

ECEN220 Tutorial 4 Questions (**SOLUTIONS**)

Block 4: Fourier Transform – conceptual and analytical questions

Q1. Solutions:

$$x(t) \cos(\omega_0 t) = \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \right) x(t).$$

Applying the transform property of complex exponentials, we have

$$\begin{aligned} \frac{1}{2} e^{j\omega_0 t} x(t) &\longleftrightarrow \frac{1}{2} \mathbf{X}(\omega - \omega_0), \\ \frac{1}{2} e^{-j\omega_0 t} x(t) &\longleftrightarrow \frac{1}{2} \mathbf{X}(\omega + \omega_0). \end{aligned}$$

Hence,

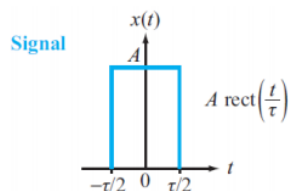
$$x(t) \cos(\omega_0 t) \longleftrightarrow \frac{1}{2} [\mathbf{X}(\omega - \omega_0) + \mathbf{X}(\omega + \omega_0)].$$

Q2.

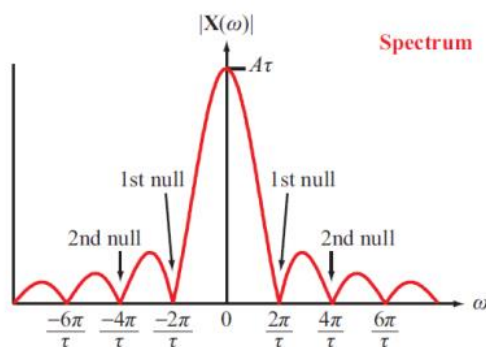
$$\text{sinc}(\theta) = \frac{\sin \theta}{\theta}.$$

$\text{sinc}(\theta)$ is an even function, and has zero-crossings at non-zero integer multiples of π . For θ a lot less than 1, $\sin(\theta)$ is approximately equal to θ . Therefore, $\text{sinc}(\theta)$ becomes $\theta/\theta = 1$.

Q3. The rectangle wave for is plotted as shown below:



And the spectrum of the Fourier Transform is shown below:



(Note: make sure you know how to derive the Fourier Transform of the rectangle wave mathematically; see lecture slides)

Q4.

The Fourier transform of a dc voltage $x(t) = A$ volts is $\mathbf{X}(\omega) = A2\pi \delta(\omega)$ (an impulse at frequency $\omega = 0$).

Q5. First nulls occur at $\pm 2\pi/\tau$. Hence, $\Delta\omega = 4\pi/\tau$. Wide τ leads to narrow spectrum.

Q6.

$$\begin{aligned}\text{sgn}(t) &\longleftrightarrow \frac{2}{j\omega}, \\ u(t) &\longleftrightarrow \pi \delta(\omega) + \frac{1}{j\omega}.\end{aligned}$$

Also,

$$\text{sgn}(t) = u(t) - u(-t).$$

Hence,

$$u(-t) = u(t) - \text{sgn}(t),$$

and the corresponding Fourier transform is

$$u(-t) \longleftrightarrow \pi \delta(\omega) + \frac{1}{j\omega} - \frac{2}{j\omega} = \boxed{\pi \delta(\omega) - \frac{1}{j\omega}}.$$

Q7. By Euler's identify, we have

$$x(t) \cos(\omega_0 t) = \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \right) x(t).$$

Applying the **frequency shift** property of Fourier Transform, we have

$$\begin{aligned}\frac{1}{2} e^{j\omega_0 t} x(t) &\longleftrightarrow \frac{1}{2} \mathbf{X}(\omega - \omega_0), \\ \frac{1}{2} e^{-j\omega_0 t} x(t) &\longleftrightarrow \frac{1}{2} \mathbf{X}(\omega + \omega_0).\end{aligned}$$

Hence,

$$x(t) \cos(\omega_0 t) \longleftrightarrow \frac{1}{2} [\mathbf{X}(\omega - \omega_0) + \mathbf{X}(\omega + \omega_0)].$$

Q8. The DTFT of $A \cos(\Omega_0 n + \theta)$ is

$$A\pi e^{j\theta} \delta((\Omega - \Omega_0)) + A\pi e^{-j\theta} \delta((\Omega + \Omega_0)),$$

where

$$\delta((\Omega - \Omega_0)) = \sum_{k=-\infty}^{\infty} \delta(\Omega + 2\pi k - \Omega_0)$$

is a chain of impulses in Ω .

Note that the DTFT of any signal is *always* periodic in Ω with period 2π .

Here, $A = 4$, $\Omega_0 = 0.15\pi$, $\theta = 1$, so the DTFT is

$$\boxed{4\pi e^{j1} \delta((\omega - 0.15)) + 4\pi e^{-j1} \delta((\omega + 0.15))}.$$

Q9. We need to write this function as $\mathbf{X}(e^{j\Omega})$ in terms of $e^{j\Omega}$, not just of Ω .

Using the definitions

$$2\cos(\Omega) = e^{j\Omega} + e^{-j\Omega}$$

and

$$2j\sin(\Omega) = e^{j\Omega} - e^{-j\Omega}$$

gives

$$\mathbf{X}(e^{j\Omega}) = [2e^{j2\Omega} + 2e^{-j2\Omega}] + [3e^{j\Omega} + 3e^{-j\Omega}] + [4e^{j2\Omega} - 4e^{-j2\Omega}] + [e^{j\Omega} - e^{-j\Omega}].$$

Summing terms,

$$\mathbf{X}(e^{j\Omega}) = 6e^{j2\Omega} + 4e^{j\Omega} + 2e^{-j\Omega} - 2e^{-j2\Omega}.$$

The DTFT is

$$\mathbf{X}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n},$$

so we can read off $\{6, 4, \underline{0}, 2, -2\}$.

Q10. The N_0 - point DFT is

$$\mathbf{X}_k = \sum_{n=0}^{N_0-1} x[n] e^{-j2\pi nk/N_0} \quad \text{for } k = 0, 1, \dots, N_0 - 1.$$

Here, $N_0 = 4$, and $e^{-j2\pi nk/4} = (-j)^{nk}$ for $k = 0, 1, 2, 3$.

Hence,

$$\mathbf{X}_0 = x[0](1) + x[1](1) + x[2](1) + x[3](1) = 4 + 3 + 2 + 1 = 10,$$

$$\mathbf{X}_1 = x[0](1) + x[1](-j) + x[2](-1) + x[3](j) = 4 - j3 - 2 + j1 = 2 - j2,$$

$$\mathbf{X}_2 = x[0](1) + x[1](-1) + x[2](1) + x[3](-1) = 4 - 3 + 2 - 1 = 2,$$

$$\mathbf{X}_3 = x[0](1) + x[1](j) + x[2](-1) + x[3](-j) = 4 + j3 - 2 - j1 = 2 + j2.$$

Since $x[n]$ is real-valued, we could also have used $\mathbf{X}_3 = \mathbf{X}_1^*$.

The DFT is $\{10, (2 - j2), 2, (2 + j2)\}$. **Check:** fft([4 3 2 1]) gives the same answer.

Q11. DTFS always has a finite number of terms, equal to the period of the signal being expanded. CTFS usually has an infinite number of terms.

Q12. DTFT applies in discrete time, and is periodic in Ω with period 2π . CTFT applies in continuous time, and is not periodic in ω .

Q13. The continuous-time sinc is $\text{sinc}(x) = \sin(x)/x$. The discrete-time sinc is $\text{sinc}(n) = \sin(\Omega n)/(\pi n)$.

Q14. DFT $X[k]$ of degree N is related to coefficients $x[k]$ of DTFS of a periodic signal with period N by $X[k] = Nx[k]$.

Q15. The DFT $X[k]$ of degree N is computed from the DTFT $X(\Omega)$ by setting $\Omega = 2\pi k/N$.