occurs when $e^{j\Omega}$ approaches a zero.

The phase of the frequency response is the sum of the angles of the zero vectors minus the sum of the angles of the pole vectors. In this case, as shown in the figure below,

$$\measuredangle H(e^{j\Omega}) = \measuredangle e^{j\Omega} - \measuredangle(e^{j\Omega} - p_1) - \measuredangle(e^{j\Omega} - p_2) \qquad (Eq.27)$$

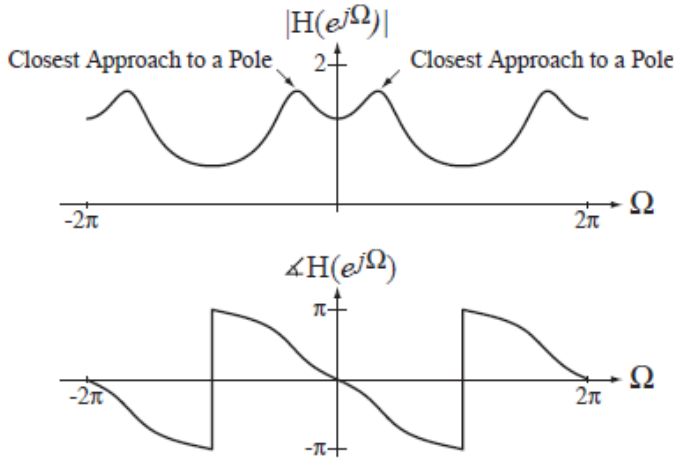


Figure 5: Magnitude and phase frequency response of the given system

The maximum magnitude frequency response occurs at approximately $z = e^{\pm j1.11}$, which are the points on the unit circle at the same angle as the finite poles of the transfer function and, therefore, the points on the unit circle at which the denominator factors $e^{j\Omega_0} - p_1$ and $e^{j\Omega_0} - p_2$ in equation (27) reach their minimum magnitudes.

An important difference between the frequency response of continuous-time and discrete-time systems is that, for discrete-time systems, the frequency response is always periodic, with period 2π in Ω.

That difference can be seen directly in this graphical technique because as Ω moves from zero in a positive direction, it traverses the entire unit circle in a counterclockwise direction and then, on its second traversal of the unit circle, retraces its previous positions, repeating the same frequency responses found on the first traversal.

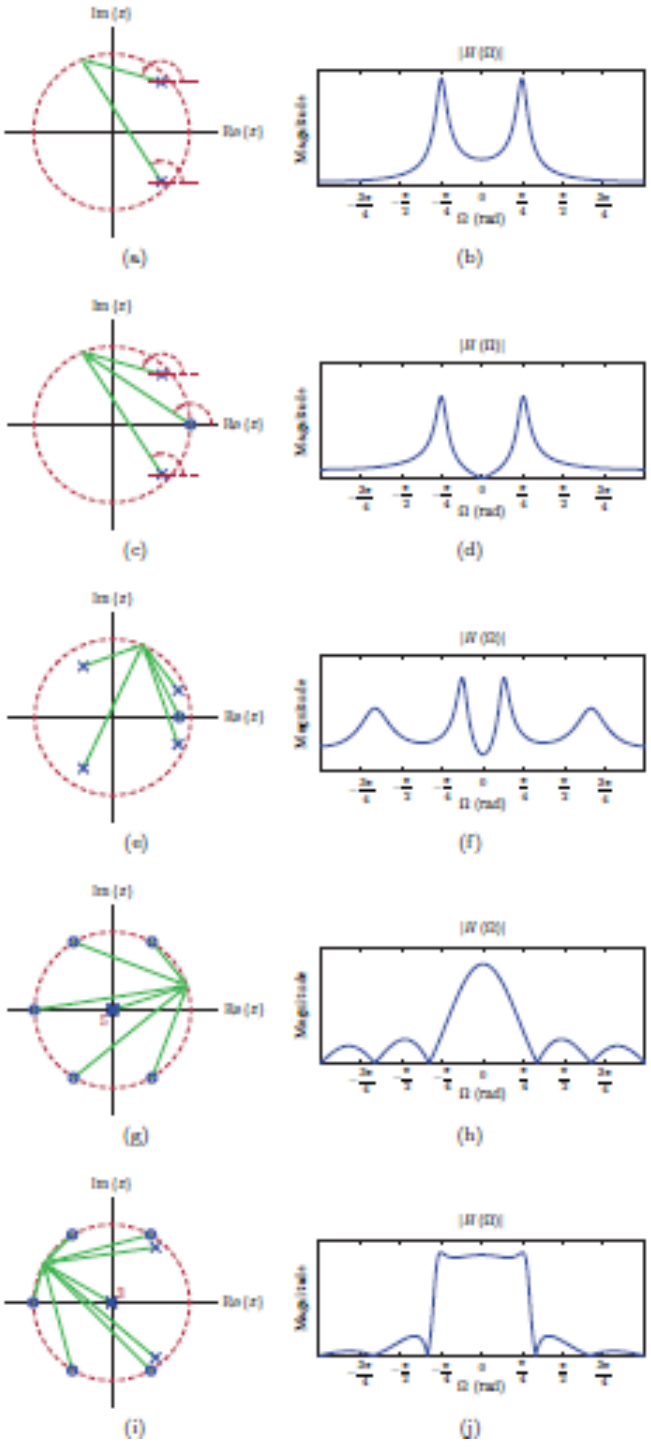


Figure 6: Several pole-zero configurations and the corresponding magnitude responses

Example 5: Frequency Response of System 1

Sketch the pole-zero diagram of the frequency response for the system with its transfer function is given as: [8 marks]

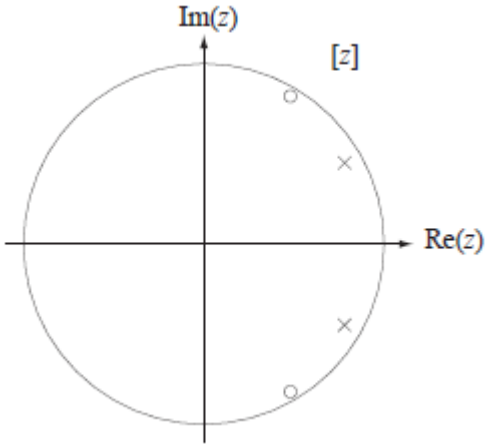
$$H(z) = \frac{z^2 - 096z + 0.9028}{z^2 - 1.56z + 0.8109}$$

Solution

The transfer function can be factored into:

$$H(z) = \frac{(z - 0.48 + j0.82)(z - 0.48 - j0.82)}{(z - 0.78 + j0.45)(z - 0.78 - j0.45)}$$

Using the roots in the polynomial equation given above, the pole-zero diagram of the system is shown in the figure below.

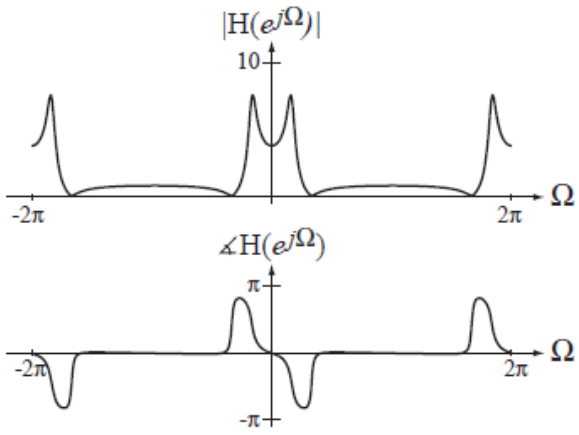


As shown in the diagram, a pair of the zeros of the system are located on $0.48 \pm j0.82$ and a pair of poles are at $0.78 \pm j0.45$.

The magnitude and phase plots of the frequency response of the system are illustrated in figure below.

Notice on the magnitude plot, that the gain of the system is maximum around the zeros at $e^{0.48 \pm j0.82}$ and around poles at $e^{0.78 \pm j0.45}$.

$$\left|H\left(e^{j\Omega}\right)\right| = \frac{\left|e^{j\Omega} - z_1\right|\left|e^{j\Omega} - z_2\right|}{\left|e^{j\Omega} - p_1\right|\left|e^{j\Omega} - p_2\right|}$$



Then, on the phase plot, the phase of the system is maximum around the zeros at $e^{0.48\pm j0.82}$ and minimum around the poles at $e^{0.78\pm j0.45}$.

$$\measuredangle H(e^{j\Omega}) = \measuredangle(e^{j\Omega} - z_1) + \measuredangle(e^{j\Omega} - z_2) - \measuredangle(e^{j\Omega} - p_1) - \measuredangle(e^{j\Omega} - p_2)$$

Example 6: Frequency Response of System 2

A system with transfer function given in the equation below is excited by a unit sequence.

$$H(z) = \frac{z}{(z-0.3)(z+0.8)} \qquad |z| > 1$$

Find the total response of the system using the following method:

a. Z-transform. [Answer: $y[n] = [-0.1169(0.3)^{n-1} + 0.3232(-0.8)^{n-1} + 0.7937]u[n-1]$] [10 marks]

b. Discrete-time Fourier transform. Comment on the difference between two methods. [Answer: $y[n] = [-0.1169(0.3)^{n-1} + 0.3232(-0.8)^{n-1} + 0.7937]u[n-1]$ and relatively more complex process with DTFT than z-transform] [32 marks]

Solution

a. The z transform of the response is:

$$Y(z) = H(z)X(z) = \left[\frac{z}{(z-0.3)(z+0.8)}\right]\left(\frac{z}{z-1}\right) \qquad |z| > 1$$

Expanding in partial fractions,

$$Y(z) = \frac{z^2}{(z-0.3)(z+0.8)(z-1)} = \frac{0.1169}{z-0.3} + \frac{0.3232}{z+0.8} + \frac{0.7937}{z-1} \qquad |z| > 1$$

Therefore, the total response is:

$$y[n] = [-0.1169(0.3)^{n-1} + 0.3232(-0.8)^{n-1} + 0.7937]u[n\ -1]$$

b. This problem can also be analysed using the discrete-time Fourier transform, but the notation is significantly clumsier, mainly because the discrete-time Fourier transform of a unit sequence is:

$$\frac{1}{1-e^{-j\Omega}} + \pi\delta_{2\pi}(\Omega)$$

The system frequency response is:

$$H(e^{j\Omega}) = \frac{e^{j\Omega}}{(e^{j\Omega}-0.3)(e^{j\Omega}+0.8)}$$

The discrete-time Fourier transform of the system response is:

$$\begin{aligned} Y(e^{j\Omega}) &= H(e^{j\Omega}).X(e^{j\Omega}) \\ &= \frac{e^{j\Omega}}{(e^{j\Omega}-0.3)(e^{j\Omega}+0.8)}\left(\frac{1}{1-e^{j\Omega}} + \pi\delta_{2\pi}(\Omega)\right) \end{aligned}$$

Or

$$\begin{aligned} Y(e^{j\Omega}) &= H(e^{j\Omega})X(e^{j\Omega}) \\ &= \frac{e^{j\Omega}}{(e^{j\Omega}-0.3)(e^{j\Omega}+0.8)(1-e^{j\Omega})} + \pi\left[\frac{e^{j\Omega}}{(e^{j\Omega}-0.3)(e^{j\Omega}+0.8)}\right]\delta_{2\pi}(\Omega) \end{aligned}$$

Expanding in partial fractions,

$$Y(e^{j\Omega}) = \frac{-0.1169}{e^{j\Omega}-0.3} + \frac{0.3232}{e^{j\Omega}+0.8} + \frac{0.7937}{e^{j\Omega}-1} + \left(\frac{\pi}{(1-0.3)(1+0.8)}\right)\delta_{2\pi}(\Omega)$$

Using the equivalence property of the impulse and the periodicity of both $\delta_{2\pi}(\Omega)$ and $e^{j\Omega}$, the equation above becomes.

$$Y(e^{j\Omega}) = \frac{-0.1169e^{-j}}{1-0.3e^{-j\Omega}} + \frac{0.3232e^{-j\Omega}}{1+0.8e^{-j\Omega}} + \frac{0.7937e^{-j\Omega}}{1-e^{-j\Omega}} + 2.4933\delta_{2\pi}(\Omega)$$

Then, manipulating this expression into a form for which the inverse discrete-time Fourier transform is direct:

$$Y(e^{j\Omega}) = \frac{0.2269e^{-j\Omega}}{1-0.3e^{-j\Omega}} + \frac{0.3232e^{-j\Omega}}{1+0.8e^{-j\Omega}} + 0.7937\left[\frac{e^{-j\Omega}}{1-e^{-j\Omega}} + \delta_{2\pi}(\Omega)\right] - 0.7937\pi\delta_{2\pi}(\Omega) + 2.4933\delta_{2\pi}(\Omega)$$

With $0.7937\pi\delta_{2\pi}(\Omega) + 2.4933\delta_{2\pi}(\Omega) = 0$, thus:

$$Y(e^{j\Omega}) = \frac{0.2269e^{-j\Omega}}{1-0.3e^{-j\Omega}} + \frac{0.3232e^{-j\Omega}}{1+0.8e^{-j\Omega}} + 0.7937\left[\frac{e^{-j\Omega}}{1-e^{-j\Omega}} + \delta_{2\pi}(\Omega)\right]$$

And, finally, taking the inverse discrete-time Fourier transform of the equation above:

$$y[n] = [-0.1169(0.3)^{n-1} + 0.3232(-0.8)^{n-1} + 0.7937]u[n-1]$$

The result is the same, but the effort and the probability of error are considerably greater.

Example 7: Frequency Response of System 3

A system with transfer function given below is excited by the sinusoid input $x[n] = \cos(2\pi n/12)$.

$$H(z) = \frac{z}{z-0.9} \qquad |z| > 0.9$$

Find the response of the system using the following transformation methods:

a. Discrete-time Fourier transform. [Answer: $y[n] = 0.1217(0.9)^n u[n] + 1.995\cos(2\pi n/12 - 1.115)\,u[n]$] [20 marks]

b. z-transform. [Answer: $y[n] = 0.1217(0.9)^n u[n] + 1.995\cos(2\pi n/12 - 1.115)\,u[n]$] [14 marks]

Solution

a. The excitation is the pure sinusoid $x[n] = \cos(2\pi n/12)$, not the causal sinusoid $x[n] = \cos(2\pi n/12)\,u[n]$. Pure sinusoids do not appear in the table of z transforms. Since the excitation is a pure sinusoid, we are finding the forced response of the system, and we can use the discrete-time Fourier transform pairs:

$$\cos((\Omega_0 n) \leftrightarrow \pi\big(\delta_{2\pi}(\Omega - \Omega_0)\big) + \delta_{2\pi}(\Omega + \Omega_0)$$

And

$$\delta_{N_0}[n] \leftrightarrow \left(\frac{2\pi}{N_0}\right)\delta_{2\pi/N_0}(\Omega)$$

and the duality of multiplication and convolution properties in z-transform

$$x[n] * y[n] \leftrightarrow X(e^{j\Omega})Y(e^{j\Omega})$$

The discrete-time Fourier transform of the response of the system is:

$$Y(e^{j\Omega}) = \left(\frac{e^{j\Omega}}{e^{j\Omega} - 0.9}\right)\left\{\pi\left[\delta_{2\pi}\left(\Omega - \frac{\pi}{6}\right) + \delta_{2\pi}\left(\Omega + \frac{\pi}{6}\right)\right]\right\}$$

And

$$Y(e^{j\Omega}) = \pi\left\{e^{j\Omega}\left[\frac{\delta_{2\pi}\left(\Omega - \frac{\pi}{6}\right)}{e^{j\Omega} - 0.9}\right] + e^{j\Omega}\left[\frac{\delta_{2\pi}\left(\Omega + \frac{\pi}{6}\right)}{e^{j\Omega} - 0.9}\right]\right\}$$

Using the equivalence property of the impulse and the fact that both $e^{j\Omega}$ and $\delta_{2\pi}(\Omega)$ have a fundamental period of 2π.

$$Y(e^{j\Omega}) = \pi\left\{e^{j\Omega/6}\left[\frac{\delta_{2\pi}\left(\Omega - \frac{\pi}{6}\right)}{e^{j\pi/6} - 0.9}\right] + e^{j\Omega/6}\left[\frac{\delta_{2\pi}\left(\Omega + \frac{\pi}{6}\right)}{e^{j\pi/6} - 0.9}\right]\right\}$$

Finding a common denominator and simplifying,

$$Y(e^{j\Omega}) = \pi\left\{\frac{\delta_{2\pi}\left(\Omega - \frac{\pi}{6}\right)\left(1 - 0.9e^{\frac{j\pi}{6}}\right) + \delta_{2\pi}\left(\Omega + \frac{\pi}{6}\right)\left(1 - 0.9e^{-\frac{j\pi}{6}}\right)}{1.81 - 18\cos\left(\frac{\pi}{6}\right)}\right\}$$

$$= \pi\left\{\frac{0.2206\left[\delta_{2\pi}\left(\Omega - \frac{\pi}{6}\right)\right] + \delta_{2\pi}\left(\Omega - \frac{\pi}{6}\right) + j0.45\left[\delta_{2\pi}\left(\Omega + \frac{\pi}{6}\right) - \delta_{2\pi}\left(\Omega - \frac{\pi}{6}\right)\right]}{0.2512}\right\}$$

$$= 2.7589\left[\delta_{2\pi}\left(\Omega - \frac{\pi}{6}\right) + \delta_{2\pi}\left(\Omega + \frac{\pi}{6}\right)\right] + j5.6278\left[\delta_{2\pi}\left(\Omega + \frac{\pi}{6}\right) - \delta_{2\pi}\left(\Omega - \frac{\pi}{6}\right)\right]$$

Recognising the discrete-time Fourier transforms of a cosine and a sine,

$$y[n] = 0.8782\cos\left(\frac{2\pi n}{12}\right) + 1.7914\sin\left(\frac{2\pi n}{12}\right)$$

Using the trigonometry identity $A\cos(x) + B\sin(x) = \sqrt{A^2 + B^2}\cos(x - \tan^{-1}(B/A))$, the equation above is now:

$$y[n] = 1.995\cos\left(\frac{2\pi n}{12} - 1.115\right)$$

b. We did not use the z transform because there is no entry in the table of z transform pairs for a sinusoid. But there is an entry for a sinusoid multiplied by a unit sequence.

$$\cos(\Omega_0 n)\,u[n] \leftrightarrow \frac{z[z - \cos(\Omega_0)]}{z^2 - 2z\cos(\Omega_0) + 1} \qquad |z| > 1$$

It is instructive to find the response of the system to this different, but similar, excitation. The transfer function is:

$$H(z) = \frac{z}{z - 0.9} \qquad |z| > 0.9$$

The z transform of the response is:

$$Y(z) = \left(\frac{z}{z - 0.9}\right)\left\{\frac{z\left[z - \cos\left(\frac{\pi}{6}\right)\right]}{z^2 - 2z\cos\left(\frac{\pi}{6}\right) + 1}\right\} \qquad |z| > 1$$

Expanding in partial fractions, the equation above becomes:

$$Y(z) = \frac{0.1217z}{z - 0.9} + \frac{0.8783z^2 + 0.1353z}{z^2 - 1.732z + 1} \qquad |z| > 1$$

To find the inverse z transform, we need to manipulate the expressions into forms similar to the table entries.

The first fraction form appears directly in the table. The second fraction has a denominator of the same form as the z transforms of $\cos(\Omega_0 n)\,u[n]$ and $\sin(\Omega_0 n)\,u[n]$ but the numerator is not in exactly the right form.

But by adding and subtracting the right amounts in the numerator we can express $Y(z)$ in the form:

$$Y(z) = \frac{0.1217}{z-0.9} + 0.8783\left[\frac{z(z-0.866)}{z^2-1.732z+1} + 2.04\left(\frac{0.5z}{z^2-1.732z+1}\right)\right] \qquad |z| > 1$$

Thus

$$y[n] = 0.1217(0.9)^n\, u[n] + 0.8783\left[\cos\left(\frac{2\pi n}{12}\right) + 2.04\sin\left(\frac{2\pi n}{12}\right)\right]u[n]$$

Knowing that $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$, as a result:

$$y[n] = 0.1217(0.9)^n u[n] + 1.995\cos\left(\frac{2\pi n}{12} - 1.115\right)u[n]$$

Notice that the response consists of two parts, a transient response $0.1217(0.9)^n u[n]$ and a forced response $1.995\cos(2\pi n/12 - 1.115)\,u[n]$ that, except for the unit sequence factor, is exactly the same as the forced response we found using the discrete-time Fourier transform.

So, even though we do not have a z transform of a sinusoid in the z transform table we can use the z transforms of $\cos(\Omega_0 n)\,u[n]$ and $\sin(\Omega_0 n)\,u[n]$ to find the forced response to a sinusoid.

4. Analysis of Discrete-Time Systems

For the analysis of the discrete-time systems, we will investigate only on two types of system characteristic i.e. causality and stability of the system.

4.1. System Function and Causality

For a discrete-time LTI system to be causal, its impulse response $h[n]$ needs to be equal to zero for $n < 0$. Thus, by changing the lower limit of the summation index to n = 0 in the definition of the z-transform, the z-domain system function for a causal discrete-time LTI system can be written as:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=0}^{\infty} h[n]z^{-n} \qquad (Eq.\,28)$$

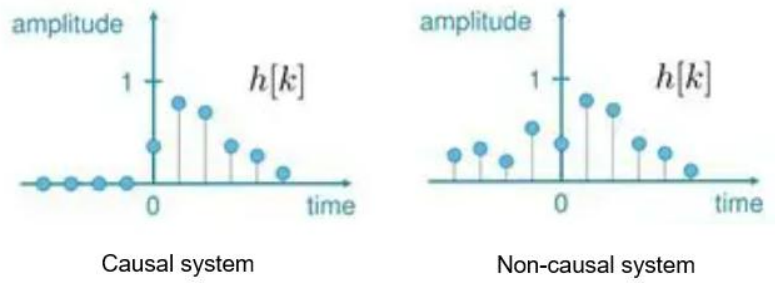
The ROC for the system function of a causal system is the outside of a circle in the z-plane. Consequently, the system function must also converge at $|z| \to \infty$. Consider a system function in the form

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_M z^M + b_{M-1} z^{M-1} + \ldots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \ldots + a_1 z + a_0} \qquad (Eq. 29)$$

For the system described by $H(z)$ to be causal we need:

$$\lim_{z \to \infty} |H(z)| = \lim_{z \to \infty} \left(\frac{b_M}{a_N}\right) z^{M-N} < \infty \qquad (Eq. 30)$$

which requires that $M - N \leq 0$ and consequently $M \leq N$.



Causal system Non-causal system

Figure 7: Causality of DTLTI systems

Thus, we arrive at an important conclusion:

Causality condition:

In the *z*-domain transfer function of a causal discrete-time LTI system the order of the numerator must not be greater than the order of the denominator.

Note that this condition is necessary for a system to be causal, but it is not sufficient. It is also possible for a non-causal system to have a system function with $M \leq N$.

4.2. System Function and Stability

A discrete-time LTI system is stable if its impulse response must be absolute summable, that is,

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \qquad (Eq. 31)$$

The discrete-time Fourier transform of a signal exists if the signal is absolute summable.

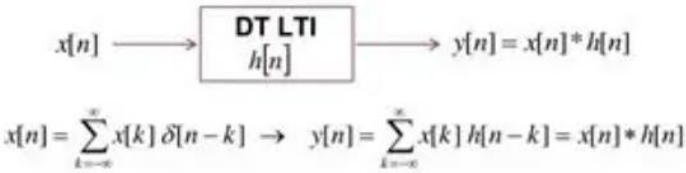


Figure 8: The output of any DTLTI signal or system is a convolution of the input signal with the unit pulse response

But the discrete-time Fourier transform of the impulse response is equal to the z-domain transfer function evaluated on the unit circle of the z-plane, that is,

$$H(\Omega) = H(z)|_{z=e^{j\Omega}} \qquad (Eq. 32)$$

provided that the unit circle of the *z*-plane is within the ROC.

Stability condition:

It follows that, for a discrete-time LTI system to be stable, the ROC of its z-domain system function must include the unit circle.

What are the corresponding conditions that must be imposed on the locations of poles and zeros for stability? We will answer this question by taking three separate cases into account as listed below.

4.2.1. Causal System

The ROC for the system function of a causal system is the outside of a circle with radius r_1, and is expressed in the form:

$$|z| > r_1 \qquad (Eq. 33)$$

For the ROC to include the unit circle, we need $r_1 < 1$. Since the ROC cannot have any poles in it, all the poles of the system function must be on or inside the circle with a radius r_1.

For a causal system to be stable, the system function must not have any poles on or outside the unit circle of the *z*-plane.

4.2.2. Anti-Causal System

If the system is anti-causal, its impulse response is equal to zero for $n \geq 0$ and the ROC for the system function is the inside of a circle with radius r_2, expressed in the form:

$$|z| < r_2 \qquad (Eq. 34)$$

For the ROC to include the unit circle, we need $r_2 > 1$. Also, all the poles of the system function must reside on or outside the circle with radius r_2 since there can be no poles within the ROC.

For an anti-causal system to be stable, the system function must not have any poles on or inside the unit circle of the z-plane.

4.2.3. Neither Causal nor Anti-Causal System

In this case the ROC for the system function, if it exists, is the region between two circles with radii r_1 and r_2, and is expressed in the form:

$$r_1 < |z| < r_2 \qquad (Eq. 35)$$

The poles of the system function may be either:

a. On or inside the circle with radius r_1
b. On or outside the circle with radius r_2

and the ROC must include the unit circle.

Example 8: Impulse response of a stable system

A stable system is characterised by the system function:

$$H(z) = \frac{z(z+1)}{(z-0.8)(z+1.2)(z-2)}$$

a. Determine the ROC of the system. [Answer: $0.8 < |z| < 1.2$] [4 marks]

b. Applying partial fraction expansion, determine the impulse response of the system. Based on the result of the partial fraction expansion, comment of the causality of the system. [Answer: $h[n] = -0.75(0.8)^n u[n] + 0.0312(-1.2)^n u[-n-1] - 0.7813(2)^n u[-n-1]$] [14 marks]

c. Simulate the impulse response of the system. Describe the system output when the sample index is increased in both directions. [6 marks]

Solution

a. The ROC for the system function is not directly stated; however, we are given enough information to deduce it.

- The zeros of the system are $z_1 = 0$ and $z_2 = -1$.
- The poles of the system are at p_1 = 0.8, p_2 = −1.2 and p_3 = 2.

Since the system is known to be stable, its ROC must include the unit circle. The only possible choice is:

$$0.8 < |z| < 1.2$$

b. Applying partial fraction expansion, the equation above becomes:

$$\frac{z(z+1)}{(z-0.8)(z+1.2)(z-2)} = \frac{A}{(z-0.8)} + \frac{B}{(z+1.2)} + \frac{C}{(z-2)}$$

The coefficients in the equation above are:

Coefficient A:

$$A = \left.\frac{z(z+1)}{(z+1.2)(z-2)}\right|_{z=0.8} = 0.75$$

Coefficient B:

$$B = \left.\frac{z(z+1)}{(z-0.8)(z-2)}\right|_{z=-1.2} = 0.0312$$

Coefficient C:

$$C = \left.\frac{z(z+1)}{(z-0.8)(z+1.2)}\right|_{z=2} = 0.7813$$

Partial fraction expansion of $H(z)$ is:

$$H(z) = -\frac{0.75z}{z-0.8} - \frac{0.0312z}{z+1.2} + \frac{0.7813z}{z-2}$$

Based on the ROC determined above, the first partial fraction corresponds to a causal signal, and the other two correspond to anti-causal signals.

The impulse response of the system is:

$$h[n] = -0.75(0.8)^n u[n] + 0.0312(-1.2)^n u[-n-1] - 0.7813(2)^n u[-n-1]$$

c. The MATLAB script and its simulated impulse response of the DTLTI system are shown below.

MATLAB script:

```
% Find left and right sided components of h[n] from the
analytical solution
nleft = [-10:-1];
hleft = 0.0312*(-1.2).^nleft-0.7813*(2).^nleft;
nright = [0:20];
hright = -0.75*(0.8).^nright;
n = [nleft,nright];
h = [hleft,hright];
clf;
stem(n,h);
```

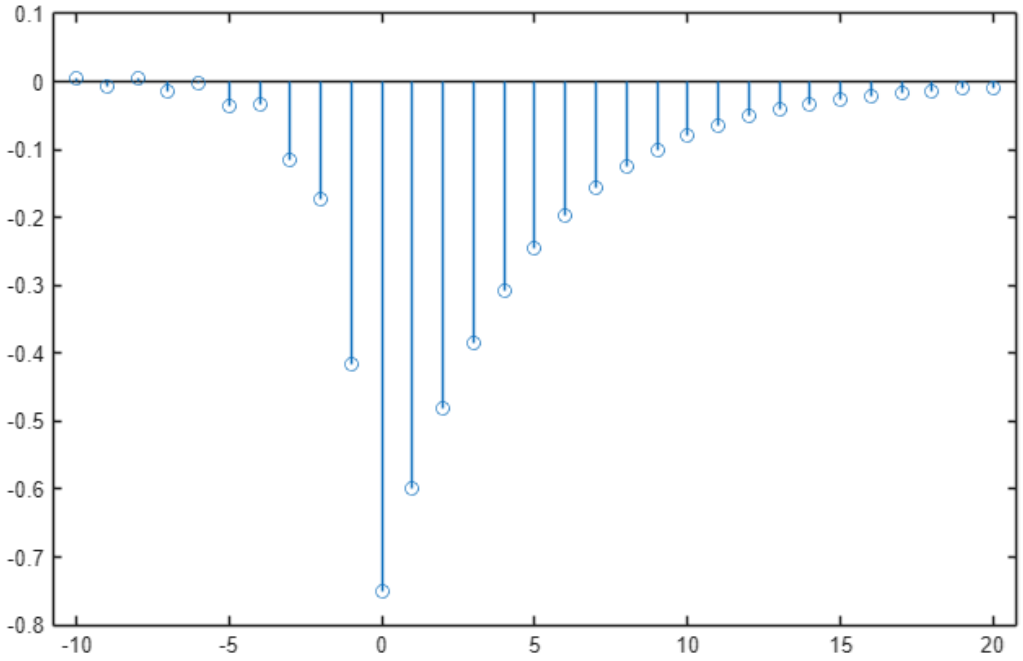


Figure: Impulse response $h[n]$

We observe that $h[n]$ tends to zero as the sample index is increased in both directions, consistent with the fact that $h[n]$ must be absolute summable.

Example 9: Analysis of a system described by a difference equation 1

An LTI system is described by the difference equation:

$$y[n] - y[n-1] + \frac{1}{4}y[n-2] = x[n] - \frac{1}{4}x[n-1] - \frac{1}{8}x[n-2]$$

a. Find the transfer function of the system. [6 marks]

b. Describe whether the system is causal and stable? [4 marks]

Solution

a. We find the transfer function of the system by applying equation:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

This obtains the following equation:

$$H(z) = \frac{1 + \left(\frac{1}{4}\right)z^{-1} - \left(\frac{1}{8}\right)z^{-2}}{1 - z^{-1} + \left(\frac{1}{4}\right)z^{-2}} = \frac{\left[1 - \left(\frac{1}{4}\right)z^{-1}\right]\left[1 + \left(\frac{1}{2}\right)z^{-1}\right]}{\left[1 - \left(\frac{1}{2}\right)z^{-1}\right]^2}$$

b. The zeros of the system are located at $z = 1/4$ and $z = -1/2$. Both zeros are inside the unit circle.

Since the double poles at $z = 1/2$ is also located inside the unit circle, so the system is causal and stable.

Example 10: Analysis of a system described by a difference equation 2

A discrete-time LTI system is characterised by the following difference equation:

$$y[n] = x[n-1] + 3x[n-2] + 2x[n-3] + 2.3y[n-1] - 2y[n-2] + 1.2y[n-3]$$

a. Describe the whether the system is causal. [Answer: causal] [4 marks]

b. Derive the transfer function equation of the system. [Answer: $H(z) = (z^2 + 3z + 2)/(z^3 - 2.3z^2 + 2z - 1.2)$] [8 marks]

c. Determine zeros and poles of the system. [Answer: z_1 = -1, z_2 = -2, $p_{1,2} = 0.4 \pm j0.8$, and $p_3 = 1.5$] [4 marks]

d. Sketch the pole and zero diagram of the system. Indicate clearly the poles and zeros of the system on the diagram. [8 marks]

e. Determine the ROC of the system. Deduce the characteristic and stability of the system. (Answer: $|z| > 1.5$ and unstable) [4 marks]

f. Verify the stability of the system from the impulse response of the system. [8 marks]

Solution

a. As in the previous example, the ROC for the system function is not directly stated.

On the other hand, the difference equation of the system is written with the current output sample $y[n]$ on the left of the equal sign. All terms on the right side of the equal sign are past samples of the input and the output signals.

The form of the difference equation suggests that the system is causal. When we find the system function, we must choose the ROC in a way that includes infinitely large values of $|z|$.

b. Rearranging terms of the difference equation we obtain:

$$y[n] - 2.3y[n-1] + 2y[n-2] - 1.2y[n-3] = x[n-1] + 3x[n-2] + 2x[n-3]$$

Taking the z-transform of each side yields:

$$(1 - 2.3z^{-1} + 2z^{-2} - 1.2z^{-3})Y(z) = (z^{-1} + 3z^{-2} + 2z^{-3})X(z)$$

Finally, solving for the ratio of $Y(z)$ and $X(z)$ we find the system function as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + 3z^{-2} + 2z^{-3}}{1 - 2.3z^{-1} + 2z^{-2} - 1.2z^{-3}}$$

Let us multiply both the numerator and the denominator of $H(z)$ by z^3 so that we express it using powers of z:

$$H(z) = \frac{z^2 + 3z + 2}{z^3 - 2.3z^2 + 2z - 1.2}$$

c. Numerator and denominator polynomials of $H(z)$ can be factorised to yield:

$$H(z) = \frac{(z+1)(z+2)}{(z-0.4+j0.8)(z-0.4-j0.8)(z-1.5)}$$

From the equation given above, the zeros and poles are determined as follows:

- The zeros are located at z_1 = -1 and z_2 = -2.
- There are complex poles at $p_{1,2} = 0.4 \pm j0.8$ and $p_3 = 1.5$.

d. The MATLAB script and its simulated pole-zero diagram for the system function are shown below.

MATLAB Script:

```
% Create vectors for the numerator and the denominator of H(z).
num = [1,3,2];          % Numerator coefficients.
den = [1,-2.3,2,-1.2];  % Denominator coefficients.
% Compute zeros and poles of H(z).
zer = roots(num)
pol = roots(den)
% Pole-zero plot.
clf;
zplane(num,den);
```

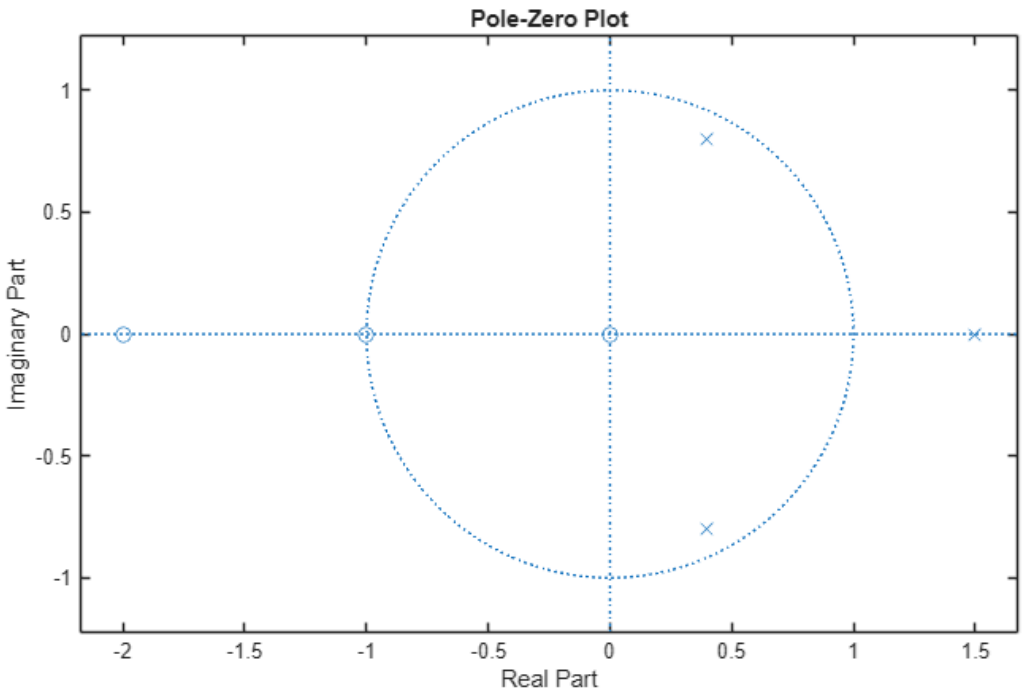
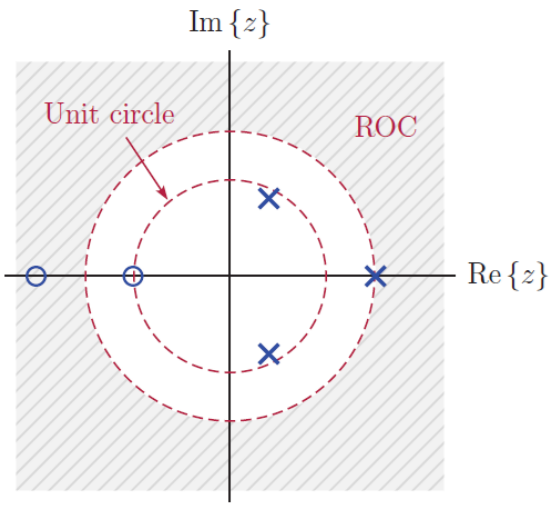


Figure: Pole-zero diagram and ROC of the system

As shown in the z-plane diagram given below, there are two zeros, one at $z_1 = -1$ and another at $z_2 = -2$. The system function also has a pole at $p_1 = 1.5$ and a conjugate pair of poles at $p_{2,3} = 0.4 \pm j0.8$.



e. Since the system is known to be causal, its ROC must be the outside of a circle. The radius of the boundary circle is determined by the outermost pole of the system which happens to be at $z = 1.5$. Consequently, the ROC for the system function is $|z| > 1.5$. Since the ROC does not include the unit circle, the system is unstable.

f. This can be verified by computing the first few samples of the impulse response $h[n]$ from the system function using long division.

$$h[1] = \frac{z^2 + 3z + 2}{z^3 - 2.3z^2 + 2z - 1.2} = \frac{(1)^2 + 3(1) + 2}{(1)^3 - 2.3(1)^2 + 2(1) - 1.2} = 1$$

$$h[2] = \frac{z^2 + 3z + 2}{z^3 - 2.3z^2 + 2z - 1.2} = \frac{(2)^2 + 3(2) + 2}{(2)^3 - 2.3(2)^2 + 2(2) - 1.2} = 5.3$$

$$h[3] = \frac{z^2 + 3z + 2}{z^3 - 2.3z^2 + 2z - 1.2} = \frac{(3)^2 + 3(3) + 2}{(3)^3 - 2.3(3)^2 + 2(3) - 1.2} = 12.19$$

$$h[4] = \frac{z^2 + 3z + 2}{z^3 - 2.3z^2 + 2z - 1.2} = \frac{(4)^2 + 3(4) + 2}{(4)^3 - 2.3(4)^2 + 2(4) - 1.2} = 18.64$$

$$h[5] = \frac{z^2 + 3z + 2}{z^3 - 2.3z^2 + 2z - 1.2} = \frac{(5)^2 + 3(5) + 2}{(5)^3 - 2.3(5)^2 + 2(5) - 1.2} = 24.85$$

$$h[6] = \frac{z^2 + 3z + 2}{z^3 - 2.3z^2 + 2z - 1.2} = \frac{(6)^2 + 3(6) + 2}{(6)^3 - 2.3(6)^2 + 2(6) - 1.2} = 34.5$$

$$h[7] = \frac{z^2 + 3z + 2}{z^3 - 2.3z^2 + 2z - 1.2} = \frac{(7)^2 + 3(7) + 2}{(7)^3 - 2.3(7)^2 + 2(7) - 1.2} = 52.02$$

$$h[8] = \frac{z^2 + 3z + 2}{z^3 - 2.3z^2 + 2z - 1.2} = \frac{(8)^2 + 3(8) + 2}{(8)^3 - 2.3(8)^2 + 2(8) - 1.2} = 80.46$$

$$h[9] = \frac{z^2 + 3z + 2}{z^3 - 2.3z^2 + 2z - 1.2} = \frac{(9)^2 + 3(9) + 2}{(9)^3 - 2.3(9)^2 + 2(9) - 1.2} = 122.42$$

$$h[10] = \frac{z^2 + 3z + 2}{z^3 - 2.3z^2 + 2z - 1.2} = \frac{(10)^2 + 3(10) + 2}{(10)^3 - 2.3(10)^2 + 2(10) - 1.2} = 183.07$$

This yields:

$$h[n] = \{1, 5.3, 12.19, 18.64, 24.85, 34.5, 52.02, 80.46, 122.42, 183.07, \dots\}$$

The impulse response keeps growing with increasing values of the index n. We can prove this with MATLAB script listed below.

MATLAB Script:

```
% Create vectors for the numerator and the denominator of
H(z).
num = [1,3,2];           % Numerator coefficients.
den = [1,-2.3,2,-1.2];   % Denominator coefficients.
% Use the function filter(...) to compute the first 10 samples
% of the impulse response h[n].
h = filter(num,den,[1,0,0,0,0,0,0,0,0,0])
```

The result of the simulation in MATLAB is shown below.

```
h =

    1.0000    5.3000   12.1900   18.6370   24.8451   34.4977   52.0190   80.4623  122.4226  183.0702
```