

ECEN220 Tutorial 2 Questions (SOLUTIONS)

Block 2: Linear Time Invariant (LTI) Systems

Q1. Solutions:

The **scaling property**, which states that if the response to input $x(t)$ is output $y(t)$, then the response to input $cx(t)$ is output $cy(t)$ for any constant c .

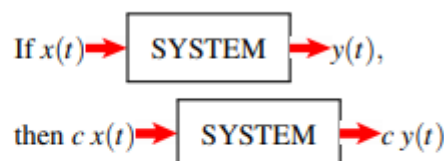
The **additivity property**, which states that if the response to input $x_1(t)$ is output $y_1(t)$, and the response to input $x_2(t)$ is output $y_2(t)$, then the response to input $(x_1(t) + x_2(t))$ is output $(y_1(t) + y_2(t))$.

The additivity and scaling properties together are called the **superposition property**. A system with these two properties is linear.

The third property of a LTI system is **time invariance**, which states that if the response to input $x(t)$ is output $y(t)$, then the response to input $x(t - T)$ is output $y(t - T)$ for any constant T . A linear system that is also time-invariant is linear time-invariant (LTI).

Q2.

Solution: The scaling property of a system is:



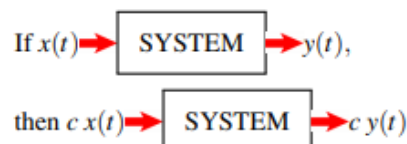
for any constant c .

The response to $[cx(t)]$ is the output $[cx(t)]^2 = c^2 x^2(t) = c^2 y(t) \neq cy(t)$.

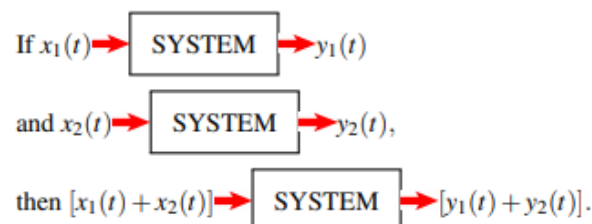
So the system does not have the scaling property.

Q3.

Solution: A system is linear if it has both the scaling property and the additivity property. The scaling property is:



for any constant c . The additivity property is:



(a) The system is $y(t) = |\sin(3t)| x(t)$.

The response to $[cx(t)]$ is the output $|\sin(3t)| [cx(t)] = c|\sin(3t)| x(t) = c y(t)$.

So the system has the scaling property.

• $x_1(t) \rightarrow \boxed{\text{SYSTEM}} \rightarrow y_1(t) = |\sin(3t)| x_1(t)$.

• $x_2(t) \rightarrow \boxed{\text{SYSTEM}} \rightarrow y_2(t) = |\sin(3t)| x_2(t)$.

• $[x_1(t) + x_2(t)] \rightarrow \boxed{\text{SYSTEM}} \rightarrow |\sin(3t)| [x_1(t) + x_2(t)]$.

The response to $[x_1(t) + x_2(t)]$ is the output

$$|\sin(3t)| [x_1(t) + x_2(t)] = |\sin(3t)| x_1(t) + |\sin(3t)| x_2(t) = y_1(t) + y_2(t).$$

So the system has the additivity property.

Since the system has both the scaling and additivity properties, it is linear.

(b) The system is $y(t) = a \frac{dx}{dt}$.

The response to $[cx(t)]$ is the output

$$a \frac{d(cx)}{dt} = ac \frac{dx}{dt} = c y(t).$$

So the system has the scaling property.

• $x_1(t) \rightarrow \boxed{\text{SYSTEM}} \rightarrow y_1(t) = a \frac{dx_1}{dt}$.

• $x_2(t) \rightarrow \boxed{\text{SYSTEM}} \rightarrow y_2(t) = a \frac{dx_2}{dt}$.

• $[x_1(t) + x_2(t)] \rightarrow \boxed{\text{SYSTEM}} \rightarrow a \frac{d}{dt} [x_1(t) + x_2(t)]$.

The response to $[x_1(t) + x_2(t)]$ is the output

$$a \frac{d}{dt} [x_1(t) + x_2(t)] = a \frac{dx_1}{dt} + a \frac{dx_2}{dt} = y_1(t) + y_2(t).$$

So the system has the additivity property.

Since the system has both the scaling and additivity properties, it is linear.

(c) The system is $y(t) = |x(t)|$.

We show that the system does not have the scaling property.

Trying $c = -1$ shows that the response to $-x(t)$ is the output

$$|-x(t)| = |x(t)| = y(t) \neq -y(t).$$

The system does not have the scaling property. So the system is not linear.

(d) The system is $y(t) = \sin[x(t)]$.

We show that the system does not have the scaling property.

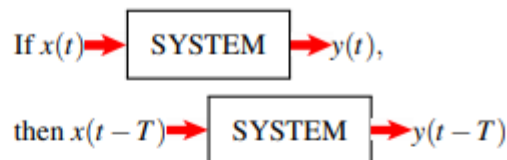
Trying $c = 2$ shows that the response to $2x(t)$ is the output

$$\sin[2x(t)] \neq 2\sin[x(t)] = 2y(t).$$

The system does not have the scaling property. So the system is not linear.

Q4.

Solution: A system is time-invariant if it has the property that:



for any constant T .

(a) The system is

$$y(t) = \frac{dx}{dt} + \sin[x(t-1)].$$

The response to $x(t-T)$ is the output

$$\frac{d(x(t-T))}{dt} + \sin[x(t-T-1)] = y(t-T).$$

So the system is time-invariant.

(b) The system is

$$\frac{dy}{dt} = 2\sin[x(t-1)] + 3\cos[x(t-1)].$$

Substituting $x(t-T)$ for $x(t)$ and $y(t-T)$ for $y(t)$ gives

$$\frac{dy(t-T)}{dt} = 2\sin[x(t-T-1)] + 3\cos[x(t-T-1)],$$

which is the system with t replaced with $t-T$.

So the system is time-invariant.

Q5.

Method #1: By measuring the response of the system to a short and high pulse.

Method #2: By measuring the response of the system to a step function, and differentiating the step response.

Q6.

By differentiating the step response.

Q7.

Solution: The impulse response is the derivative of the step response:

$$h(t) = \frac{dy_{\text{step}}}{dt}.$$

For the given $y_{\text{step}}(t)$, we have:

$$h(t) = \frac{dy_{\text{step}}}{dt} = \begin{cases} \frac{d0}{dt}, & t \leq 0 \\ \frac{dt}{dt}, & 0 \leq t \leq 1 \\ \frac{d1}{dt}, & t \geq 1 \end{cases} = \begin{cases} 0, & t \leq 0 \\ 1, & 0 \leq t \leq 1 \\ 0, & t \geq 1 \end{cases}$$

This rectangular pulse can be written succinctly as $h(t) = u(t) - u(t - 1)$.

Q8.

One of the two signals to be convolved is reversed in time, and time shifted by time t . The unreversed and unshifted signal is multiplied point-by-point by the shifted and reversed signal, and the area under the product is computed. The result is the convolution at time t .

Q9.

Time-shifting each of two signals to be convolved leads to time-shifting their convolution by the sum of the time shifts, as described in Eq. (2.72):

If $h(t) * x(t) = y(t)$, then

Eq. (2.72)

$$h(t - T_1) * x(t - T_2) = y(t - T_1 - T_2).$$

Q10.

Convolving a signal with a step integrates the signal up to the time of the convolution.

$$\begin{aligned}
 x(t) * u(t) &= \int_{-\infty}^{\infty} x(\tau) u(t - \tau) d\tau \\
 &= \int_{-\infty}^t x(\tau) d\tau, \\
 &\text{(ideal integrator)}
 \end{aligned}$$

Convolving a signal with an impulse gives the original signal. If the impulse is time-shifted, the original signal is time-shifted by the same amount.

$$\begin{aligned}
 x(t) * \delta(t - T) &= \int_{-\infty}^{\infty} x(\tau) \delta(t - T - \tau) d\tau \\
 &= x(t - T).
 \end{aligned}$$

Q11.

The area of the convolution of two signals is the product of the areas of the two signals.

$$\begin{aligned}
 \text{Area of } y(t) &= \int_{-\infty}^{\infty} y(t) dt \\
 &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau \right] dt \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau) x(t - \tau) dt d\tau \\
 &= \int_{-\infty}^{\infty} h(\tau) \left[\int_{-\infty}^{\infty} x(t - \tau) dt \right] d\tau \\
 &= \left[\int_{-\infty}^{\infty} h(\tau) d\tau \right] \left[\int_{-\infty}^{\infty} x(t - \tau) dt \right] \\
 &= \text{area of } h(t) \times \text{area of } x(t).
 \end{aligned}$$

Q12. (refer to the Table provided in lecture ppt)

Solution: Using convolution property #6 in Table 2-1, $u(t) * \delta(t-3) = u(t-3)$ and $u(t) * \delta(t) = u(t)$. Then using convolution property #5 with $T_1 = 4$ and $T_2 = -1$, we obtain

$$u(t-4) * \delta(t+1) = u(t-4+1) = u(t-3).$$

So

$$u(t) * \delta(t-3) - u(t-4) * \delta(t+1) = u(t-3) - u(t-3) = \boxed{0}.$$

Q13. (refer to the Table provided in lecture ppt)

Solution: Let $y(t) = e^{-3t} u(t) * u(t)$.

Using convolution property #9 in Table 2-1,

$$y(t) = \int_{-\infty}^t e^{-3\tau} u(\tau) d\tau = \int_0^t e^{-3\tau} d\tau = \frac{1}{3}(1 - e^{-3t}) u(t).$$

Then

$$\lim_{t \rightarrow \infty} (e^{-3t} u(t) * u(t)) = \lim_{t \rightarrow \infty} y(t) = \boxed{\frac{1}{3}}.$$

Q14.

Solution: The system is BIBO stable if and only if it is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(t)| dt$$

is finite. Here,

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} \left| \frac{u(t-1)}{t^2} \right| dt = \int_1^{\infty} \frac{dt}{t^2} = \left. \frac{-1}{t} \right|_1^{\infty} = 1 < \infty.$$

So the system is BIBO stable.

Q15.

Solution: The system is BIBO stable if and only if it is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(t)| dt$$

is finite. Here,

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} \left| \frac{u(t-1)}{t} \right| dt = \int_1^{\infty} \frac{dt}{t} = \log(|t|) \Big|_1^{\infty} \rightarrow \infty.$$

So the system is not BIBO stable.

Q16.

The amplitude and phase can change, but the frequency cannot change.

Q17.

Solution:

$$A \cos(\omega t + \phi) \rightarrow \hat{\mathbf{H}}(\omega) \rightarrow |\hat{\mathbf{H}}(\omega)| A \cos(\omega t + \phi + \theta)$$

where $\theta = \angle \hat{\mathbf{H}}(\omega)$. Here, $A = 1$, $\omega = 1$ rad/s, $\phi = 0$, and the frequency response function $\hat{\mathbf{H}}(\omega)$ is

$$\hat{\mathbf{H}}(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt = \int_0^{\infty} e^{-t} e^{-j\omega t} dt = \frac{1}{j\omega + 1}.$$

At $\omega = 1$ rad/s,

$$\hat{\mathbf{H}}(1) = \frac{1}{j1 + 1} = \frac{1}{\sqrt{2}} e^{-j45^\circ}.$$

So the output is $\frac{1}{\sqrt{2}} \cos(t - 45^\circ)$.

Q18.

Solution: No. The response of an LTI system to a sinusoid at a given frequency is another sinusoid at that same frequency. This is a crucial property of LTI systems.

An LTI system cannot create a sinusoid at a frequency different from that of its input.

This system has created sinusoids at frequencies $\omega = 0$ and $\omega = 2$, so it is not LTI.

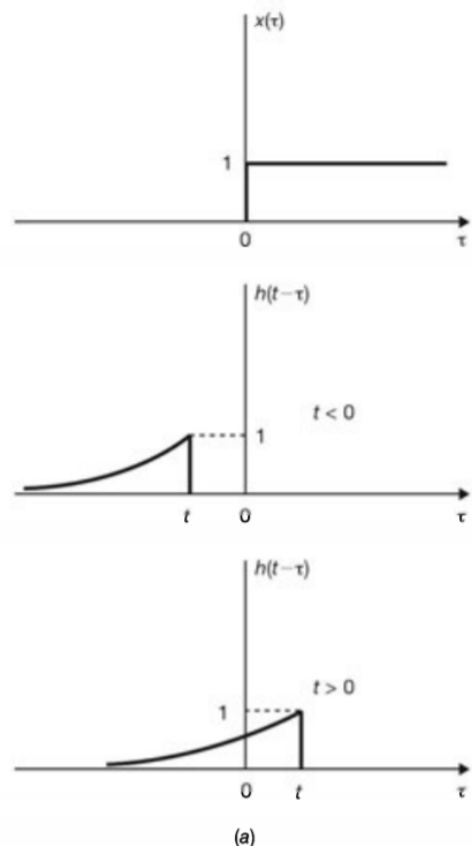
Q19.

- (a) Functions $x(\tau)$ and $h(t - \tau)$ are shown in (a) for $t < 0$ and $t > 0$. From (a) we see that for $t < 0$, $x(\tau)$ and $h(t - \tau)$ do not overlap, while for $t > 0$, they overlap $\tau = 0$ to $\tau = t$. Hence, for $t < 0$, $y(t) = 0$. For $t > 0$, we have

$$\begin{aligned} y(t) &= \int_0^t e^{-\alpha(t-\tau)} d\tau = e^{-\alpha t} \int_0^t e^{\alpha\tau} d\tau \\ &= e^{-\alpha t} \frac{1}{\alpha} (e^{\alpha t} - 1) = \frac{1}{\alpha} (1 - e^{-\alpha t}) \end{aligned}$$

So, we can write $y(t)$ as

$$y(t) = \frac{1}{\alpha} (1 - e^{-\alpha t}) u(t)$$



(b)

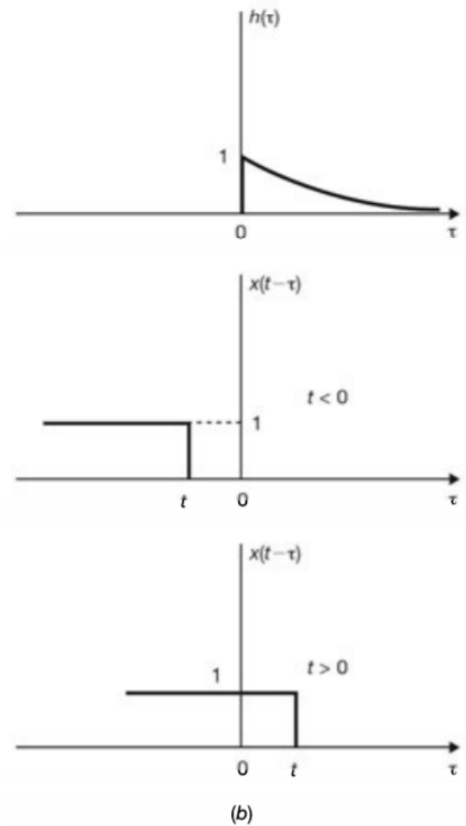
Functions $h(\tau)$ and $x(t - \tau)$ are shown in (b) for $t < 0$ and $t > 0$. From (a) we see that for $t < 0$, $h(\tau)$ and $x(t - \tau)$ do not overlap, while for $t > 0$, they overlap $\tau = 0$ to $\tau = t$. Hence, for $t < 0$, $y(t) = 0$. For $t > 0$, we have

$$y(t) = \int_0^t e^{-a\tau} d\tau = \frac{1}{a}(1 - e^{-at})$$

So, we can write $y(t)$ as

$$y(t) = \frac{1}{a}(1 - e^{-at})u(t)$$

Which is the same as in (a).



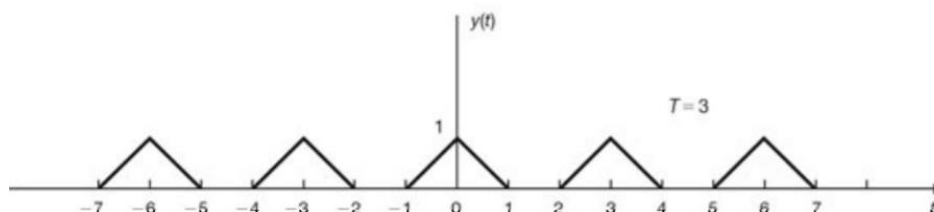
Q20. Using convolutional sum, we have

$$\begin{aligned} y(t) &= h(t) * \delta_T(t) = h(t) * \left[\sum_{n=-\infty}^{\infty} \delta(t - nT) \right] \\ &= \sum_{n=-\infty}^{\infty} h(t) * \delta(t - nT) = \sum_{n=-\infty}^{\infty} h(t - nT) \end{aligned}$$

(a) For $T = 3$, we then have

$$y(t) = \sum_{n=-\infty}^{\infty} h(t - 3n)$$

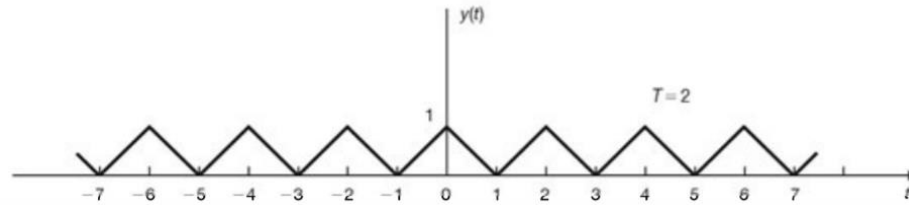
which is plotted here



(b) For $T=2$, we have

$$y(t) = \sum_{n=-\infty}^{\infty} h(t - 2n)$$

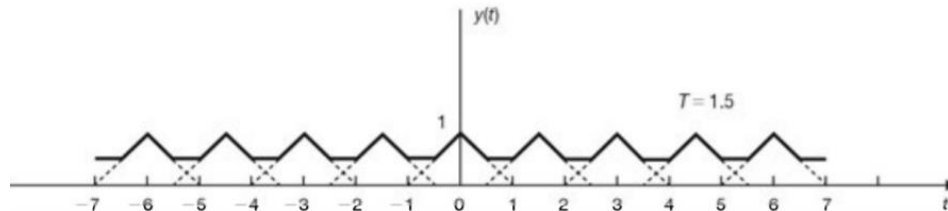
which is plotted here



(c) For $T=1.5$, we have

$$y(t) = \sum_{n=-\infty}^{\infty} h(t - 1.5n)$$

which is plotted here



Q21.

Solution: An LTI system is BIBO stable if and only if $\sum_{n=-\infty}^{\infty} |h[n]|$ is finite.

Here,

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6}.$$

This is finite, so the system is BIBO stable.

Q22.

Solution: An LTI system with impulse response $h[n] = C p^n u[n]$ is BIBO stable if and only if $|p| < 1$, since

$$\sum_{n=-\infty}^{\infty} |h[n]| = |C| \sum_{n=0}^{\infty} |p|^n = \frac{|C|}{1 - |p|}$$

if and only if $|p| < 1$.

Here, $p = 0.5$ and $|p| = |0.5| < 1$, so the system is BIBO stable.

Q23.

Solution:

$$y[n] = \sum_{i=0}^{\infty} h[i] x[n-i] = h[0] x[n] + h[1] x[n-1] + \dots$$

$h[0] \neq 0$ means that $y[n]$ depends on $x[n]$ as well as on past values of $x[n]$.

The output of a causal system at time n can depend on the input at the same time n and at previous times.

Hence, the system is causal.

Q24. $y[n] = \{(2)(0)+(1)(0), (2)(0)+(1)(3), (2)(3)+(1)(4), (2)(4)+(1)(0)\} = \{0, 3, 10, 8\}$