

ECEN220 Tutorial 3 Questions (SOLUTIONS)

Block 3: Fourier Series

Q1. Solutions:

With $T_0 = 4$ s and $\omega_0 = 2\pi/T_0 = \pi/2$ rad/s,

$$\begin{aligned}a_0 &= \frac{1}{T_0} \int_{-2}^2 x(t) dt \\&= \frac{1}{4} \left[\int_{-2}^0 5t dt + \int_0^2 (10 - 5t) dt \right] = 0, \\a_n &= \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cos n\omega_0 t dt \\&= \frac{1}{2} \left[\int_{-2}^0 5t \cos \frac{n\pi t}{2} dt + \int_0^2 (10 - 5t) \cos \frac{n\pi t}{2} dt \right].\end{aligned}$$

Using the integral relationship :

$$\int x \cos ax dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax,$$

we have

$$a_n = \frac{20}{n^2 \pi^2} (1 - \cos n\pi).$$

Similarly, using the relation

$$\int x \sin ax dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax,$$

we have

$$\begin{aligned}b_n &= \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \sin n\omega_0 t dt \\&= \frac{1}{2} \left[\int_{-2}^0 5t \sin \frac{n\pi t}{2} dt + \int_0^2 (10 - 5t) \sin \frac{n\pi t}{2} dt \right] \\&= \frac{10}{n\pi} (1 - \cos n\pi).\end{aligned}$$

Hence,

$$x(t) = \sum_{n=1}^{\infty} \left[\frac{20}{n^2 \pi^2} (1 - \cos n\pi) \cos \frac{n\pi t}{2} + \frac{10}{n\pi} (1 - \cos n\pi) \sin \frac{n\pi t}{2} \right].$$

Q2. Solutions:

$$\begin{aligned}c_n &= \sqrt{a_n^2 + b_n^2} \\&= \left\{ \left[\frac{20}{n^2 \pi^2} (1 - \cos n\pi) \right]^2 + \left[\frac{10}{n\pi} (1 - \cos n\pi) \right]^2 \right\}^{1/2} \\&= (1 - \cos n\pi) \frac{20}{n^2 \pi^2} \sqrt{1 + \frac{n^2 \pi^2}{4}}, \\ \phi_n &= -\tan^{-1} \left(\frac{b_n}{a_n} \right) \\&= -\tan^{-1} \left(\frac{n\pi}{2} \right).\end{aligned}$$

We note that $c_n = 0$ when $n = \text{even}$.

Q3. Solutions:

(a)

$$x(t) \neq x(-t) \rightarrow \text{no even symmetry}$$

$$x(t) \neq -x(-t) \rightarrow \text{no odd symmetry}$$

(b)

$$a_0 = \frac{2 \times 1 + 3 \times 1 + (-1) \times 1}{4} = 1.$$

(c) $y(t) = [x(t) - a_0]$ has odd symmetry.

Q4.

Periodic functions, which can be expanded using a Fourier series expansion, into a weighted sum of sinusoids. Then the sinusoidal steady-state response to each sinusoid can be computed using phasor analysis.

Q5.

The angular frequency of the n th harmonic is $n\omega_0$ with $\omega_0 = 2\pi/T$.

Q6.

1. Compute the Fourier series expansion of the (periodic) input.
2. Compute the sinusoidal steady-state response to each harmonic using phasor analysis.
3. Sum the sinusoidal steady-state responses to the harmonics.

Q7.

For the complex exponential representation, the limits are $n = -\infty$ to $n = \infty$.

Q8.

If the periodic function has even symmetry, all of the sine terms in its Fourier series expansion are zero.

If the periodic function has odd symmetry, all of the cosine terms, and the constant term, in its Fourier series expansion are zero.

Q9.

If a periodic function has a discontinuity at time t (and therefore at times $t + kT$, where k is any integer and T is the period) the Fourier series expansion of the function exhibits ringing in the vicinity of the discontinuity at time t . See Fig. 5-19(b).

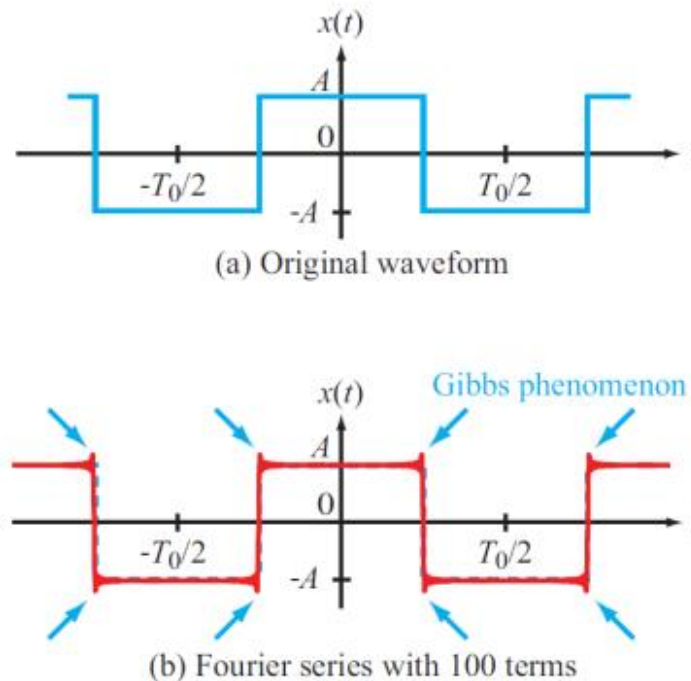


Figure 5-19: Square-wave waveform and its Fourier-series representation computed for $n_{\max} = 100$ terms.

Q10.

For any t_0 we have

$$\begin{aligned} \int_{t_0}^{t_0+T_0} e^{jm\omega_0 t} dt &= \frac{1}{jm\omega_0} e^{jm\omega_0 t} \Big|_{t_0}^{t_0+T_0} = \frac{1}{jm\omega_0} (e^{jm\omega_0(t_0+T_0)} - e^{jm\omega_0 t_0}) \\ &= \frac{1}{jm\omega_0} e^{jm\omega_0 t_0} (e^{jm2\pi} - 1) = 0 \quad m \neq 0 \end{aligned}$$

since $e^{jm2\pi} = 1$. When $m = 0$, we have $e^{jm\omega_0 t} \Big|_{m=0} = 1$ and

$$\int_{t_0}^{t_0+T_0} e^{jm\omega_0 t} dt = \int_{t_0}^{t_0+T_0} dt = T_0$$

From the above two equations, we can conclude that

$$\int_{t_0}^{t_0+T_0} e^{jm\omega_0 t} (e^{jk\omega_0 t})^* dt = \int_{t_0}^{t_0+T_0} e^{j(m-k)\omega_0 t} dt = \begin{cases} 0 & m \neq k \\ T_0 & m = k \end{cases}$$

which shows that the set $\{e^{jk\omega_0 t}; k = 0, \pm 1, \pm 2, \dots\}$ is orthogonal on any interval over a period T_0 .

Also, check out details in our lecture sides.

Q11.

(a) Using Euler's Identity, we can rewrite cosine as:

$$\cos \omega_0 t = \frac{1}{2} (e^{j\omega_0 t} + e^{-j\omega_0 t}) = \frac{1}{2} e^{-j\omega_0 t} + \frac{1}{2} e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Thus, the complex Fourier coefficients for $\cos \omega_0 t$ are

$$c_1 = \frac{1}{2} \quad c_{-1} = \frac{1}{2} \quad c_k = 0, |k| \neq 1$$

(b) In a similar fashion, we have

$$\sin \omega_0 t = \frac{1}{2j} (e^{j\omega_0 t} - e^{-j\omega_0 t}) = -\frac{1}{2j} e^{-j\omega_0 t} + \frac{1}{2j} e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Thus, the complex Fourier coefficients for $\sin \omega_0 t$ are

$$c_1 = \frac{1}{2j} \quad c_{-1} = -\frac{1}{2j} \quad c_k = 0, |k| \neq 1$$

(c)

The fundamental angular frequency ω_0 of $x(t)$ is 2. Thus,

$$x(t) = \cos\left(2t + \frac{\pi}{4}\right) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Now

$$\begin{aligned} x(t) &= \cos\left(2t + \frac{\pi}{4}\right) = \frac{1}{2} (e^{j(2t+\pi/4)} + e^{-j(2t+\pi/4)}) \\ &= \frac{1}{2} e^{-j\pi/4} e^{-j2t} + \frac{1}{2} e^{j\pi/4} e^{j2t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt} \end{aligned}$$

Thus, the complex Fourier coefficients for $\cos(2t + \pi/4)$ are

$$\begin{aligned} c_1 &= \frac{1}{2} e^{j\pi/4} = \frac{1}{2} \frac{1+j}{\sqrt{2}} = \frac{\sqrt{2}}{4} (1+j) \\ c_{-1} &= \frac{1}{2} e^{-j\pi/4} = \frac{1}{2} \frac{1-j}{\sqrt{2}} = \frac{\sqrt{2}}{4} (1-j) \\ c_k &= 0 \quad |k| \neq 1 \end{aligned}$$

Q12.

Let

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad \omega_0 = \frac{2\pi}{T_0}$$

We then have:

$$\begin{aligned} c_k &= \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0} \int_0^{T_0/2} A e^{-jk\omega_0 t} dt \\ &= \frac{A}{-jk\omega_0 T_0} e^{-jk\omega_0 t} \Big|_0^{T_0/2} = \frac{A}{-jk\omega_0 T_0} (e^{-jk\omega_0 T_0/2} - 1) \\ &= \frac{A}{jk2\pi} (1 - e^{-jk\pi}) = \frac{A}{jk2\pi} [1 - (-1)^k] \end{aligned}$$

since $\omega_0 T_0 = 2\pi$ and $e^{-jk\pi} = (-1)^k$. Thus,

$$\begin{aligned} c_k &= 0 & k = 2m \neq 0 \\ c_k &= \frac{A}{jk\pi} & k = 2m + 1 \\ c_0 &= \frac{1}{T_0} \int_0^{T_0} x(t) dt = \frac{1}{T_0} \int_0^{T_0/2} A dt = \frac{A}{2} \end{aligned}$$

Hence,

$$c_0 = \frac{A}{2} \quad c_{2m} = 0 \quad c_{2m+1} = \frac{A}{j(2m+1)\pi}$$

and we obtain

$$x(t) = \frac{A}{2} + \frac{A}{j\pi} \sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\omega_0 t}$$

Q13. First, we expand and write the signal as

$$\begin{aligned} x[n] &= 4 \cos(0.15\pi n + 1) \\ &= 4 \cos\left(2\pi\left(\frac{3}{40}\right)n + 1\right) \\ &= 2 \cos\left[e^{j(2\pi(\frac{3}{40})n+1)} + e^{-j(2\pi(\frac{3}{40})n+1)}\right] \\ &= 2e^{j1} e^{j2\pi(\frac{3}{40})n} + 2e^{-j1} e^{-j2\pi(\frac{3}{40})n} e^{j2\pi} \\ &= 2e^{j1} e^{j2\pi(\frac{3}{40})n} + 2e^{-j1} e^{j2\pi(1-\frac{3}{40})n} \\ &= 2e^{j1} e^{j2\pi(\frac{3}{40})n} + 2e^{-j1} e^{j2\pi(\frac{37}{40})n} \end{aligned}$$

We conclude from the expression for $x[n]$ that $N_0 = 40$ samples. Next, we use the formula for DTFS representation of the same signal:

$$\begin{aligned}
 x[n] &= \sum_{k=0}^{N_0-1} \mathbf{x}_k e^{jk\Omega_0 n} \\
 &= \sum_{k=0}^{N_0-1} \mathbf{x}_k e^{j2\pi nk/N_0},
 \end{aligned}$$

Then by comparison of the two forms of equations above, leads to the conclusion that the first term corresponds to $k = 3$ and the second term corresponds to $k = 37$. Hence,

$$\mathbf{x}_3 = 2e^{j1}, \quad \mathbf{x}_{37} = 2e^{-j1},$$

and all other terms for $k = 0$ to 39 are zero.

Q14.

Solution: Parseval's theorem states that the average power of a periodic signal is the same whether it is computed in the time domain or in the frequency (DTFS) domain.

Time domain: The average power of the *periodic* sinusoid is

$$\frac{4^2}{2} = 8.$$

Frequency domain: The average powers of the two *periodic* complex exponentials are

$$|2e^{j1}|^2 + |2e^{-j1}|^2 = 8.$$

Hence, the average powers are identical.